VCSMS PRIME

Program for Inducing Mathematical Excellence

Session 8: Algebraic Manipulation

October 6, 2017

Lecture problems

1. (AI14) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (2x - y, x + 2y)$. Let $f^0(x, y) = (x, y)$ and, for each $n \in \mathbb{N}$, $f^n(x, y) = f(f^{n-1}(x, y))$. Determine the distance between $f^{2016}(4/5, 3/5)$ and the origin.
2. Solve the equation $x^4 + (x - 2)^4 + 16 = 0$.
3. Factorize $x^3 + y^3 - 3xy + 1$.
4. (Sipnayan 2016) Find all x satisfying $\sqrt[3]{20x + \sqrt[3]{20x + \sqrt[3]{20x + 16}}} = 16$.
5. (AIME 1990/4) Solve $\dfrac{1}{x^2 - 10x - 29} + \dfrac{1}{x^2 - 10x - 45} - \dfrac{2}{x^2 - 10x - 69} = 0$.
6. Factorize $(x + y - 2z)^3 + (y + z - 2x)^3 + (z + x - 2y)^3$.
7. (ARML 2016) Factorize $13^4 + 16^5 - 172^2$, given it is the product of three distinct primes.
8. (AI9) Find the integer which is closest to the value of $\left(\sqrt[6]{5^6 + 1} - \sqrt[6]{5^6 - 1}\right)^{-1}$.
9. Given $x^2 - 3x + 1 = 0$, find the value of $x^5 + x^{-5}$.
10. Solve the equation $x^4 - 6x^3 - 11x^2 - 6x + 1 = 0$.
11. Prove that the product of four consecutive integers plus one is always a perfect square.
12. (MMC) Given $6x^2 + 47x + 77 = (2x + 11)(3x + 7)$, factorize 64,777.
13. Suppose $x + 3y = 3$, $y + 3z = 4$, and $z + 3x = 5$. Find x.
14. (AII1) Let x and y satisfy $\dfrac{x}{x^2y^2 - 1} - \dfrac{1}{x} = 4$ and $\dfrac{x^2 y}{x^2y^2 - 1} + y = 2$. Find all possible values of xy.
15. (AIME 1989/8) Find $16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7$ given that
$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123 \end{aligned}$$
16. (AIME 1990/15) Let $f(n) = ax^n + by^n$. Given $f(1) = 3, f(2) = 7, f(3) = 16, f(4) = 42$, find $f(5)$.
17. (AIME 2014/14) Find all real solutions to $\dfrac{3}{x - 3} + \dfrac{5}{x - 5} + \dfrac{17}{x - 17} + \dfrac{19}{x - 19} = x^2 - 11x - 4$.

Abusing symmetry again

- Algebraic manipulation – substitution, factorization, manipulation – usually only has one end goal. To create symmetry. (Almost) all of the problems today deal with this.

Substitution

- Problem 1: Substituting $x, y \to 2x - y, x + 2y$ directly to $\sqrt{x^2 + y^2}$ shows that it's multiplied by $\sqrt{5}$ each time. The symmetry arises when the terms in $(2x - y)^2$ and $(x + 2y)^2$ cancel.
- Problem 2: We force symmetry by substituting $x \to y + 1$. The terms in $(y + 1)^2$ and $(y - 1)^2$ cancel.
- Problem 3: Exchanging any two variables keeps the expression the same, which means it's *symmetric*. When dealing with symmetric expressions, we usually try substituting $x + y, xy \to a, b$, because of the Fundamental Theorem of Symmetric Polynomials. This is $a^3 - 3ab - 3b + 1$, and there's an $a + 1$ factor.
- Problem 4: Substitution again. You substitute the whole expression into the 16 in the innermost infinitely many times, to get $\sqrt[3]{20x + \sqrt[3]{20x + \cdots}} = 16$, which is more symmetric. Then substitute 16 for the whole expression to get $\sqrt[3]{20x + 16} = 16$.
- Problem 5: By letting $x^2 - 10x - 29 \to a$, and *then* multiplying out, it's much easier.

Factorization

- Problem 6: There are a few factorizations you are expected to know, and one of them is $a^3+b^3+c^3-3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$. But when $a + b + c = 0$, this becomes $a^3 + b^3 + c^3 = 3abc$.
- Problem 7: Use Sophie–Germain: $a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$. 16^5 is 2^{20} and $13^4 - 172^2$ is $1 - 2^{10}$ via difference of two squares. Multiply both sides by $2^{10} + 1$ and then factor using Sophie–Germain on $1^4 + 4 \cdot \left(2^7\right)^4$.
- Problem 8: Factor $2 = (5^6 + 1) - (5^6 - 1)$ using difference of two squares, then two cubes.

Manipulation

- Problem 9: Divide both sides by x to get $x + 1/x = 3$. There are several ways to get $x^5 + x^{-5}$. We can take the fifth power and use the smaller powers, or we can recurse more generally by relating $x^n + x^{-n}$ and $x^{n+1} + x^{-(n+1)}$.
- Problem 10: Divide both sides by x^2 and use the substitution $x + 1/x \to a$.
- Problem 11: Suppose the smallest was n. Then $n(n + 1)(n + 2)(n + 3) + 1$. To make it easier, pair multiply $n(n + 3)$ and $(n + 1)(n + 2)$, then substitute $n^2 + 3n + 2 \to a$.
- Problem 12: Substitute $x = 100$.
- Problem 13: Add all the equations and divide by 5 to find $x + y + z = 12/5$. Subtract from the second to eliminate y and use the third to find x.
- Problem 14: Force symmetry, multiply the first by xy and subtract from the second. Solve for y in terms of x, substitute, simplify.
- Problem 15: Remember the method of finite differences? The perfect squares are quadratic, so the second difference is constant.
- Problem 16: We want to relate $f(n)$ and $f(n + 1)$. Like we did in Problem 9 for $x^n + x^{-n}$ and $x^{n+1} + x^{-(n+1)}$, we can see $f(n)(x + y) = f(n + 1) + xyf(n - 1)$.
- Problem 17: The key idea is to add 4 to both sides. The fraction $3/(x - 3)$ becomes $x/(x - 3)$, etc. Cancel out x and substitue $x \to y + 11$ for symmetry. Add opposite terms and cancel out y.