Sipnayan 2018

Junior High School

October 27, 2018

Thanks to Nathanael Joshua Balete for the figures. If you have any corrections or questions, please contact me at `cj@cjquines.com`.

Written round, two hours

Easy, two points each

1. Black Widow has three identical daggers and five identical firearms. If she grabs four weapons at random, what is the chance that all of them are firearms? $\left[\dfrac{1}{14}\right]$

2. In an alternate universe, six Avengers, participated in the Grandmaster's tournament (and were the only ones to participate). All we know for certain about the final rankings is that Thor placed higher than the Hulk, who placed higher than Hawkeye. Assuming there were no ties, how many possible outcomes of the tournament could there have been? [120]

3. Find the value(s) of x that satisfies the equation $\log_2(x-2)+\log_2 x-\log_2(3x-2)=1$. $[4+2\sqrt{3}]$

4. In the Avengers' training course, there are 6 flags labeled from 0 to 5. From any Flag to the next (besides the last one), there are two different paths that one could possibly take. Captain America and Iron Man start at Flag 0 and need to make it to Flag 5, and once they've gotten a Flag, they will never backtrack to the previous one. Also, the two move at the same speed but refuse to ever take the same path as the other person. How many different ways are there for the two to complete the course? [32]

5. Find the number of ordered pairs of integer (x, y) that satisfy the equation $x^2+15^2=y^2$. [18]

6. Spiderman is searching for Ebony Maw on the Cartesian Plane using his Spider Senses. When he is on $(-7,0)$, he detects that Ebony Maw is 13 units away from him. When he is on $(7,0)$, he detects that Ebony Maw is 15 units away from him. What are all the possible points where Ebony Maw could be located? $[(-2,-12),(-2,12)]$

7. Find the smallest integer n such that the expression $40!\times 5^n$ has the maximum number of trailing zeros. [29]

8. Triangle ABC has side lengths $AB=AC=13$, and $BC=10$. Find the radius of the circle that circumscribes $\triangle ABC$. $\left[\dfrac{169}{24}\right]$

Average, three points each

1. Thor meets a group of 10 Asgardian warriors. Each warrior is either male or female, with equal probability. Each warrior is also a sword user or a lance user, with equal probability. Thor heard from Heimdall that this group contains exactly 6 males and exactly 4 sword users. What is the probability that the group contains at least one female lance user? $\left[\dfrac{209}{210}\right]$

2. Let AB be the diameter of circle O, where $AB = 2$. Circle P is internally tangent to circle O at point B, and $PB = \frac{2}{3}$. Two different chords AX and AY are drawn tangent to circle P. Let R be the region bounded by AX, AY, and arc XBY. What is the area of the region inside R but outside circle P? $\left[\frac{4\sqrt{3}}{9} - \frac{4\pi}{27}\right]$

3. Evaluate the following sum, giving your answer as a decimal: $\sum_{n=1}^{199} \frac{n^3+n^2+1}{n^2+n}$. [19900.995]

4. How many values of $\theta \in \{0, 2\pi\}$ satisfy the following equation
$$\frac{2-\cos 2\theta}{\sin\theta} + \frac{\sin\theta}{1+\tan^2\theta} = \cot^2\theta \cdot \sin\theta?$$
[0]

5. Find all ordered triples of integers (x, y, z) that satisfy the following system of equations:
$$\begin{aligned}(x+y)(x+y+z) &= 384\\ (y+z)(x+y+z) &= 288\\ (x+z)(x+y+z) &= 480.\end{aligned}$$
$[(12, 4, 8), (-12, -4, -8)]$

Difficult, five points each

1. Find the last 4 digits of 7^{65}. [8807]

2. What is the remainder when $5\,000\,000^{500\,000\,000\,000}$ is divided by the prime number $10^6 + 3$? [225]

3. Let
$$A = \sum_{n=1}^{3027} \sin\left(\frac{n\pi}{2018}\right), \qquad B = \sum_{n=1}^{3027} \cos\left(\frac{n\pi}{2018}\right).$$
Evaluate
$$A\left(1 - \cos\frac{\pi}{2018}\right) + B\sin\frac{\pi}{2018}.$$
$[-1]$

