VCSMS PRIME

Session 1: Algebra 1
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Domain and range

1. (11AI9) Find the range of 2^{x^2-4x+1} as x ranges over the real numbers.
2. (13QII6) Find the domain and range of $f(x) = \dfrac{6}{5\sqrt{x^2-10x+29}-2}$.
3. (13NA3) Find the area of the domain of $f(x,y) = \sqrt{25-x^2-y^2} - \sqrt{|x|-y}$.
4. (11NA4) Find the domain of $f(x) = \dfrac{1}{\lfloor x^2-x-2 \rfloor}$.
5. (13QIII3, 14QII1) Find the range of $f(x) = \dfrac{2\cdot 3^{-x}-1}{3^{-x}-2}$ and $g(x) = \dfrac{4^{x+1}-3}{4^x+1}$.
6. (15AI12) Suppose that $1-y = \dfrac{9e^x+2}{12e^x+3}$. Find the integer m such that $m < \dfrac{1}{y} < m+1$ for all real x.
7. (16NE3) Let $f(x) = \ln x$. What are the values of x in the domain of $(f\circ f\circ f\circ f\circ f)(x)$?
8. (13QIII5) Find the range of the following function, where a, b, c are distinct real numbers.

$$f(x) = \frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)}.$$

Logarithms

1. (13QII1) Find $\log_2\left(2^3\cdot 4^4\cdot 8^5\cdot\dots\cdot(2^{20})^{22}\right)$.
2. (9QI7) How many real roots does $\log_{(x^2-3x)^3} 4 = \dfrac{2}{3}$ have?
3. (10NE14) How many times does the graph of $y = |\log_{\frac{1}{2}}|x|| - 1$ cross the x-axis?
4. (15AI13) The product of two roots of $\sqrt{2014}x^{\log_{2014}x} = x^{2014}$ is an integer. Find its units digit.
5. (13QI3) Given $xy = 10^a, yz = 10^b, zx = 10^c$, find $\log x + \log y + \log z$.
6. (10NE10) Given $a = \log_{14} 16$, express $\log_8 14$ in terms of a.

Exponents

1. Solve for x:
   a) (11QI3) $2^{2^x} = 4^3$
   b) (9QI14) $x^x = x^2$
   c) (9AI6) $x^{x^x} = (x^x)^x$
   d) (10NE3) $x^{x^{2010}} = x^{2010}$
2. (10NE1) Find the smallest integer n such that $n^{300} > 3^{500}$.
3. (16QII7) Arrange from least to greatest: $25^{12}, 16^{14}, 11^{16}$.
4. (11QII9) Given $9^{2x} - 9^{2x-1} = 8\sqrt{3}$, find $(2x-1)^{2x}$.

More logarithms

1. (15AII1) Arrange in ascending order: $\log_3 2, \log_5 3, \log_{625} 75, \dfrac{2}{3}$.
2. (13QI12) Given $\dfrac{\log_2 x}{\log_2 2x - \log_8 2} = 3$, find $1 + x + x^2 + \cdots$.
3. (9NA6) Suppose that $a \geq b > 1$. Find the maximum value of $\log_a \dfrac{a}{b} + \log_b \dfrac{b}{a}$.
4. (11NE6) There exists positive integers k, m, n whose greatest common divisor is 1 such that $k \log_{400} 5 + m \log_{400} 2 = n$. Find $k + m + n$.
5. (11NE12) Let $f(m) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$, where there are m twos. Find the least integer m such that $\log f(m) > 6$.
6. (14AI9) Solve for $x : \log(5^{\frac{1}{x}} + 5^3) < \log 6 + \log 5^{\left(1+\frac{1}{2x}\right)}$.
7. (11QI15) Solve for $x : \log x \geq \log 2 + \log(x - 1)$.

Floor, ceiling, fractional

1. (11NE5) Solve for $x : 2\lfloor x \rfloor = x + 2\{x\}$.
2. (14ND5) Solve for $x : 2x(x - \lfloor x \rfloor) = \lfloor x \rfloor^2$.
3. (13AI17) The number x is chosen randomly from the interval $(0, 1]$. Define $y = \lceil \log_4 x \rceil$. Find the sum of the lengths of all subintervals of $(0, 1]$ for which y is odd.

Value-finding

1. (11QI6) Given $f(1) = 5, f(x + 1) = 2f(x) + 1$, find $f(7) - f(0)$.
2. (13QII8) Suppose $f : \mathbb{R}^* \to \mathbb{R}^*$ and $f(a) + \dfrac{1}{f(b)} = f\left(\dfrac{1}{a}\right) + f(b)$. Find all possible values of $f(1) - f(-1)$.
3. (16NE2) A function $f(x)$ satisfies $(2 - x)f(x) - 2f(3 - x) = -x^3 + 5x - 18$ for all real x. Find $f(0)$.

Cauchy functional equation

1. (14QII5) Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x + y) = f(x)f(y), f(xy) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$. Find $f(\pi^{2013})$.
2. (9QIII3) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $f(a + b) = f(a) + f(b)$. Given $f(2008) = 3012$, find $f(2009)$.
3. (13NE6) Given $f : \mathbb{R} \to \mathbb{R}$ and $f(a + b) = f(a)f(b)$. If $f(4) = 625$, what is $3f(-2)$?

Other functional equations

1. (10NE5) Given $f : \mathbb{R} \to \mathbb{R}^*$, and for all $x, y \in \mathbb{R}, f(x - y) = 2009f(x)f(y)$, find $f(\sqrt{2009})$.
2. (14QIII2) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$ and $f(2xy - 1) = f(x)f(y) - f(x) - 2y - 1$.
3. (16QIII2) If $f : \mathbb{R} \to \mathbb{R}, f(5) = 3$ and $f(4xy) = 2y[f(x + y) + f(x - y)]$, find $f(2015)$.
4. (10N3) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $x + f(x) + 2f\left(\dfrac{x + 2009}{x - 1}\right) = 2010$, for all $x \in \mathbb{R}$.
5. (11N4) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) + xf(x) = 1$, for all $x \in \mathbb{R}$.