Survival Round I

Easy

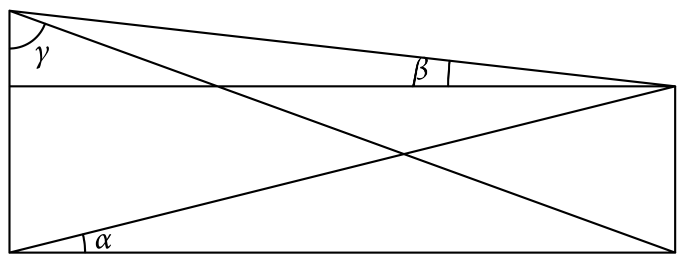
SR1-E1 *(20s)* Find the value of x that satisfies the equation $3^{x-3} + 3^{x-4} + 2(3^{x-1}) = 522$. [$x = 6$]

SR1-E2 *(15s)* The measures of the angles of a quadrilateral form an arithmetic progression. If the smallest angle has a measure 33°, find the measure of the three other angles of the quadrilateral. [71°, 109°, 147°]

SR1-E3 *(20s)* Mark and James are playing a game with a fair die. If the die lands on an even number, Mark gets a point. Otherwise, James gets a point. In a race to 9 game, the current score of Mark is 5 and James is 8. What is the probability that James will win the game? [$\frac{15}{16}$]

SR1-E4 *(15s)* Farmer Jed has 9 golden spheres of radius 3 inches. He melted these, molded them into a right circular cylinder of the same radius, and offered it to Princess Leah. What is the height of this cylinder in inches? [36 inches]

SR1-E5 *(20s)* Two vertical poles of unequal length are at a distance apart. Angles α, β and γ are given as follows. If $\tan\alpha = \frac{1}{8}$ and $\tan\gamma = \frac{11}{2}$, determine the value of $\tan\beta$. [$\frac{5}{88}$]



SR1-E6 *(15s)* The triangular region bounded by the lines $y = 0$, $x = 9$ and $y = mx$ has an area of 1296 square units. Find the value of m, if m is positive. [32]

SR1-E7 *(15s)* A line segment in the Cartesian plane has endpoints $P(5,5)$ and $Q(x,y)$. The coordinates of the midpoint of line segment PQ are positive integers with a product of 100. What is the maximum possible value of x? [195]

SR1-E8 *(20s)* Below is an excerpt from Stan's diary:

> *07 December 2009*
> *10:00 AM*
>
> *"Today is Monday. 2009 hours from now, I shall eat rice once again."*

At what time and day of the week shall Stan eat rice once again? [3:00 AM, Monday]

Average

SR1-A1 *(40s)* Evaluate $\dfrac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}}$. [$\sqrt{2}$]

SR1-A2 *(40s)* If x is a positive integer, find the smallest value of x, such that $\cos^{-1}(\cos(2009^\circ)) = \cos^{-1}(\cos((2009+x)^\circ))$. [302°]

SR1-A3 *(30s)* How many integers satisfy the inequality $1 < \log_3(\log_2 x) < 2$? [503]

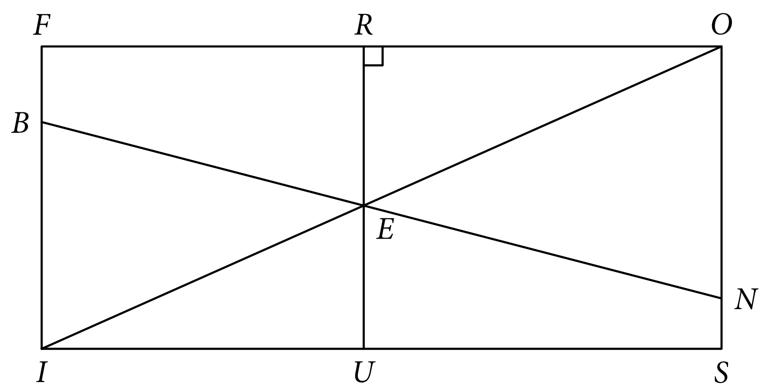
SR1-A4 *(30s)* Ruth and Tara are putting letters inside envelopes. Alone, Ruth can finish the task in 3.5 hours. Tara, on the other hand, can finish the task in 3 hours and 42 minutes. Unknown to them, Geldof is removing letters from the envelopes and is putting them in the pile of papers still to be put inside envelopes. Geldof can do this at a rate of 99 letters an hour. If there are 777 letters to be put inside envelopes, how long will the two finish the job with Geldof disrupting them?

