Semifinals A

Easy Round: 2pts, 30s [Answer]

SFA E-1 A common misconception in mathematics is that $(x+y)^2 = x^2 + y^2$. This is not the right way of expanding $(x+y)^2$, although this does not necessarily mean that it is wrong for all values of x and y. How many ordered pairs of integers (x, y) are there such that $-18 \le x \le 18$, $-18 \le y \le 18$, and the equation stated is true? [73]

SFA E-2 Let there be 2016 points on a circle. What is the maximum number of intersection points in the interior of the circle that may be formed by chords defined by these points. Factorial notation may be used. $\left[\dfrac{2016!}{2012! \cdot 4!}\right]$

SFA E-3 The side lengths of a trapezoid are $\sqrt[4]{6}$, $\sqrt[4]{6}$, $\sqrt[4]{6}$, and $2\sqrt[4]{6}$. Its area can be expressed in the form $\sqrt{\dfrac{a}{b}}$, where a and b are coprime positive integers. What is the value of $a+b$? [89]

SFA E-4 Let x and y be rational numbers such that $x + \sqrt{y} = 40$, $\sqrt{x} + y = 22$. Find $x + y$. [52]

SFA E-5 Find the difference between the radius of the circumcircle and the diameter of the incircle of the right triangle with legs 9 and 40. $\left[\dfrac{25}{2}\right]$

Average Round: 3pts, 60s

SFA A-1 When a right triangle is rotated about one leg, the volume of the generated cone is $800\pi\,\text{cm}^3$. If the other leg is rotated, the volume of the generated cone becomes $1920\pi\,\text{cm}^3$. Find the length of the hypotenuse of the triangle. [26 cm]

SFA A-2 If a is the number of pairs of odd primes p, q such that $p^q - q^p = p + q$, find $|(a-3)^{a+3}|$. [27]

SFA A-3 Sophie is a chef who is trying to make iced tea with the element Sipnayium. Mixing two water and one part Sipnayium yields a mix with the ratio 3 : 1 of Scoride and waste. Scoride is an ingredient in Highland Iced Tea, which requires 5 parts Scoride for every 7 parts eater. If Sophie wants to make 2 L of Highland Iced Tea, how many liters of Sipnayium should she prepare? $\left[\dfrac{10}{27}\,\text{L}\right]$

SFA A-4 Find the maximum value of $8 \cdot 27^{\log_6 x} + 27 \cdot 8^{\log_6 x} - x^3$ as x varies over the positive real numbers. [216]

SFA A-5 The sum of all perfect squares less than 2016^2 that do not divide 2016 can be expressed in the form $\dfrac{2016ab}{c} - d$, where a, b, c, d are non-zero positive integers, a and b have no common factors, and c does not divide a or b. Find the least value of $a+b+c+d$. [6266]

Difficult Round: 5pts, 90s

SFA D-1 If $\cos^2\theta + \cos\theta = 1$, express $\sin^8\theta + \sin^6\theta + \sin^4\theta + \sin^2\theta - 2$ as a single term in terms of $\sin\theta$. $[-\sin^2\theta]$

SFA D-2 Determine the number of points on the Cartesian plane that lie on $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{11}$, where x, y are integers. [5]

SFA D-3 A basic Sharp calculator has had all nine of its nonzero single-digit buttons jumbled. You randomly pressed two of these buttons (not necessarily distinct) and got a two-digit number. What is the probability that the number is prime? $\left[\frac{7}{27}\right]$

SFA D-4 Find all solutions of $\sin^4 x + \cos^2 x = \frac{15}{16}$ where $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. $\left[x = \pm\frac{\pi}{12}, \pm\frac{5\pi}{12}\right]$

SFA D-5 Let a, b, c be positive integers such that $abc + bc + c + 1 = 1272$. Find the minimum value of $a + b + c$. [43]

Semifinals B

Easy Round: 2pts, 30s

SFB E-1 How many positive integers from 1 to 2016 are divisible by 20 or 16? [201]

SFB E-2 If $\sum_{n=1}^{101}(-1)^{n+1}n^3$ can be expressed in the form $a^2(b^2 - c^2)$, where a, b, c are coprime positive integers, find the minimum value of $a + b + c$. [252]

SFB E-3 Find set of all real numbers x such that $j(x) = \ln\left(\sin^{-1}\left(\sqrt{x^2 - 3}\right)\right)$ is real. $\left[\left[-2, -\sqrt{3}\right) \cup \left(\sqrt{3}, 2\right]\right]$

SFB E-4 If $\log_3 \log_{27} m = \log_{27} \log_3 m$, find m. $\left[3^{3\sqrt{3}}\right]$

