VCSMS PRIME

Session 3: Number theory
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September 28, 2016

Ad hoc

1. Each factor of 5 in 126! has a corresponding factor of 2 to produce a trailing zero, so we only need to count the number of factors of 5. It is well-known to be, by Legendre's formula, $\left\lfloor \frac{126}{5} \right\rfloor + \left\lfloor \frac{126}{25} \right\rfloor + \left\lfloor \frac{126}{125} \right\rfloor = 25 + 5 + 1 = 31$.

2. By Legendre's, $\left\lfloor \frac{27}{2} \right\rfloor + \left\lfloor \frac{27}{4} \right\rfloor + \left\lfloor \frac{27}{8} \right\rfloor + \left\lfloor \frac{27}{16} \right\rfloor = 13 + 6 + 3 + 1 = 23$.

3. The answer is 26. Selecting all 25 even numbers has no two relatively prime; by Pigeonhole, selecting 26 will guarantee two consecutive numbers are selected, which are relatively prime.

4. It is easy to verify the cases $n = 0, 1$ to not produce perfect squares. Suppose $n \geq 2$ and factor out 7^2 to produce $7^2\left(7^{n-2} + 9\right)$. Since the first factor is a perfect square, so should the second.

   Let $m^2 = 7^{n-2} + 9$, so $7^{n-2} = m^2 - 3^2 = (m-3)(m+3)$. Then both $m - 3$ and $m + 3$ are two powers of 7 differing by 6, and since the difference between consecutive powers of 7 increases, the only possible choice is $m = 4$, giving $n = 3$. The answer is 1.

5. Factoring out 2^8 gives $2^8\left(1 + 2^3\right) + 2^n = 2^n + 2^8 3^2$. Let $m^2 = 2^n + 2^8 3^2$, and transposing and using the difference of two squares gives $2^n = (m - 48)(m + 48)$. Then $m - 48$ and $m + 48$ are two powers of two that differ by 96, the only possible pair being 32 and 128, giving $n = 12$.

6. Since $abcde$ is divisible by 5, the only choice for e must be 5. There are only three even-numbered digits, and b, e, f must all be even, so they match to b, e, f in some order. This leaves 1 and 3 for a and c.

   Wishing to maximize, we try $a = 3$. Then $c = 1$, and the number so far is $3b1d5f$. The condition of ab being divisible by 2 is guaranteed, and so is the condition of $abcdef$ being divisible by 6; we are concerned about abc being divisible by 3 and $abcd$ being divisible by 4. The first forces $b = 2$ and the second forces $d = 6$, so the number is 321654.

7. The number N should be the largest power of 2 dividing 10!. By Legendre's formula, the largest power is $\left\lfloor \frac{10}{2} \right\rfloor + \left\lfloor \frac{10}{4} \right\rfloor + \left\lfloor \frac{10}{8} \right\rfloor = 5 + 2 + 1 = 8$, so $N = 2^8$. Thus $2x + y = 2^8$, and we maximize x^2y^2, or $\left(x(2^8 - 2x)\right)^2$. The base is a quadratic with vertex at $x = 2^6$, with value 2^{13}, and its square is thus 2^{26}

8. Since P is divisible by all prime numbers less than 90, for $P + n$ to have a prime factor less than 90, so must n. All $n < 90$ work for trivial reasons, and so do $90, \ldots, 96$, failing at $n = 97$ since it is a prime. Thus the largest N is 96.

Factors

1. The fifth largest divisor corresponds to the fifth smallest divisor upon division. $2{,}015{,}000{,}000 = 2015 \cdot 10^6 = 5 \cdot 13 \cdot 31 \cdot 2^6 \cdot 5^6$, and its smallest divisors are, in order, $1, 2, 4, 5, 8$. Dividing the number by 2^3 leaves $5 \cdot 13 \cdot 31 \cdot 2^3 \cdot 5^6 = 251{,}875{,}000$.

2. The even positive divisors of 1152 are precisely the positive divisors of $1152 \div 2 = 576$ times two, so it remains to find the sum of all its divisors. Since $576 = 2^6 3^2$, the well-known formula for the sum of divisors gives $\left(1 + 2 + \cdots + 2^6\right)\left(1 + 3 + 3^2\right) = \left(2^7 - 1\right)(13) = 1651$, multiplying by 2 gives 3302.