[2 hours and 20 minutes]

SR1-A5 *(45s)* A right triangle has sides of integral length in inches. One of its sides has length 11 inches. Find the area of the circle that is inscribed in the triangle. [25π inches^2]

SR1-A6 *(45s)* How many infinite arithmetic progressions whose terms are positive integers contain the integers 1, 61, and 211? [8]

SR1-A7 *(60s)* In the figure below, $\square FOSI$ is a rectangle. If $FB : BI = 1 : 3$ and $ON : NS = 5 : 1$, what is $RE : EU$? [10 : 9]



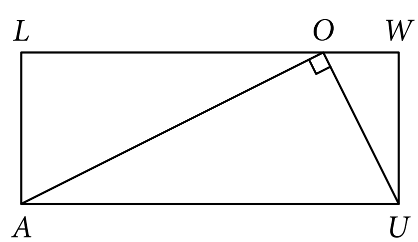
Difficult

SR1-D1 *(150s)* The lengths of the sides of a triangle are consecutive positive integers, and the largest angle is twice the smallest angle. Find the value of the cosine of the smallest angle. [$\frac{3}{4}$]

SR1-D2 *(90s)* Twenty-two points are placed evenly along the circumference of a circle. Ramon randomly picks 2 distinct points and forms the line segment connecting these points. He then picks another pair of distinct points from the remaining 20 points and forms the line segment connecting them. What is the probability that the 2 line segments intersect inside the circle? [$\frac{1}{3}$]

SR1-D3 *(120s)* What is the least integer n such that $n!$ has more than 500 terminal zeros? [2010]

SR1-D4 *(100s)* In rectangle $LWUA$, $LW = 5$ and $WU = 2$. A point O is selected on side LW such that $\triangle OUA$ is a right triangle. Find the perimeter of $\triangle OUA$. [$3\sqrt{5} + 5$]



SR1-D5 *(120s)* Given that $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$, find the value of $\sqrt{2\sqrt[3]{2\sqrt[4]{2\sqrt[5]{2\cdots}}}}$. [$2^{e-2}$]

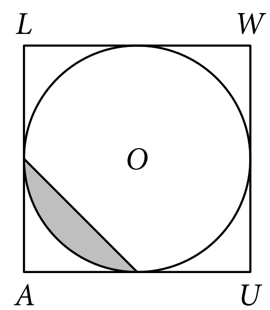
Survival Round II

Easy

SR2-E1 *(15s)* If $f(x+y) = f(x)f(y) > 0$ for all $x, y \in \mathbb{R}$ and $f(2) = \frac{1}{4}$, find the value of $f(2009)$. $\left[\dfrac{1}{2^{2009}}\right]$

SR2-E2 *(10s)* Let x, y and z be natural numbers less than 100 such that x, y and z are consecutive odd numbers that are all prime. Find the largest possible value of the sum $x + y + z$. [15]

SR2-E3 *(25s)* A circle O is inscribed in $\square LWUA$. If the area of the shaded region is $(4\pi - 8)\,\text{cm}^2$, find the perimeter of the square. [48 cm]



