Problem 1. Find all nonempty finite sets X of real numbers with the following property:

$$x + |x| \in X \quad \text{for all } x \in X.$$

Solution. Let $X = \{x_1, x_2, \ldots, x_n\}$, $n \geq 1$, where $x_1 < x_2 < \cdots < x_n$.

If $x_n > 0$, then $x_n + |x_n| = 2x_n \in X$, which is a contradiction because $x_n < 2x_n$ but x_n is the largest element of X.

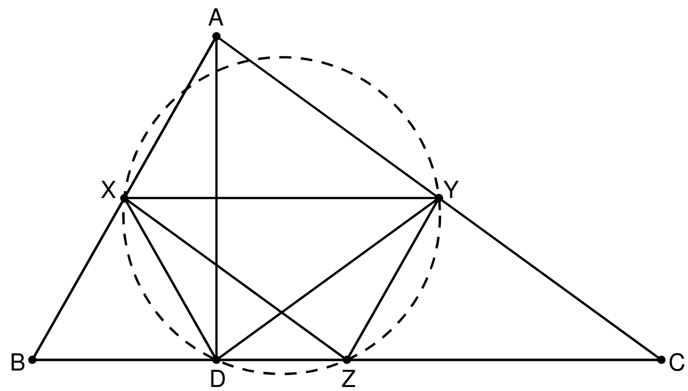
The contradiction in the previous paragraph implies that $x_n \leq 0$. If $x_1 < 0$, then $x_1 + |x_1| = x_1 - x_1 = 0 \in X$. Hence, we must have $x_n = 0$, so that $x_i + |x_i| = x_i - x_i = 0 \in X$ for any $x_i \in X$, $i = 1, 2, \ldots, n$.

Hence, in order for the desired property to be satisfied, X must be a finite subset of the interval $(-\infty, 0]$ and it must contain 0. On the other hand, such subsets satisfy the said property.

The only nonempty finite sets that satisfy the desired property are those finite subsets of $(-\infty, 0]$ containing 0. Q.E.D.

Problem 2. In $\triangle ABC$, let X and Y be the midpoints of AB and AC, respectively. On segment BC, there is a point D, different from its midpoint, such that $\angle XDY = \angle BAC$. Prove that AD is perpendicular to BC.

Solution. Let Z be the midpoint of BC. Without loss of generality, we assume that D is between B and Z. Then $XY \| BC$, $XZ \| AC$, and $YZ \| AB$, and so $\angle ABC = \angle XYZ$ and $\angle XDY = \angle BAC = \angle XZY$. It follows that quadrilateral $XYZD$ is cyclic.



Since opposite angles of a cyclic quadrilateral are supplementary, we have

$$\angle XDB = 180^\circ - \angle XDZ = \angle XYZ = \angle ABC,$$

which implies that $XA = XB = XD$. Therefore, AB is a diameter of the circumcircle of $\triangle ABD$, and so $\angle ADB = 90^\circ$. Q.E.D.

Problem 3. The 2011th prime number is 17483, and the next prime is 17489.

Does there exist a sequence of 2011^{2011} consecutive positive integers that contains exactly 2011 prime numbers? Prove your answer.

Solution. Let $N = 2011^{2011}$. Since $N > 17489$, there are more than 2011 primes in the sequence $1, 2, 3, \ldots, N$.

CLAIM. There exists a sequence of N consecutive positive integers that are all composite.

Proof of the Claim. The sequence

$$(N+1)! + 2, (N+1)! + 3, (N+1)! + 4, \ldots, (N+1)! + N, (N+1)! + (N+1)$$

consists of N consecutive composite numbers, and so the claim follows.

Back to the solution of the problem, let

$$a, a+1, a+2, \ldots, a+N-1 \tag{$\star$}$$

be a sequence of N consecutive positive integers with no prime. We do repeatedly the following operation to the numbers in ($\star$). Delete the far-right number $a + N - 1$, and append to the far-left the number $a - 1$. The resulting sequence

$$a-1, a, a+1, \ldots, a+N-2$$

has at most one prime. Repeating this operation until we reach the sequence $1, 2, 3, \ldots, N$, which has more than 2011 primes.