3. By the formula for the number of divisors, the number must either be a product of two primes or the cube of a prime. The first three numbers are $6, 8, 10$, and the fourth is 14.

4. The power of 5 in the LHS is 2, which means that the power of 5 in the RHS is 2 as well, so $y = 2$. Then power of 3 in the RHS is 2, so the power of 3 in the LHS, $2x$, should equal to 2. Thus $x = 1$.

5. The highest power of 7 less than one million is 7^7, so there are 8 factors smaller than a million. The rest of the 10,000 factors are larger, so there are 9992 such factors.

6. Factoring out 5^x gives $5^x(1 + 2 \cdot 5) = 5^x 11$. The number of factors formula gives $(x + 1)2 = 2x + 2$ factors.

7. Multiplying the two equations and taking the square root gives $p = 2^2 \cdot 5^3 \cdot 7^2 \cdot 11$, which has $(2 + 1)(3 + 1)(2 + 1)(1 + 1) = 72$ divisors.

8. For each factor of n^2 less than n, dividing through n^2 gives a corresponding factor greater than n. Thus the number of factors of n^2, minus one to account for n, divided by 2, gives the number of its factors less than n. Then we subtract the number of factors of n.

   In this case, $n^2 = 2^{62}3^{38}$ which has $(62 + 1)(38 + 1) = 2457$ factors, $\dfrac{2457 - 1}{2} = 1228$ of which are less than n. The number n itself has $(31 + 1)(19 + 1) = 640$ factors, so subtracting gives $1228 - 640 = 588$ factors.

9. The number is $300^3 + 1 = (300 + 1)(300^2 - 300 + 1)$. The former, 301, factors as $7 \cdot 43$. The latter factor is $300^2 - 300 + 1 = 300^2 + 600 + 1 - 900 = (300 + 1)^2 - 30^2 = (301 - 30)(301 + 30)$, and both 271 and 331 are prime. The sum is $7 + 43 + 271 + 331 = 652$.

10. Factor out 3^19 from the first two terms to leave $3^19\,(3 + 1) - 12$. Factor out 12 to leave $12\left(3^18 - 1\right)$, which factors by repeatedly using difference of two squares and cubes as $(3 - 1)(3^2 + 3 + 1)(3^6 + 3^3 + 1)(3 + 1)(3^2 - 3 + 1)(3^6 - 3^3 + 1)$. After tedious checking, the factorization is $2^5 \cdot 3 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 757$.

11. The number $360{,}000 = 2^6 \cdot 3^2 \cdot 5^4$ has $(6 + 1)(2 + 1)(4 + 1) = 105$ factors. Since the factors of 360,000 pair up, each of them multiplying to 360,000, and there being $\dfrac{105}{2}$ pairs, the product of all the factors is $(360{,}000)^{\frac{105}{2}}$. Expanding, $\left(2^6 \cdot 3^2 \cdot 5^4\right)^{\frac{105}{2}}$ has sum of exponents $\dfrac{105}{2}(6 + 2 + 4) = 630$.

12. Suppose $f(r) = 0$ for some integer r, and then $f(x) = (x - r)g(x)$ for some polynomial $g(x)$. Let the four integers be a, b, c, d. Substituting a gives $f(a) = p = (a - r)g(a)$, so $a - r$ is a factor of p. Similarly, $b - r, c - r, d - r$ are all factors of p. Since these are all distinct, they must be $-p, -1, 1, p$ in some order.

    Then, from above, $f(-p + r) = p = (-p)g(-p + r)$ implies $g(-p + r) = -1$; similarly, $f(p + r) = p = pg(p + r)$, so $g(p + r) = 1$. However, it is well-known that $a - b$ is a factor of $f(a) - f(b)$; applying this shows $(p + r) - (-p + r) = 2p$ is a factor of $1 - (-1) = 2$, which is impossible.

Divisibility

1. Dividing gives $\dfrac{n + 3}{n - 1} = 1 + \dfrac{4}{n - 1}$, so we must have $n - 1 | 4$. Since 4 has factors $-4, -2, -1, 1, 2, 4$, the number of possible values of n is the same, 6.

2. Dividing gives $2n^2 - n + 1\dfrac{31}{3n + 1}$. As 31 is a prime, $3n + 1$ must equal either $-31, -1, 1$ or 31, which happens only for integers $n = 0, 10$.