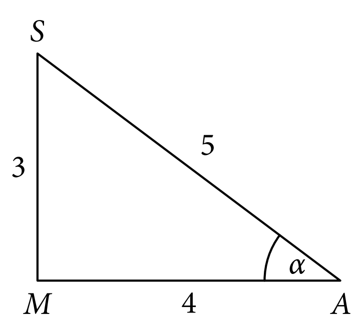
SR2-E4 *(15s)* Let $1 \le x, y \le 100$. Find the maximum value of $\dfrac{x^2 - y^2}{(x-y)^2 - y(y-x)}$. [101]

SR2-E5 *(25s)* The units of volume, High and Sierra, are related by the following equation, $H = \frac{7}{11}(S - 17)$, where H and S denote the magnitude of the object's volume in High and Sierra respectively. If the volume of an object increases by 3.5 Highs, by how much will the volume of the object increase in Sierras? [5.5 Sierras]

Average

SR2-A1 *(20s)* Simplify $1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$. $\left[\dfrac{1+\sqrt{3}}{2}\right]$

SR2-A2 *(45s)* Consider the figure below. A new triangle is constructed from this triangle by removing side MS and replacing it with a longer line segment, such that $m\angle MAS = 2\alpha$. Find the length of the new side. $\left[\dfrac{\sqrt{745}}{5}\right]$



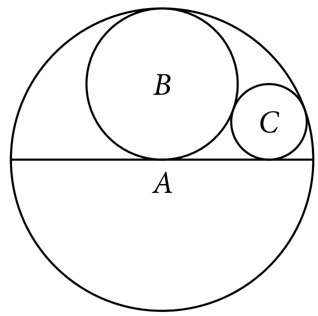
SR2-A3 *(20s)* Suppose a and b are the roots of $x^2 + 2x - 1 = 0$. Evaluate $a^4 + b^4$. [34]

SR2-A4 *(45s)* A number is selected at random from the set $\{1!, 2!, \ldots, 100!\}$. What is the probability that the number is divisible by 2009? $[\frac{3}{5}]$

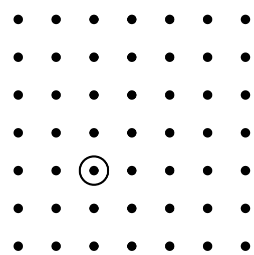
Difficult

SR2-D1 *(60s)* If a, b and c are three numbers chosen randomly and with replacement from the set $\{1, 2, 3, 4, 5\}$, find the probability that $a + bc$ is even. $[\frac{59}{125}]$

SR2-D2 *(120s)* Find the ratio of the area of circle C to the area of circle A. $[\frac{1}{16}]$



SR2-D3 *(120s)* Consider the grid system where point P is encircled as shown below. How many unique lines are determined by point P and other points on the grid system? [22]



Quaterfinal Round

Easy

QFR-E1 *(15s)* What is the sum of the digits of $2^{2009} \times 7^2 \times 5^{2011}$? [10]

QFR-E2 *(15s)* Billy's High Sierra bag contains 14 blue ballpens, 8 red ballpens, 10 black ballpens, 12 green ballpens and 16 violet ballpens. For their drawing class, he takes out a ballpen randomly from inside his bag. If he got a red ballpen, how many more ballpens must he take out to get at least four ballpens of different colors if he takes out the ballpens at random? [38]

QFR-E3 *(15s)* For what values of b will the system of equations $x^2 + bx + 1 = 0$, $x^2 - x - b = 0$ have a common real solution? [2]

QFR-E4 *(25s)* Let D be the set of all integers b such that $\dfrac{2009}{b}$ is an integer. Find the sum of all elements of D. [0]

QFR-E5 *(10s)* The four-digit $HAHA$ is divisible by 7. Find the least value of $H + A$. [3]

Average

QFR-A1 *(45s)* Consider the figure below:

A coin with radius r rolls clockwise without slipping. The coin has already rotated 5 times and some more, before it ends up in position 2. The labels A and B correspond to unique points along the circumference of the coin. Find the total area of the rectangular structure. $[13\pi r^2 + 4r^2]$

