VCSMS PRIME

Program for Inducing Mathematical Excellence

Session 2: Functions

September 15, 2017

Lecture problems

1. (QI1) If $27^3 + 27^3 + 27^3 = 27^x$, what is the value of x?
2. (AI11) How many real numbers x satisfy the equation

$$\left(\left|x^2 - 12x + 20\right|^{\log x^2}\right)^{-1+\log x} = \left|x^2 - 12x + 20\right|^{1+\log(1/x)}?$$

3. (QI2) Let $a, b > 0$. If $|x - a| \leq a + b$, what is the minimum value of x?
4. (QIII4) Let $f(x) = \sqrt{-x^2 + 20x + 400} + \sqrt{x^2 - 20x}$. How many elements in the range of f are integers?
5. (AI2) Let f be a real-valued function such that $f(x - f(y)) = f(x) - xf(y)$ for any real numbers x and y. If $f(0) = 3$, determine $f(2016) - f(2013)$.
6. (AI8) For each $x \in \mathbb{R}$, let $\{x\}$ be the fractional part of x in its decimal representation. For instance, $\{3.4\} = 3.4 - 3 = 0.4$, $\{2\} = 0$, and $\{-2.7\} = -2.7 - (-3) = 0.3$. Find the sum of all real numbers x for which $\{x\} = \frac{1}{5}x$.

Exponents

- $b^e = x$. If $b > 0$ (and not 1) then $e \in \mathbb{R}$. If $b = 0$, then $e > 0$. If $b = 1$, then range is just 1. The negative case is very complicated. Range is all real numbers, except $b \leq 0$ and $b = 1$. Monotonic, so if $e \in [c, d]$ then $x \in [b^c, b^d]$.
- Write everything in the same base and hope it works!
- If we can't make the bases the same, we can make the exponents the same: 11^8 and 16^7.
- If $a, b, c \in \mathbb{R}$, and $a \geq 0$ then $a^b = a^c$ implies one of either: **a)** $a = 0, b, c > 0$, **b)** $a > 0, b = c$, **c)** $a = 1$. The case of negative base is complicated again.

Logarithms

- $\log_b x = e$. Must have $b > 0$ and $x > 0$, but range is any $e \in \mathbb{R}$. Monotonic, so if $x \in [c, d]$ then $e \in [\log_b c, \log_b d]$.
- Write everything in the same base and hope it works!
- Spam $\log_b x = e \iff x = b^e$. Think of "raising both sides to the bth power" and "cancelling the logarithm:" $b^{\log_b x} = x$. Since logarithms are monotonic, inequalities work too.
- Recall the rules of logarithms: the most important are $\log_b x + \log_b y = \log_b xy$, $c \log_b x = \log_b x^c$, and $\log_b x = \frac{\log_c x}{\log_c b}$, the rest can be derived.

Surds

- $y = \sqrt{x}$. Must have $x \geq 0$. Monotonic, so if $x \in [a, b]$ then $y \in [\sqrt{a}, \sqrt{b}]$.
- Rationalize the denominator, often with $x^2 - y^2 = (x-y)(x+y)$ or $x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$.
- If you have $\sqrt{a + \sqrt{b}}$, maybe you can simplify it to $x + \sqrt{y}$. Equate and square both sides. Same thing with cube roots.
- If you have conjugates, like $x = \sqrt{a} + \sqrt{b}$ and $y = \sqrt{a} - \sqrt{b}$, you can often write y in terms of x.

Floor, ceiling, fractional

- $\lfloor x \rfloor$ is the integer part of x. If $\lfloor x \rfloor = c$, then $c \leq x < c + 1$. Monotonic.
- $\lceil x \rceil$ is ceiling, if $\lceil x \rceil = c$ then $c < x \leq c + 1$. Monotonic.
- $\{x\}$ is fractional part or $x - \lfloor x \rfloor$. *Not monotonic*. Always has $0 \leq \{x\} < 1$.
- One technique is to substitute $x = n + r$ where $n = \lfloor x \rfloor$ and $r = \{x\}$. Use the fact that $0 \leq r < 1$ to find values of n.
- Another technique is to replace all $\{x\}$ with $x - \lfloor x \rfloor$.

Absolute value

- $y = |x|$ is always split into two cases: when $x < 0$, $|x| = -x$ and when $x > 0$, $|x| = x$.
- *Not monotonic*, so we have to be careful with inequalities: if you have $|x| \leq y$ then you split it into $-y \leq x \leq y$. If $|x| \geq y$ then $x \leq -y$ or $x \geq y$.
- Sums of absolute values: if you're minimizing $|x - a| + |x - b| + |x - c|$, the minimum value is when x is the median of a, b, c. If even number of values, then any x between the two median values works.

Rational functions and limits

- $y = \frac{f(x)}{g(x)}$. Must have $g(x) \neq 0$.
- Very common to find the range, as in $\frac{x^4+3}{2x^4+1}$. Find the fastest growing term and consider that. What happens if $x \to \infty$, or $x \to -\infty$? What makes it the smallest value?
- From slow growing to fast: constants, logarithms, polynomials, exponents. (This is towards positive infinity.)

Functional equations

- Treat it as a system of equations machine and find stuff.
- Substitution: To find $f(0)$ or $f(1)$ or whatever, get stuff to cancel. Try substituting all 0 or all 1.
- Involutions: if we have $f(x)$ and $f(a - x)$ and we're finding $f(c)$, then substituting $x = c$ and $x = a - c$ gives two equations. Similar: $f(x)$ and $f(1/x)$ means substituting $x = c$ and $1/c$. Functions where $f \circ f(x) = x$ are called *involutions*.
- Induction: if we have $f(x)$ and $f(x + 1)$ and you know $f(0)$, you can find any $f(n)$ for any natural n.
- Cheat: if only one function satisfies the conditions (i.e. there's only one possible answer), then just find one and use that. Try linear functions, constants, etc.
- Cauchy FE: if $f(x + y) = f(x) + f(y)$ for $x, y \in \mathbb{Q}$ then $f(x) = kx$ for some constant k. Making it reals is harder, it works if you have either bounding, monotonicity, or continuity.