3. The greatest common factor of $7^4 - 1 = 2^5 \cdot 3 \cdot 5^2$ and $11^4 - 1 = 2^4 \cdot 3 \cdot 5 \cdot 61$ is $2^4 \cdot 3 \cdot 5$. We show that all $p^4 - 1$ are divisible by $2^4 \cdot 3 \cdot 5$. Note that $p^4 - 1 = (p^2 + 1)(p - 1)(p + 1)$.

   Since p is odd, $p^2 + 1$ is even, and $p - 1, p + 1$ are consecutive even integers, so their product is divisible by 8. When divided by 3, p gives a remainder of 1 or 2; in the former, $3 | p - 1$, in the latter, $3 | p + 1$. Similarly, it is always divisible by 5, as $5 | p - 1$ and $5 | p + 1$ when it has remainder 1 or 4, and $5 | p^2 + 1$ otherwise. The greatest common factor is thus $2^4 \cdot 3 \cdot 5 = 240$.

4. Rationalizing the denominator gives $\dfrac{2013ab - bc + \left(b^2 - ac\right)\sqrt{2013}}{2013b^2 - c^2}$. For this to be rational, the irrational part must be zero, so $b^2 = ac$. Thus a, b, c are in geometric sequence. Rewrite a, b, c as a, ar, ar^2.

   Then $\dfrac{a^2 + b^2 + c^2}{a + b + c} = \dfrac{a^2 + a^2r^2 + a^2r^4}{a + ar + ar^2} = a(r^2 - r + 1)$ after long division. Similarly, $\dfrac{a^3 - 2b^3 + c^3}{a + b + c} = a^2(r^4 - r^3 - r + 1)$. These are both integers.

5. Multiply both sides by $x + y$ and transpose to obtain $xy - 1000x - 1000y = 0$. Add 1,000,000 to both sides and factor to get $(x - 1000)(y - 1000) = 1{,}000{,}000$. It is easy to rule out the case where both factors in the LHS are negative: they cannot both be -1000, and one must be smaller than -1000, meaning either x or y must be negative.

   Thus both are positive, and each factor of $1{,}000{,}000 = 2^6 \cdot 5^6$ corresponds to one positive integer pair. Since it has $6 + 1)(6 + 1) = 49$ factors, then there are 49 pairs.

6. It is well-known that all primes greater than 3 are either 1 or -1 modulo 6. Note that a number that is -1 modulo 6 cannot be divisible by 2 or 3. If none of its prime factors were -1 modulo 6, then all of its prime factors are 1, and their product would be 1 as well, contradiction. Therefore there must be a prime that is -1 modulo 6 that divides it.

   Suppose finitely many primes existed that are -1 modulo 6; multiplying them and adding either 4 or 6 (depending on number of primes) produces a new number that is also -1 modulo 6. This number must be composite, and by the above, divisible by a prime that is -1 modulo 6. But when divided by any such prime, it leaves a remainder of either 4 or 6, contradiction.

Diophantine equations

1. Since both $2x$ and 100 are even, so is $5y$, and thus y is even as well. Any even y produces an integer solution, the ones that give positive solutions are $y = 2, 4, \ldots, 18$. Thus there are 9 ordered pairs.

2. Since $2^{3x} + 5^{3y} = (2^x + 5^y)\, 2^{2x} - 2^x \cdot 5^y + 5^{2y} = 189$. The factors of 189 are $1 \cdot 189, 3 \cdot 63, 7 \cdot 27, 9 \cdot 21$. The only pair that works is $9 \cdot 21$, giving the only values $x = 2$, $y = 1$.

3. Adding twice the second equation to the first gives $5x = 56 - 3a$, and subtracting the second equation from twice the first gives $5y = 4a - 13$. Since $56 - 3a$ and $4a - 13$ are integers divisible by 5, their sum, $a + 43$, is divisible by 5, so a is an integer as well, and it is 2 modulo 5. Both $56 - 3a$ and $4a - 13$ have to be positive, so a is at least 4 and at most 18. The only integers in this range that are 2 modulo 5 are $7, 12, 17$.

4. This is $2xy - 2x + y = 43$ and subtracting 1 to both sides completes the rectangle, giving $(2x+1)(y-1) = 42$. Then $2x + 1$ is an odd factor of 42, so it is either $3, 7, 21$, giving $x = 1, 3, 10$, with corresponding $y = 15, 7, 3$. The largest $x + y$ is thus 16.