QFR-A2 *(60s)* Evaluate $\frac{1}{4(7)} + \frac{1}{5(8)} + \frac{1}{6(9)} + \frac{1}{7(10)} + \cdots$. $\left[\frac{37}{180}\right]$

QFR-A3 *(45s)* Let $\sin x + \cos x = \frac{1}{4}$. Evaluate $\sin^3 x + \cos^3 x$. $\left[\frac{47}{128}\right]$

QFR-A4 *(45s)* A line is tangent to the circle $x^2 + y^2 = 4$ at the point (a, b) in the first quadrant. If the line passes through $(4, 0)$, find the value of $a + b$. $[1 + \sqrt{3}]$

Difficult

QFR-D1 *(120s)* Find the 2009th term of the sequence $1, 1, 3, 1, 3, 5, 1, 3, 5, 7, 1, 3, 5, 7, 9, \ldots$. $[111]$

QFR-D2 *(120s)* An ant is situated at point E in a regular hexagon $ABCDEF$ with side of length 10 cm. If the ant must travel to side $\overline{BC}$ and go to point F, what is the least possible distance which the ant can travel? $[10\sqrt{13}\,\text{cm}]$

QFR-D3 *(120s)* Evaluate $\cos(\csc^{-1}(2) - \cot^{-1}(2))$. $\left[\frac{\sqrt{15} + \sqrt{10}}{10}\right]$

Semifinal Round

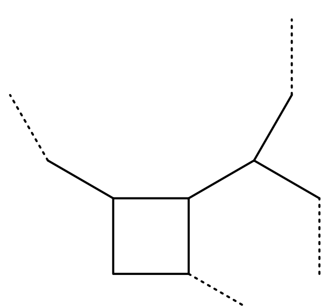
Easy

SFR-E1 *(15s)* Let a, b, c, d be complex numbers satisfying the following system of the equations:

$$\begin{cases} a^2 + b^2 + d^2 = 0 \\ a^2 + c^2 + d^2 = 1 \\ b^2 + c^2 + d^2 = 2 \\ a^2 + b^2 + c^2 = 3 \end{cases}$$

Find all possible values of $c + d$. $[\sqrt{2} \pm i, -\sqrt{2} \pm i]$

SFR-E2 *(25s)* A square, a regular m-gon, and a regular n-gon, whose side lengths are equal, are arranged as shown below. Find the value of n if $m > n$ and $m + n = 18$. $[6]$



SFR-E3 *(20s)* Unique digits are assigned to each of the letters in the set $\{A, B, G, H, I, M, N, R, T, Y\}$. What is the largest possible value for the sum $MATHIRANG + MATHIBAY$? $[1\,086\,420\,201]$

SFR-E4 *(15s)* A die is in the shape of a cuboctahedron, a semi-regular polyhedron having 8 equilateral triangle faces and 6 square faces. The triangle face is worth 3 points while the square is worth four. The probability that a triangle face shows up is $\frac{2}{3}$. After three rolls, what is the probability that the total points is at least 11? $\left[\frac{7}{27}\right]$

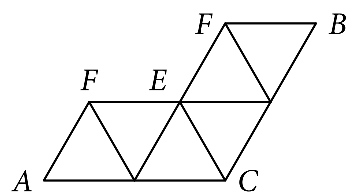
SFR-E5 *(25s)* Find all values of x satisfying the equation $16\sin^2 x + \cos^2 x - 20 = 0$. [No solution]

Average

SFR-A1 *(30s)* What is the greatest integer n such that 12^n divides $1 \times 2 \times \cdots \times 24$? [10]

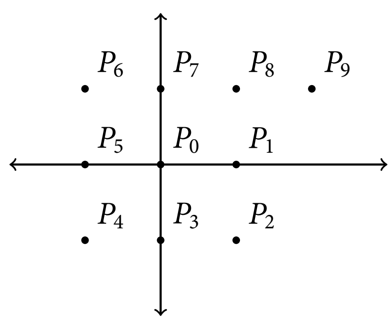
SFR-A2 *(30s)* Determine the smallest integer n for which $\log_7(\log_5(\log_3(\log_2 n)))$ is a real number. [9]

SFR-A3 *(45s)* Consider six equilateral triangles arranged as follows.