4. Find the largest prime factor of $1\times2\times3 + 2\times3\times4 + 3\times4\times5 + \cdots + 50\times51\times52$. [53]

5. Let P be a point in the interior of triangle ABC. The lengths of AP, BP, and CP are 3, 4, and 5, respectively. Find the value of AB^2 if $AC = 7$ and $CB = 8$. $\left[\frac{181+9\sqrt{357}}{10}\right]$

Very difficult, eight points each

1. Consider 100 dots arranged in a 10×10 square grid. How many ways can I choose 4 of these dots such that the dots can be used to form a square? The squares need not be perpendicular to the axes (i.e. diagonally tilted squares count too). [825]

2. If $f(x) = x^3 - 4x$ and $g(x) = x^4 + 1005x^2 - 4036$, find the value of $\sum_{i=0}^{2017} \frac{f(1009-i)}{g(i-1009)}$.[1] $\left[\frac{1}{1010}\right]$

3. Consider a 7×7 grid where each cell can contain any integer from 7 to 100. How many ways are there to fill up the grid such that the sums of each of the rows and each of the columns all result in an even number? Express your answer as a product of prime powers. $\left[2^{36} \cdot 47^{49}\right]$

Semifinals A

Easy, two points, thirty seconds

1. Ten bells which make different sounds are lined up in a row, which a priest can ring in a certain order to play a greeting. If he rings 5 distinct bells, it is a traditional greeting. If the order in which he rings the bells is from left to right, then it is a proper greeting. How many proper and traditional greetings are there? [252]

2. Tony Stark has a sphere with radius 2 cm. Obviously, the numerical value of the surface area and volume (in cm^2 and cm^3 respectively) are not equal. Therefore, Tony defined a new unit of measurement: Tm (tonimeters). When his sphere is measured with Tm, the numerical values of the surface area and volume become equal. How many Tm equal 1 cm? [1.5 Tm]

3. Find the number of even factors of 2016. [30]

4. When Kobie goes to school, he walks half the time and runs half the time. When he comes home from school, he walks half the distance and runs half the distance. If he runs twice as fast as he walks, find the ratio of the time it takes for him to get to school, to the time it takes for him to come home from school. [8 : 9]

5. Let $\sin\theta + \cos\theta = \frac{1}{\sqrt{2}}$. Evaluate $\sin^4\theta + \cos^4\theta$. $\left[\frac{7}{8}\right]$

Average, three points, sixty seconds

1. Let $f(1) = 2016$, $f(2) = 2018$, $f(n) = [f(n-1)]^2 + [f(n-2)]^2$ for all $n \geq 3$. What is the units digit of $f(2018)$? [2]

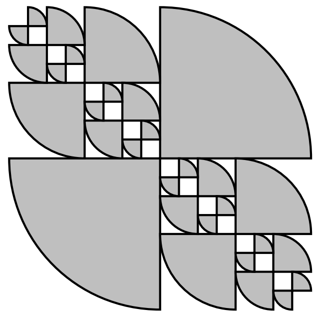
2. How many ordered triples of integers (a, b, c) are there such that the least common multiple of a, b, and c is 2016? [12103]

3. Find the remainder when A is divided by 7, where $A = \sum_{i=0}^{2018} i!$. [6]

4. Let a and b be real numbers such that $2 \cdot \sqrt{2 - \sqrt[4]{7}} \cdot \sqrt{2 + \sqrt[4]{7}} = \sqrt{a} - \sqrt{b}$. Find the value of $|a - b|$. [12]

[1] Not sure if this was voided, because one of the terms is 0/0.

