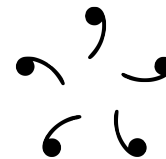


VCSMS PRIME

Program for Inducing Mathematical Excellence

Week 4 Homework

Due October 11, 2017



Homework

Due on Wednesday, October 11. We have five sets this week, solve the two assigned to you as usual.

Set A (12) **S4**: Ad hoc 1, 4. **S5**: Equations 1–2; Systems of equations 1–2; Polynomials 1; Polynomial factors 3. **S8**: Manipulation 1–2; Sequences 1; Series 1.

Set B (12) **S4**: Ad hoc 6. **S5**: Equations 3; Systems of equations 3–4. **S8**: Manipulation 5; Sequences 2–4; Series 2–5.

Set C (12) **S4**: Ad hoc 7. **S5**: Equations 5–6. **S8**: Manipulation 3–4; Series 6, 8; Inequalities 1; Single-variable extrema 2; Multi-variable extrema 1–2, 5.

Set D (12) **S4**: Ad hoc 8. **S5**: Equations 4; Systems of equations 5. **S8**: Manipulation 6–7; Sequences 5; Series 7; Single-variable extrema 1, 4; Multi-variable extrema 3–4, 7.

Set E (13) **S5**: Vieta's 7. **S8**: Manipulation 8; Sequences 6–7; Series 9; Inequalities 2–5; Single-variable extrema 5; Multi-variable extrema 7–9.

Additional problems

- (AIME 2005/7) Let $x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$. Find $(x+1)^{48}$.
- (SMO 2011) Find $a^2 + b^2 + c^2 - ab - bc - ca$ if $a = 2011x + 9997$, $b = 2011x + 9998$ and $c = 2011x + 9999$.
- (Titu 1997) Prove that $\frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{5} + \sqrt{7}} + \cdots + \frac{1}{\sqrt{9997} + \sqrt{9999}} > 24$.
- (AIME 1989/7) Find an integer k such that $36 + k$, $300 + k$ and $596 + k$ are the squares of three consecutive terms of an arithmetic series.
- (SMO 2006) Let a, b be positive reals such that $\frac{1}{a} - \frac{1}{b} - \frac{1}{a+b} = 0$. Find $\left(\frac{a}{b} + \frac{b}{a}\right)^2$.
- Evaluate $1!(1^2 + 1 + 1) + 2!(2^2 + 2 + 1) + \cdots + 2017!(2017^2 + 2017 + 1)$.
- (AIME I 2013/5) Find the real root of $8x^3 - 3x^2 - 3x - 1 = 0$.
- Prove that $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{99}{100} < \frac{1}{10}$.
- Find the maximum of $2^x + 3^x - 4^x + 6^x - 9^x$.
- How many nonempty subsets of $\{1, 2, \dots, 1000\}$ have sum divisible by 3?
- (OMO Spring 2014/25) Compute $\sum_{n=1}^{\infty} \frac{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}}{\binom{n+100}{100}}$.

Additional reading

- Summations (Evan Chen), <http://web.evanchen.cc/handouts/Summation/Summation.pdf>.
- A Potpourri of Algebra, <https://www.scribd.com/document/82663491/A-Potpourri-of-Algebra>.