If the distance between points A and B is $10\sqrt{3}$ centimeters, what is the area of hexagon $ABCDEF$? $\left[\frac{75\sqrt{3}}{2}\text{ cm}^2\right]$

SFR-A4 *(45s)* In the accompanying figure, the lattice points are ordered in a clockwise manner. $P_0 = (0,0)$, $P_1 = (1,0)$, $P_2 = (1,-1)$, $P_3 = (0,-1)$, and so on. Find the coordinates of P_{82}. [53]



Difficult

SFR-D1 *(120s)* Let $g(xy) = g(x) + g(y)$ for positive integers x and y. Evaluate $g(1) + g(2) + \cdots + g(d) + \cdots + g(72) + g(144)$ where d is a positive divisor of 144 in terms of p and q where $p = g(2)$ and $q = g(3)$. $[30p + 15q]$

SFR-D2 *(150s)* Find A_{100} if $a_{n+1} = \dfrac{A_n}{1 + n^3 A_n}$ for $n = 0, 1, 2, \ldots$ and $A_0 = 1$. $\left[\dfrac{1}{24\,502\,501}\right]$

SFR-D3 *(75s)* A bendable wire of fixed length is first folded into an equilateral triangle, then a square, then a regular pentagon, and so on, with the two endpoints joined together as one of the vertices of the resulting polygon. Every fold made on the wire makes a permanent mark on the wire. After bending the wire into an octagon, how many marks are there? [21]

Final Round

Easy

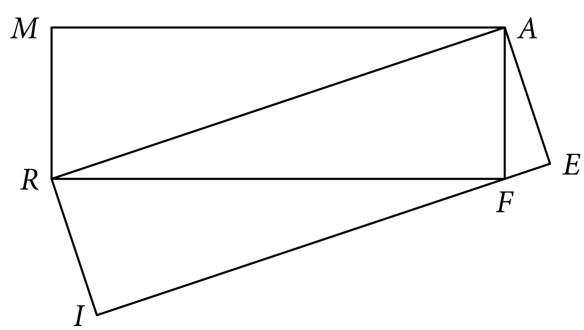
FR-E1 *(15s)* Let $m, s, a > 0$, $m + s - a = 4$ and $ms = 30$, evaluate $(\sqrt{m} + \sqrt{s} + \sqrt{a})(\sqrt{m} + \sqrt{s} - \sqrt{a})(\sqrt{m} - \sqrt{s} + \sqrt{a})(-\sqrt{m} + \sqrt{s} + \sqrt{a})$. $[104]$

FR-E2 *(20s)* Blusom, Batercap, and Babols are working on a project to rebuild MSA Tutorial Center and it will take them x, $2x$, and $3x$ hours respectively to finish the work alone. When they divided the work equally, the sum of the individual work time is 12 hours. How many hours will it take them to finish it working together? $[\frac{36}{11}$ hours$]$

FR-E3 *(20s)* Define a sequence of real numbers $a_1, a_2, a_3, \ldots, a_n$ by $a_1 = 1$ and $an + 1^4 = 2a_n^4$ for all $n \geq 1$. Find a_{2009}. $[\pm 2^{502}]$

FR-E4 *(25s)* Te challenged Baron to a game of dice. Three fair dice are tossed. If one number is the mean of the other two numbers, Te wins the game and Baron leaves the org, otherwise Baron wins the game and Te leaves the org. What is the probability that Baron leaves the org? $[\frac{7}{36}]$

FR-E5 *(25s)* In the figure below, □$RMAF$ and □$RAEI$ are rectangles. If $RM = 1$ and $MA = 3$, what is the perimeter of □$RAEI$? $\left[\frac{13\sqrt{10}}{5}\right]$