5. A square with area 64 m^2 is divided into 4 small squares. Quarter circles are inscribed in the lower left and upper right squares, as shown in the figure. The remaining squares are then divided again into 4 smaller squares with quarter circles inscribed in the lower left and upper right squares. If this process is continued indefinitely, find the total area of all the quarter circles formed in this manner. $\left[16\pi \text{ m}^2\right]$



Difficult, five points, ninety seconds

1. Black Panther has 45 boxes, each of which contains N Vibranium weapons. If he tried to distribute all the weapons evenly to his 2026 soldiers, he would have 2016 weapons left over. What is the smallest positive value of N? [450]

2. Let $S = \{1, 2, 3, \ldots, 20\}$ and let a_n denote the number of two-element subsets of S such that the sum of the two elements is divisible by n. Find $\displaystyle\sum_{i=2}^{5} a_i$. [237]

3. When an integer N has its last digit moved to the beginning of the number, its value is multiplied by 4. Suppose N has 6 as its last digit. What is the smallest possible value of N? [153846]

4. Find the sum of $\dfrac{1^2}{1^2 - 10 + 50} + \dfrac{2^2}{2^2 - 20 + 50} + \dfrac{3^2}{3^2 - 30 + 50} + \cdots + \dfrac{9^2}{9^2 - 90 + 50}$. [9]

5. Evaluate $\sin^2 1^\circ \sin^2 89^\circ + \sin^2 2^\circ \sin^2 88^\circ + \cdots + \sin^2 89^\circ \sin^2 1^\circ$. $\left[\dfrac{45}{4}\right]$

Semifinals B

Easy, two points, thirty seconds

1. Let s be the number of positive factors of 2016. Find the number of factors of s. [9]

2. Solve for the exact numerical value of the following continued fraction $2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}$. $\left[1 + \sqrt{2}\right]$

3. A cylinder has the same surface area as a sphere. If a sphere and the base of the cylinder have the same radius, what is the ratio of the volume of the sphere to that of the cylinder? [4 : 3]

4. Matt has 20 identical stickers he wants to give to his 4 friends. What is the probability that all of his friends gets at least one sticker? $\left[\dfrac{969}{1771}\right]$

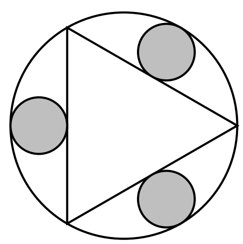
5. Two numbers have a sum of 195. If the greatest common factor of the numbers is 15 and their least common multiple is 540, find the sum of their squares. [21825]

Average, three points, sixty seconds

1. Thor is on his way from Knowhere to Nidavellir. Because he foresees that the fuel remaining in his ship is not enough to make it back to Knowhere after getting his hammer to Nidavellir, he calls for a refueling ship. The refueling ship then leaves from Knowhere, catches up to where Thor is, then returns to Knowhere without any fuel left. Suppose the refueling ship and Thor's spaceship can carry the same amount of fuel, enough for 150 light-years, and consume the same amount of fuel per light-year traveled. What is the maximum distance possible from Knowhere to Nidavellir? [100 light-years]

2. Define $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$ with $a_0 = 1$, $a_1 = 4$, and $S_n = a_n + a_{n-1} + \cdots + a_1 + a_0$. Find S_{1999}. [5999000]

3. Suppose an equilateral triangle is inscribed in a circle P, and three circles, A, B, and C, each tangent to a side of the triangle and circle P, are constructed. What is the ratio of the sum of the areas of the smaller circles, A, B, and C to the area of circle P? [3 : 16]



4. What is the remainder when $10^{10^{10^{2018}}}$ is divided by $1\,000\,000\,001$? [999 999 991]

5. If $\sin\theta - \cos\theta = \dfrac{\sqrt{2}}{2}$ and $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$, find all possible values of θ. $\left[\dfrac{5\pi}{12}\right]$

Difficult, five points, ninety seconds

1. Suppose K is a finite set of distinct real numbers having the following properties:
   - the mean of the elements of $K \cup \{2\}$ is 12 less than the mean of the elements of K
   - the mean of the elements of $K \cup \{2018\}$ is 51 more than the mean of the elements of K.