SFB E-5 What is the least six-digit palindrome that is a multiple of 45? [504 405]

Average Round: 3pts, 60s

SFB A-1 There are 2016 lines on a plane. What is the maximum number of regions these lines can divide the plane into? [2033137]

SFB A-2 Find the exact value of $1 + \frac{1}{1+}\frac{2}{1+}\frac{1}{1+}\frac{2}{1+}\cdots$*. $\left[\sqrt{2}\right]$

SFB A-3 The sum $\sum_{n=1}^{2016} n! \cdot n$ can be expressed in the form $a! - b!$, where a and b are positive integers. What is $a - b$? [2016]

SFB A-4 Find $f(2016)$ if $f(x)$ and $g(x)$ are linear functions such that for all x, $f(g(x)) = g(f(x)) = x$, and have values $f(11) = 5$ and $g(20) = 16$. [6020]

SFB A-5 In the back seat of Halsey's rover were three boxes. The first box contains 6 sheets (pulled right off the corner). The second box contains 9 sheets (from her roommate back in Boulder)†. How many sheets should the third box contain if the number of ways of rearranging all the sheets in the boxes is equal to the number of ways you can choose two distinct days on October? Assume that all the sheets are identical, and that rearranging them means putting all the sheets into any of the three boxes. [14]

Difficult Round: 5pts, 90s

SFB D-1 What is the value of $\log_{1/2}(\sin 1° \cdot \sin 3° \cdots \sin 89°)$? $\left[\frac{89}{2}\right]$

SFB D-2 Evaluate $6\sum_{n=0}^{\infty}\frac{n^2 - 1}{7^n}$. $\left[-\frac{49}{9}\right]$

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* See (2) in http://mathworld.wolfram.com/ContinuedFraction.html

† Rover, corner and Boulder? Cf. http://genius.com/The-chainsmokers-closer-lyrics

SFB D-3 An equilateral triangle that has an area of 4 is divided into four congruent equilateral triangles, with the center triangle shaded. Each of the three smaller unshaded equilateral triangles is then divided into 4 even smaller equilateral triangles in the same manner. Suppose this goes on forever; what is the total area of the shaded region? Note that shaded triangles are no longer subdivided into smaller triangles. [4 units^2]

SFB D-4 Mivan was counting the total number of seats in the auditorium and noticed that the rows followed some sort of pattern. The first row had 8 seats, the second had 15 seats, the third row had 28 seats, the fourth row has 49 seats and the fifth row had 80 seats. If this pattern were to continue, how many rows would Mivan expect to see in the 9th row? [344]

SFB D-5 Find $n > 0$ such that $\sqrt[3]{\sqrt[3]{5\sqrt{2}+n}+\sqrt[3]{5\sqrt{2}-n}} = \sqrt{2}$. [7]

Finals‡

Easy Questions: 2pts, 30s

F E-TETRIS Let $f(x)$ be defined as the least positive integer that can be added to x to yield a perfect square. For example, $f(5) = 4$ and $f(9) = 7$. Starting from 1, the 52nd occurrence of 99 as a function value corresponds to what value of x? [10 102]

F E-PONG Find the positive integer c such that the reciprocal of $\dfrac{18\sqrt{5}-3\sqrt{c}}{c}$ is $\dfrac{18\sqrt{5}+3\sqrt{c}}{c}$. [36]

F E-PACMAN Suppose that $f(x)$ is a quartic whose leading coefficient is 1. If $f(1) = 4$, $f(2) = 9$, $f(3) = 16$, and $f(4) = 25$, find $f(5)$. [60]

F E-INVADERS If $\dfrac{\sin 2\alpha}{\cos^2\alpha} = \dfrac{\sin^2\alpha}{\cos 2\alpha}$, find the non-zero value of $\tan 2\alpha$. [4]

F E-SONIC Suppose that a regular pentagon $SONIC$ is drawn. The intersection of OC and SN is A. What is $\angle SAC$? [72°]

Average Questions: 3pts, 60s

F A-TETRIS Define a positive integer x to be a *Master Facial Wash number* if it satisfies $\lfloor x^{1/2}\rfloor + \lfloor x^{1/4}\rfloor + \lfloor x^{1/8}\rfloor + \lfloor x^{1/16}\rfloor + \lfloor x^{1/32}\rfloor = 100$. The greatest Master Facial Wash number can be expressed in the form $c^2 - 1$, where c is a positive integer. What is c? [86]

F A-PONG The graph of the equation $y = 2x^3 - x^2 - 7x + 6$ intersects the graph of the equation of $y = x + 2$ at three points A, B, and C. What is the maximum value of $\overline{AB} + \overline{AC}$? $\left[\dfrac{13\sqrt{2}}{2}\right]$