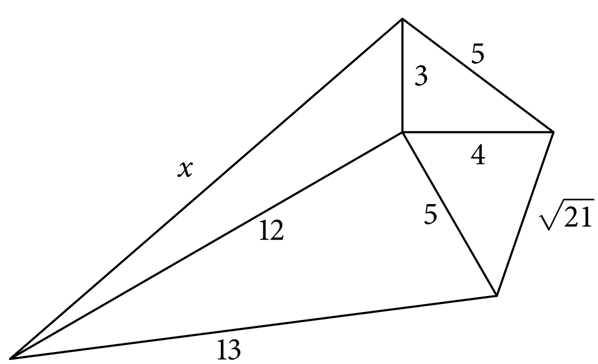
Average

FR-A1 *(60s)* Let $R = \{14, 28, 42, 56, 70, \ldots\}$ and $M = \{15, 30, 45, 60, 75, \ldots\}$. An integer n is randomly picked from $R \cup M$. What is the probability that 21 divides n? $[\frac{3}{14}]$

FR-A2 *(60s)* Let n be a natural number. Find the sum of all n such that $n^2 + 1$ divides $n^5 + n^4 + n^3 + n^2 + 100$. $[13]$

FR-A3 *(50s)* $ABCD$ is a parallelogram and F is a point on $\overline{BC}$. Let G be the point of intersection of $\overline{AF}$ and $\overline{BD}$ and let E be the point of intersection of the extensions of $\overline{AF}$ and $\overline{DC}$. If $AG = 3$ and $GF = 2$, find FE. $[\frac{5}{2}]$

FR-A4 *(90s)* Consider the quadrilateral shown below. Find the value of x. $[3\sqrt{21}]$



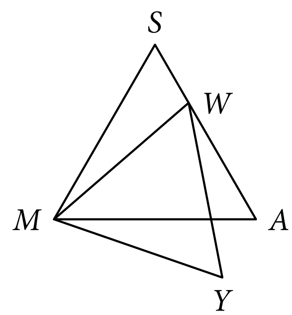
FR-A5 *(45s)* Find the 21st term in the sequence of Roman numerals XX, XLVIII, LXXXIV, CXXVIII, CLXXX, ... [MMC]

Difficult

FR-D1 *(150s)* Let MSA be a triangle and let D be a point on $\overline{MA}$ such that $\overline{SD}$ bisects $\angle S$. If $\sin M = \frac{4}{5}$ and if the ratio of the area of $\triangle ADS$ and $\triangle MDS$ is $8:5$, determine $\sin A$. $\left[\frac{1}{2}\right]$

FR-D2 *(150s)* Let $S_n = n\left(18 - \dfrac{n-1}{2}\right)$. Evaluate $S_1 + S_3 + S_5 + \cdots + S_{35}$. [2109]

FR-D3 *(150s)* Let $\triangle MSA$ be equilateral. Let W be a point on $\overline{SA}$ such that $SW = 1$ and $WA = 2$. Let Y be a point outside $\triangle MSA$ such that $\triangle MWY$ is equilateral, as shown in the figure. Let X be the intersection of $\overline{MA}$ and $\overline{WY}$. Find WX. $\left[\dfrac{2\sqrt{7}}{3}\right]$



FR-D4 *(150s)* In a party, seven guests Do, Re, Mi, Fa, Sol, La and Ti want to have their pictures taken with at least one of the other six. However, Do and Ti, as well as Mi and Fa, do not go along well with each other, and because of this, they don't want to be seen in any picture with the other. How many pictures are possible? [64]

FR-D5 *(100s)* The night of December 6, 2009, Heart gave birth to quintuplets. It is known that the probability of Heart giving birth to a boy is $\frac{1}{2}$. What is the probability that on that fateful night, Heart gave birth to two boys and three girls given that one of her children is a boy? $\left[\frac{10}{31}\right]$