VCSMS PRIME

Session 9: Geometry 2
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Ad hoc

1. Let $BC = 1$, $AB = 2$. Then $AC = \sqrt{5}$, and $CD = DA = BD = \dfrac{\sqrt{5}}{2}$ by Thales's. Since $CD = DA$ and they share the same altitude from B, $[BCD] = [BDA] = \dfrac{1}{2}[ABC] = \dfrac{1}{2}$. But $[BCD] = \dfrac{1}{2}CE \cdot BD$, so $CE = \dfrac{2\sqrt{5}}{5}$. Using the Pythagorean theorem gives BE and ED, then $BE : ED = 2 : 3$.

2. Let the center of the circle be O, the intersection of the diagonals of the square $ABCD$ be E. Let the tangents from A to the circle be AR and AS, with R lying on AE. Let EO intersect the square at T, and let $RE = x$.

   Then $AR = \sqrt{2} - x$ as $AE = \sqrt{2}$, and $AS = AR$ as they are both tangents from A. But clearly $ATOS$ is a rectangle, so $AS = TO$, whence $EO = ET + TO = 1 + \sqrt{2} - x$. From Pythagorean on ERO, we have $EO^2 = RE^2 + RO^2$ or $\left(1 + \sqrt{2} - x\right)^2 = x^2 + 1$, giving $x = 1$ by inspection.

   Then $TO = \sqrt{2} - 1$, and $PO = 1$, so by Pythagorean $PT = \sqrt{2\sqrt{2} - 2}$. PQ is double this, or $2\sqrt{2\sqrt{2} - 2} = \sqrt{8\left(\sqrt{2} - 1\right)}$.

3. Let the perpendicular bisectors of AP and BP intersect at O, and let OP intersect CD again at F. Then $\angle CPF = \angle APO$ due to vertical angles. However, $\angle ABP = \dfrac{1}{2}\angle AOP = \dfrac{1}{2}\left(180^\circ - 2\angle APO\right)$ since $AO = OP$ due to it being the circumcenter, and thus AOP is isosceles. This makes $\angle ABP = 90^\circ - \angle APO = 90^\circ - \angle CPF$. But $\angle ABP = \angle DCP$ since $\triangle ABP \cong \triangle DCP$ by SAS. Thus $\angle DCP = \angle FCP = 90^\circ - \angle CPF$, so $\angle FCP + \angle CPF = 90^\circ$ and thus $\angle PFC = 90^\circ$, which is what we wanted.

4. Let $AB = a, BC = b, CD = c, DA = d, PD = p$. Then $[CPD] = \dfrac{1}{2}cp\sin D$, and $[ABCP] = [ABC] + [ACD] - [CPD] = \dfrac{1}{2}ab\sin B + \dfrac{1}{2}cd\sin D - \dfrac{1}{2}cp\sin D$, but $\sin B = \sin D$ since it is a cyclic quadrilateral. Factoring out, $[CPD] = [ABCP]$ implies $cp = ab + cd - cp$, or $2cp = ab + cd$. Equal perimeters imply $2p = a + b - c + d$, substituting yields $ac + bc - c^2 + cd = ab + cd$, which factors as $(c - a)(c - b) = 0$. Thus either $c = a$ or $c = b$.

5. There is a solution using similar triangles, as the official solution: from $PBC \sim PDB$ implies $BC/BD = BP/DP$ and from $PAC \sim PDA$ implies $AC/AD = AP/DP$. Since $AP = BP$, we get $BC/AC = BD/AD$. But from $AEB \sim ABC$, $BC/AC = BE/AB$ and from $AFB \sim ABD$ we get $BD/AD = BF/AB$. Thus $BE/AB = BF/AB$ and $BE = BF$.

   But projective is much nicer. Since AA, BB and CD concur, then $ACBD$ is a harmonic quadrilateral, and $-1 = (A, B; C, D)$. Taking a perspectivity through A to line EF gives us $-1 = (T, B; E, F)$, where T is the point on infinity on EF, from whence B is the midpoint.

Triangles

1. We can construct a lot of altitudes, but trigonometry is cleaner: $DE^2 = DC^2 + EC^2 - 2DC \cdot EC\cos\angle DCE$, but $\cos\angle DCE = \cos\angle ACB = \dfrac{4}{5}$. Thus $CE = \dfrac{8}{3}$, so the perimeter of $ABED$ is $\dfrac{28}{3}$.

2. Let $BC = x$, from which $AB = AF = 2x$ as they are both tangents, $BC = CD = x$ as they are both tangents. For the perimeter to be 36, we must have $EF = DF = 18 - 3x$. Using Pythagorean on ACE gives $x = 0, 3$, where 0 is obviously extraneous. Then $CE = 18 - 2x = 12$.

3. Since $AQC \sim QEC$, we get $AC/QC = QC/EC$, or $QC^2 = EC \cdot AC$. Similarly, $PC^2 = DC \cdot BC$. As $\angle AEB = \angle ADB = 90°$ then $ABDE$ is cyclic and $EC \cdot AC = DC \cdot BC$ by power of a point through C, whence $PC^2 = QC^2$ and $PC = QC$.

4. WLOG $AB < AC$. Use Ptolemy's, Pythagorean, and the given identity to show that $2 \cdot DF(AB+AC) = BC \cdot AC - BC \cdot AB$. Since $EF = EC - FC$, we can find EC using angle bisector theorem and FC is half of BC. Simplifying shows $DF = EF$.

5. Let Z be the midpoint of BC. Since $XYZ \sim ABC$, then $\angle XZY = \angle BAC = \angle XDY$ so $XDZY$ is cyclic. But $\angle XDB = 180° - \angle XDZ = \angle XYZ = \angle ABC$ again since $XYZ \sim ABC$. This implies $XA = XB = XD$, and thus AB is a diameter of (ABD), from which $\angle ADB = 90°$.

Coordinate geometry

1. Let the center of the circle be $Q(0, 2)$ and let P be a point on the circle. From the equation, it has radius 1. When P is on the upper semicircle, the tangent line clearly intersects the y-axis above the circle, so it has a positive y-intercept.

   Consider the point P such that the tangent line through Q passes through the origin $O(0, 0)$. Since it is a tangent, $\angle QPO = 90°$, since it is a radius, $QP = 1$ and we know the distance $QO = 2$. Thus triangle QPO is a $30 - 60 - 90$ triangle. Then $\angle PQO = 60°$.

   There is a $60°$ arc from either side in the lower half, and in this arc everything has non-negative y-intercept. There is the whole upper half from earlier, which makes a total of $60° + 60° + 180° = 300°$. The length of the arcs is thus $\frac{300°}{360°} 2\pi r = \frac{5}{3}\pi$.

2. Shoelace formula gives 144.

3. Assign a mass of $1A$, $1B$ and $2C$. Let E be the midpoint of AB, and G be the intersection of CE and AP. Then $1A + 1B = 2E$, and since $BP : PC = 2 : 1$, we have $1B + 2C = 3P$. Then $4G = 1A + 3P = 2E + 2C$, making G the midpoint of EC.