   What is the mean of the elements of K? [386]

2. A lattice point in the Cartesian plane is a point (x, y) where both x and y are integers. Find the area of the convex polygon whose vertices are all the lattice points on the circle $x^2 + y^2 = 100$. [296]

3. Suppose n is a positive integer such that $n^2 + 19n + 48$ is a perfect square. Find the value of n. [33]

4. Let $ABCD$ be a rectangle such that $AB = DC = 12$ and $AD = BC = 6$. Let O be a point in the interior of the rectangles such that $\angle DOA = 45°$ and $DO = 3\sqrt{6}$. Find the value of OB^2. $\left[72 - 18\sqrt{3}\right]$

5. Find the last two digits of the number $7^{7^{1000}}$. [07]

Finals

Teams sequentially choose question order, and can skip between rounds. If not stated, mechanics are as usual.

Mind Stone (MN) gives teams the choice to get twice the alloted points if correct. Can only be used twice.

Power Stone (PW) gives teams the choice to get twice the alloted points if correct and deducted the number of points if wrong or blank.

Time Stone (TM) gives teams a flag to raise: raising before 1/3 of the time is up gives twice if correct, deducted if wrong, none if blank; raising before 1/2 the time is up gives 1.5 times if correct, deduct 0.5 times if wrong, none if blank. Answers cannot be changed after raising the flag.

Reality Stone (RL) questions are normal.

Soul Stone (SL) gives teams the choice to allow a randomly chosen team member to sit out (decided by the roll of a die for each team) for 2.5 times the points if correct.

The question order this year is E-RL, A-RL, D-RL, V-RL, E-SL, E-PW, E-TM, E-MN, A-MN, D-MN, V-MN, V-SL(a), A-PW, A-TM, D-TM, D-SL, A-SL, V-TM, D-PW, V-PW, V-SL(b).

Easy, 2 points, 30 seconds

E-MN. Black Panther has 200 kg of the Heart-Shaped Herb, and 99% of its mass is water-weight. After a few days, some of the water evaporated and now only 98% of its mass is water-weight. What is the new mass of the Heart-Shaped Herb? [100 kg]

E-PW. Loki and Unloki were independently solving the roots of a quadratic function. Loki got the rooks -17 and 15, while Unloki got the roots 8 and -3. Loki realized he miscopied the constant term, while Unloki realized he miscopied the coefficient of the first degree term. What are the actual roots of the quadratic function? $[-6, 4]$

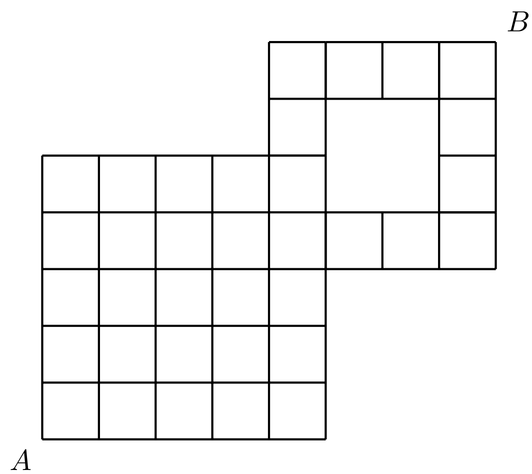
E-TM. Let $T = (2+1)\left(2^2+1\right)\left(2^4+1\right)\cdots\left(2^{256}+1\right)+1$. Find $T^{\frac{1}{64}}$. [256]

E-RL. In planet Titan, plate numbers are 2018-digit numbers that consist of only 2's and 7's. How many of these plate numbers do not contain a '27'? [2019]

E-SL. How many ways are there to choose 4 distinct integers a, b, c, and d from the set $[100, 125]$ such that $a < b < c < d$? [14950]

Average, 3 points, 60 seconds

A-MN. Sally wants to travel from point A to point B using the shortest path possible and traversing only along the edges of the map shown below. In how many ways can she do this? [2646]

