PMO 1999

7th Philippine Mathematical Olympiad

March 4, 1999

Pre-Final National Competition Phase
Level III Individual Stage

Time allotment: 2.5 hours
Each question is worth 8 points.

1. Let r_1 and r_2 be the roots of the quadratic equation

$$x^2 + ax + \frac{a^2 - 1}{2} = 0$$

   Find $r_1^3 + r_2^3$ in terms of a.

2. The formula for surface area and volume of a sphere are resepctively $4\pi r^2$ and $\frac{4}{3}\pi r^3$ where r is the radius of the sphere. If the area and volume of a sphere are both 4-digit multiples of π, what is the radius of the sphere?

3. Solve the following inequality:

$$\frac{5}{6(x+2)} + \frac{49}{6(x-4)} \leq -2$$

4. If $\triangle ABC$ is a right triangle with right angle at A and if the medians to AC and AB are drawn and extended to meet the line through A parallel to BC at points D and E respectively, prove that

$$BD^2 + CE^2 = 5BC^2$$

5. If a, b and c are positive numbers such that $a + b + c = 2k$, show that $ab + bc$ is never greater than k^2.

Pre-Final National Competition Phase
Level IV Individual Stage

Time allotment: 2.5 hours
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1. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function; that is, if $x_1 < x_2$ then $f(x_1) < f(x_2)$. Prove that x_0 is a root of $x = f(f(x))$ if and only if x_0 is a root of $x = f(x)$.

2. For which real numbers x does the inequality

$$2\log_x(a+b) \le \log_x a + \log_x b + \log_x 4$$

   hold for every positive numbers a and b.

3. Consider two circles that intersect at points A and B. Let a line through B meet the circles at K and M respectively. Let E and F be the midpoints of arcs AK and AM respectively (the arcs that don't contain B), and let L be the midpoint of segment KM. Prove that $\angle ELF$ is a right angle.

4. For every real number x, define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. If $x > 2$, what is $\left\lfloor \dfrac{1999x - 1998}{2x - 2} \right\rfloor$?

5. Let $p_1, p_2, p_3, \ldots, p_n$ be a sequence of n distinct primes such that $p_1 < p_2 < p_3 < \cdots < p_n$. Find the value of n such that

$$\left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{p_2}\right)\cdots\left(1 + \frac{1}{p_n}\right)$$

   is an integer.

Grand Final National Competition Phase
Level III Individual Stage

Time allotment: 2.5 hours
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1. The interior angles of a convex polygon of 9 sides are in arithmetic progression with positive common difference d. Find the least integer k such that $d < k$.

2. Find two numbers such that their sum multiplied by the sum of their squares is 405 and their difference multiplied by the difference of their squares is 81.

3. An integer m lies between 2 consecutive perfect squares and differs from them by a and b. Prove that $m - ab$ is also a perfect square.

4. D and E are respective points of sides AB and BC of $\triangle ABC$, so that

   $$\frac{AD}{DB} = \frac{2}{3} \text{ and } \frac{BE}{EC} = \frac{1}{4}$$

   If AE and DC meet at P, find $\dfrac{PC}{DP}$.

5. Three men, Arnel, Ben and Carlo went with their wives to the market to buy some fruits. The wives' names are Lita Minda and Nancy, though not necessarily in that order. By a strange chance, the average price that each person paid for his (her) fruits was the same as the actual number of fruits that he (she) bought. (Thus, if Arnel bought A fruits, he paid an average of A pesos per fruit.)

   Arnel bought 73 more fruits than Minda and Carlo bought 11 more fruits than Lita. Each man spent 175 pesos more than his wife.

   Who was married to whom?

Grand Final National Competition Phase
Level IV Individual Stage

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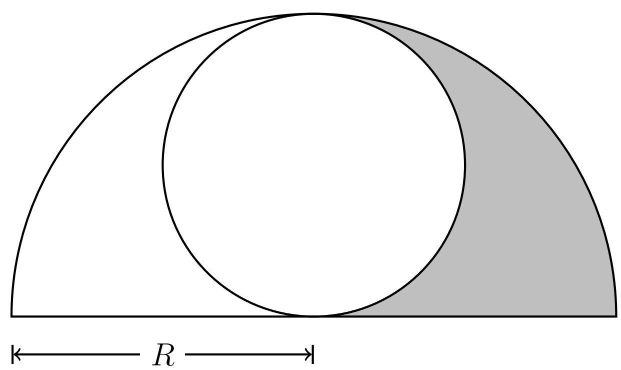
1. Find all integers $n \geq 1$ such that $\log_{2n-1}(n^2+2)$ is a rational number.
2. What is the largest positive integer n for which there is a unique integer k such that $\dfrac{8}{15} < \dfrac{n}{n+k} < \dfrac{7}{13}$?
3. A set of 1999 distinct positive integers is given such that none of these numbers is a sum of any two distinct numbers from the set. What is the last possible value of the greatest number from such a set.
4. A circle of maximum radius is inscribed in a semicircle of radius R as shown in the figure.



   What is the radius of the largest circle that can be drawn in the shaded region?

5. An elevator starts on the top floor of a 100-floor building and in its descent to the bottom (first floor) stops at least 40 floors, counting both the top and bottom floors as stops. Show that somewhere in its travel the elevator had to stop at two floors that were 9, 10, or 19 floors apart. (Note: In this problem, two floors a and b are c floors apart if $|a-b| = c$.)