F A-PACMAN There are positive integers b and c such that the polynomial $2x^2 + bx + c$ has two real roots which differ by 20. Find the least possible value of $b + c$. [86]

F A-INVADERS It is well-known that if you place five points inside an equilateral triangle of side length 1, there will always be two points whose distance from each other is at most $\frac{1}{2}$. How many points should you place inside equilateral triangle of side length 2 so that there will always be two points whose distance from each other is at most $\frac{1}{3}$? [37]

‡ INVADERS = SPACE INVADERS

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F A-SONIC Your math teacher decided to have a surprise quiz today about Unilever that you were absolutely unprepared for. The quiz was a 10-point multiple choice quiz with 10 questions worth one point each and three possible answers for each item: Ponds, Cream Silk, or Sunsilk. What is the probability that you would score exactly 7 points based on pure random guessing alone? $\left[\dfrac{320}{19\,683}\right]$

Difficult Questions: 5pts, 90s

F D-TETRIS In $(y-1)(y-3)(y-5)\cdots(y-99)$, what is the coefficient of y^{49}? $[-2500]$

F D-PONG What is the circumradius of a triangle with side lengths 104, 112, and 120? $[65]$

F D-PACMAN Harry asked Nora when her birthday was. She answered that she was exactly 7571 days old. Harry then checked his calendar and noted that today is November 5, 2016. How many days is it until her next birthday? [100 days]

F D-INVADERS If $\dfrac{xy}{y-1} = 3$ and $\dfrac{x^2y^2}{y^2-1} = 5$ then find $\dfrac{x^3y^3}{y^3-1}$. Express your answer in the form $\dfrac{a^3}{b^3-c^3}$, where a, b, c are positive coprime integers. $\left[\dfrac{15^3}{7^3-2^3}\right]$

F D-SONIC The vertices of hexagon $ABCDEF$ lie on a circle with sides $AB = CD = EF = 6$, and $BC = DE = FA = 10$. The area of the hexagon is $m\sqrt{3}$. Find m. $[94]$

Very Difficult Questions: 8pts, 120s

F VD-TETRIS Laurel is tired of listening to her normal music, so she decides to speed things up. She has three types of songs — three identical slow songs each 5 minutes long, five identical medium songs each 4 minutes long, and seven identical fast songs each 3 minutes long. She speeds up slow songs to 1.5 times the playing speed, medium songs to 1.25 times the playing speed, and she retains the speed of fast songs. If she wants to make a playlist of five songs, what is the probability that the playlist will be under 16 minutes long? $\left[\dfrac{11}{18}\right]$

F VD-PONG Find the value of $2014\left(\dfrac{1}{3!} - 2013\left(\dfrac{1}{4!} - 2012\left(\dfrac{1}{5!} - \cdots - \cdots 3\left(\dfrac{1}{2014!} - 2\left(\dfrac{1}{2015!} - \dfrac{1}{2016!}\right)\right)\cdots\right)\right)\right)$. $\left[\dfrac{1007}{2016}\right]$

F VD-PACMAN A rectangle is formed from the points $A(0,0)$, $B(5,0)$, $C(5,3)$, and $D(0,3)$ on the Cartesian plane. Two perpendicular lines are also drawn on the Cartesian plane, as shown. If the ratio of the area of DFI to the area of CGH is $\dfrac{1}{4}$, $\angle EIG = 45°$, and $HB = FD$, what is the intersection point of the perpendicular lines? $[(2,4)]$

Figure will be added in the next version. A description: *E* is the intersection point of the two perpendicular lines, is outside the rectangle, and is nearest to side *CD* of the rectangle. The line that intersects *AD* at *F* intersects *CD* at *I*. The other intersects *BC* at *H* and *CD* at *G*, such that collinear points *D*, *I*, *G*, and *C* are arranged in that order.

F VD-INVADERS An integer is a *Space Invader number* if it contains a pair of adjacent even digits that are equal. For example, 2254 and 1500 are Space Invader numbers; on the other hand, 2016, 808, and 116 are not. How many three-digit numbers are not Space Invader numbers? [819]

F VD-SONIC For positive integers m and n, let $\mathsf{UNIVERSIDAD}(m, n)$ be the remainder when m is divided by n. Find the minimum value of m such that $\mathsf{UNIVERSIDAD}(m, 1) + \mathsf{UNIVERSIDAD}(m, 2) + \mathsf{UNIVERSIDAD}(m, 3) + \cdots + \mathsf{UNIVERSIDAD}(m, 10) = 5$. [540]

Clincher Problem

How many ways can you arrange the letters in the word "HARAMBE" such that no two vowels are adjacent to one another? [720]

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