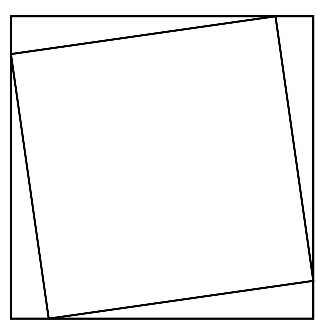


A-PW. Find the sum of the reciprocals of the factors of 1530. $\left[\dfrac{234}{85}\right]$

A-TM. Thanos and Doctor Strange are having a number battle. Thanos chooses the number 325! while Doctor Strange rolls 120 fair six-sided dice and his number is the product of all the results. To win, the number of trailing zeroes in Doctor Strange's number must be greater than or equal to the trailing zeroes in Thanos's number. If the probability that Doctor Strange wins is $\dfrac{n}{6^{120}}$, find n. You may express your answer in terms of factorials. $\left[\dfrac{120!}{40!80!}\right]$

A-RL. How many positive integers less than 999 999 999 999 contain only the digits 0 or 1? [4095]

A-SL. Four congruent right-angled triangles are arranged such that their legs form the shape of a square, and their hypotenuses form another square inscribed in the larger one, as shown in the figure. If the ratio of the area of the inscribed square to the area of the larger square is 25 : 32, what is the ratio of the shorter leg of the right-angle triangle to its longer leg? [1 : 7]



Difficult, 5 points, 90 seconds

D-MN. Find the number of factors of the coefficient of x^{17} in the expansion of $(x-2)^{20}+(2+x)^{21}$. [60]

D-PW. We define the sequence $\{a_n\}$ as follows: $a_n = \dfrac{10n^2-n-3}{10n^2+19n+6}$, $n \geq 1$. Find the product of the first 100 terms of the sequence $\{a_n\}$. $\left[\dfrac{3}{50953}\right]$

D-TM. In how many ways can the numbers $1, 2, 3, 4, 5, 6$ be arranged in a row if the product of every pair of adjacent numbers must be even? [144]

D-RL. Find the greatest common divisor of $2^{2018} + 2072$ and $2^{2019} + 2128$. [24]

D-SL. Given $\sin A = \dfrac{5}{13}$ and $\cot B = \dfrac{3}{4}$, A and B in Quadrant I, find the exact value of $\sin(2A - B)$. $\left[-\dfrac{116}{845}\right]$

Very Difficult, 8 points, 120 seconds

V-MN. Given the system

$$\begin{aligned} x^2 + y^2 + \frac{\sqrt{3}}{2}xy &= 32 \\ x^2 + z^2 + \frac{1}{2}xz &= 16 \\ y^2 + z^2 &= 16. \end{aligned}$$

Find the value of $xy + \sqrt{3}xz + 2yz$. [32]

V-PW. Consider an 80×80 chessboard which you wish to tile with 2×1 featureless dominoes. How many ways can you completely tile the chessboard with dominoes such that exactly 2 dominoes are vertically aligned, while the rest are horizontally aligned? [64780]

V-TM. Find the sum

$$\sum_{x=0}^{2018} \left(6x^5 + 45x^4 + 140x^3 + 225x^2 + 186x + 63\right).$$

Express your answer in the form $a^b - 1$ where a is minimized. $\left[2020^6 - 1\right]$

V-RL. Define $f(n) = n! \times (n-1)! \times \cdots \times 3! \times 2! \times 1!$. How many trailing zeroes are there to the right of $f(256)$ when it is evaluated? [7776]

V-SL. a) Evaluate $3 \cdot 3^{1!} + 9 \cdot 3^{2!} + 27 \cdot 3^{3!} + \cdots + 3^{2018} \cdot 3^{2018!}$.[2] $\left[3^{2019!} - 3\right]$

b) Solve for the value(s) of $\theta \in \{0, \pi\}$ in the equation $\sin 2\theta - \left(2 - \sqrt{3}\right)\cos 2\theta = \sqrt{3} - 1$. $[30^\circ, 75^\circ]$

[2] Voided and replaced with the next question.