5. Adding 1 to both sides in each equation completes the rectangle, making $(a + 1)(b + 1) = 16, (b + 1)(c + 1) = 100$, and $(c + 1)(a + 1) = 400$. Taking the product of all equations and its square root gives $(a + 1)(b + 1)(c + 1) = 800$. Dividing with second equation gives $a + 1 = 8$, so $a = 7$. Similarly, $b = 1$ and $c = 49$.

6. Adding twice the first equation to the second gives $16x + 13y = 77$, which has only one nonnegative integer solution, $x = 4, y = 1$. Substituting to either equation gives $z = 2$.

7. Cheat: it must be constant. One such soution is $(3, -4)$, and $\lfloor y/x \rfloor = -1$. In fact, the rest of the solutions are $(3 - 4k, 7k - 4)$ for integral k, and indeed $\lfloor y/x \rfloor = -1$.

8. Dividing both sides by xyz gives $x^{y^z - 1} y^{z^x - 1} z^{x^y - 1} = 3$. One of x, y, z must be 3, so WLOG $x = 3$. Then $y^z - 1 = 1$, which only happens for $y = 2$ and $z = 1$, giving $(3, 2, 1)$, which works, and so does its cycles, giving 3 triples.

9. Note that $\frac{15}{2013} = \left(1 - \frac{1}{x_1}\right) \cdots \left(1 - \frac{1}{x_n}\right) \geq \left(1 - \frac{1}{2}\right) \cdots \left(1 - \frac{1}{n+1}\right) = \frac{1}{n+1}$, showing $n \geq 134$. To prove this is achievable, set $x_1, \ldots, x_{133}$ to $2, \ldots, 134$ and $x_{134} = 671$. This gives us the value $\left(1 - \frac{1}{2}\right) \cdots \left(1 - \frac{1}{134}\right)\left(1 - \frac{1}{671}\right) = \frac{1}{134} \cdots \frac{670}{671} = \frac{15}{2013}$. The minimum value is thus 134.

Modulo

1. The highest power of 5 dividing 16 is, by Legendre's, $\left\lfloor \frac{16}{5} \right\rfloor = 3$, so we take out 8 and three factors of 5 and compute modulo 100 the product $1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 6 \cdot 7 \cdot 1 \cdot 9 \cdot 2 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 3 \cdot 16 = 96$.

2. (The remainder when divided by 5 should be 4.) Since $n + 5 \equiv 3 \pmod 4$, $n \equiv 3 - 5 \equiv -2 \equiv 2 \pmod 4$. Similarly, $n \equiv 0 \pmod 5$. We check $5, 10, 15$ if any give a remainder of 2 when divided by 4, and 10 works. Then $10 + 6 \equiv 16 \pmod 20$, so the remainder is 16.

3. Note that $n = 1$ works, but we require it to be greater than one. By CRT, the solutions to any linear system of moduli differ by the LCM of the moduli. The LCM of $3, 4, 5, 6$ is 60, so the next solution is $1 + 60 = 61$.

4. Taking modulo 11, by Fermat's Little Theorem, we only need to consider the exponent modulo 10. However, $5! \equiv 0 \pmod 10$, so by Fermat's Little Theorem, $3!^{5!^{\cdots}} \equiv \left(3!^{10}\right)^{\cdots} \equiv 1^{\cdots} \equiv 1 \pmod{11}$. The remainder is 1.

5. Since $96 = 3 \cdot 32$, we take modulo 3 and modulo 32. Modulo 3 the expression is $1^{15} - (-1)^{15} - 1^{15} - (-1)^{15} - 1^{15} \equiv 1$. Modulo 32, everything evaporates except for $-1^{15} \equiv -1$. It is 1 modulo 3 and -1 modulo 32, combining both gives the expression as 31 modulo 96.

6. Since $7, 8, 9$ are relatively prime, $739ABC$ is divisible by 504. It is $739000 + ABC \equiv 136 + ABC \equiv 0 \pmod{504}$, giving only the choices $ABC = 368, 872$.

7. If $p \mid a^p$, then $p \mid a^p \mid a^q$. Suppose $p \mid a^q$ and $p \nmid a^p$, then there exists some prime power r^n such that $r^n \mid p$ and $r^n \nmid a^p$. Then $r^n \mid p \mid a^q$ so $r \mid a$, and $r^n \mid a^n$. However, since $r^n \nmid a^p$, then $p < n$. Then $r^p \mid r^n \mid p$, but this implies $r^p < p$, contradiction.