VCSMS PRIME

Program for Inducing Mathematical Excellence
Week 4 Homework
Due October 11, 2017

Homework

Due on Wednesday, October 11. We have five sets this week, solve the two assigned to you as usual.

Set A (12) **S4**: Ad hoc 1, 4. **S5**: Equations 1–2; Systems of equations 1–2; Polynomials 1; Polynomial factors 3. **S8**: Manipulation 1–2; Sequences 1; Series 1.

Set B (12) **S4**: Ad hoc 6. **S5**: Equations 3; Systems of equations 3–4. **S8**: Manipulation 5; Sequences 2–4; Series 2–5.

Set C (12) **S4**: Ad hoc 7. **S5**: Equations 5–6. **S8**: Manipulation 3–4; Series 6, 8; Inequalities 1; Single-variable extrema 2; Multi-variable extrema 1–2, 5.

Set D (12) **S4**: Ad hoc 8. **S5**: Equations 4; Systems of equations 5. **S8**: Manipulation 6–7; Sequences 5; Series 7; Single-variable extrema 1, 4; Multi-variable extrema 3–4, 7.

Set E (13) **S5**: Vieta's 7. **S8**: Manipulation 8; Sequences 6–7; Series 9; Inequalities 2–5; Single-variable extrema 5; Multi-variable extrema 7–9.

Additional problems

1. (AIME 2005/7) Let $x = \dfrac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$. Find $(x+1)^{48}$.
2. (SMO 2011) Find $a^2+b^2+c^2-ab-bc-ca$ if $a = 2011x+9997$, $b = 2011x+9998$ and $c = 2011x+9999$.
3. (Titu 1997) Prove that $\dfrac{1}{\sqrt{1}+\sqrt{3}} + \dfrac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \dfrac{1}{\sqrt{9997}+\sqrt{9999}} > 24$.
4. (AIME 1989/7) Find an integer k such that $36+k, 300+k$ and $596+k$ are the squares of three consecutive terms of an arithmetic series.
5. (SMO 2006) Let a, b be positive reals such that $\dfrac{1}{a} - \dfrac{1}{b} - \dfrac{1}{a+b} = 0$. Find $\left(\dfrac{a}{b} + \dfrac{b}{a}\right)^2$.
6. Evaluate $1!(1^2+1+1) + 2!(2^2+2+1) + \cdots + 2017!(2017^2+2017+1)$.
7. (AIME I 2013/5) Find the real root of $8x^3 - 3x^2 - 3x - 1 = 0$.
8. Prove that $\dfrac{1}{2} \cdot \dfrac{3}{4} \cdots \dfrac{99}{100} < \dfrac{1}{10}$.
9. Find the maximum of $2^x + 3^x - 4^x + 6^x - 9^x$.
10. How many nonempty subsets of $\{1, 2, \ldots, 1000\}$ have sum divisible by 3?
11. (OMO Spring 2014/25) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}}{\binom{n+100}{100}}$.

Additional reading

- Summations (Evan Chen), `http://web.evanchen.cc/handouts/Summation/Summation.pdf`.
- A Potpourri of Algebra, `https://www.scribd.com/document/82663491/A-Potpourri-of-Algebra`.