Performing such operation either retains, increases by one, or decreases by one the number of primes of the previous sequence. Since the starting sequence has no prime at all, while the last sequence has more than 2011 primes, there exists a sequence (after applying the operation a number of times) that contains exactly 2011 primes. Q.E.D.

Problem 4. Find all (if there is one) functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following functional equation:

$$f(f(x)) + xf(x) = 1 \quad \text{for all } x \in \mathbb{R}. \tag{1$\star$}$$

Solution. We prove that no such function exists.

Suppose there exists a function f that satisfies (1$\star$). Let $f(0) = a$.

Set $x = 0$ in (1$\star$) to get

$$f(a) = 1, \tag{2$\star$}$$

which implies, after setting $x = a$ in (1$\star$), that

$$f(1) = 1 - a. \tag{3$\star$}$$

This last equation and setting $x = 1$ in (1$\star$) yield

$$f(1-a) = f(f(1)) = 1 - f(1) = a.$$

Further, with the preceding equation and setting $x = 1 - a$ in (1$\star$), we have

$$f(a) = f(f(1-a)) = 1 - (1-a)f(1-a) = 1 - (1-a)a = 1 - a + a^2.$$

This last equation and (2$\star$) give $a = 0$ or $a = 1$.

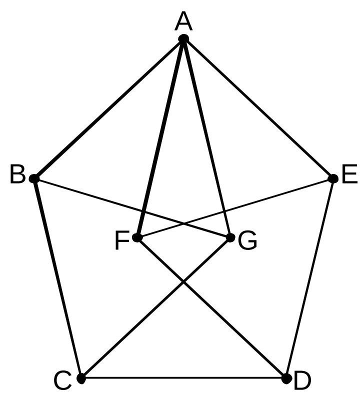
If $a = 0$, then $f(0) = 0$, which contradicts (2$\star$).

If $a = 1$, then setting $x = 0$ in (1$\star$) gives $f(1) = 1$, which contradicts (3$\star$).

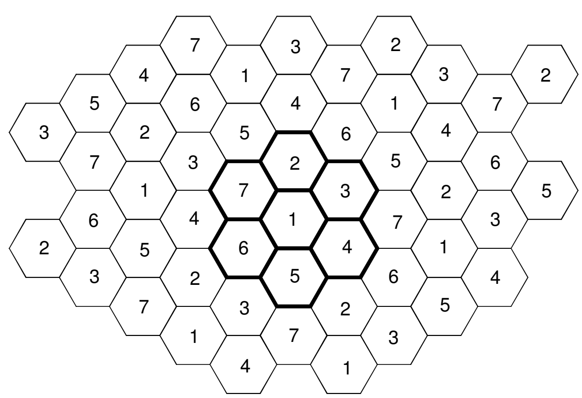
Problem 5. The *chromatic number of the (infinite) plane*, denoted by χ, is the smallest number of colors with which we can color the points on the plane in such a way that no two points of the same color are one unit apart.

Prove that $4 \leq \chi \leq 7$.

Solution. Suppose $\chi \leq 3$. Consider the following configuration, where each segment has unit length. Then the points A, B, and G must receive different colors, and so are the points A, E, and F. This will force points C and D to receive the same color as A, which is a contradiction. Thus, we obtain $\chi \geq 4$.



On the other hand, we will exhibit a coloring of the points on the plane using 7 colors in such a way that points one unit apart have different colors. We first tile the plane by regular hexagons with unit sides. Now, we color one hexagon with color 1, and its six neighbors with colors $2, 3, \ldots, 7$, as highlighted in the following diagram.



The union of the seven highlighted hexagons forms a symmetric polygon P of 18 sides. Translates of P also tile the plane and determine how we color the plane using 7 colors.

It is easy to compute that each color does not have monochromatic segments of any length d, where $2 < d < \sqrt{7}$. Thus, if we proportionally shrink the configuration by a factor of, say, 2.1, we will get 7-coloring that has no monochromatic segments of unit length, which implies that $\chi \leq 7$. Q.E.D.