Angle Chasing

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MOSC

March 24, 2019

Let's warm up!

1. Do twenty jumping jacks. (This warm-up does not involve angle chasing. But exercise is good for you!)
2. Quadrilateral $WXYZ$ has $WY \perp XZ$, $\angle WZX = 30^\circ$, $\angle XWY = 40^\circ$, and $\angle WYZ = 50^\circ$. Find $\angle WXY$.
3. In triangle ABC, let D, E, F be the feet of the altitudes from A, B, C. Let H be its orthocenter.
    1. Identify six cyclic quadrilaterals with vertices among $\{A, B, C, D, E, F, H\}$.
    2. Show that $\triangle AEF \sim \triangle ABC$. What about other triangles?
    3. Prove that H is the incenter of triangle DEF.
    4. Let X be the reflection of H over BC. Show that X lies on (ABC) (the circumcircle of triangle ABC.)
    5. Let Y be the reflection of H over the midpoint of BC. Show that AY is a diameter of (ABC).

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These last two results are known as "reflecting the orthocenter".

Example

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In parallelogram $ABCD$, $AC > BD$. Let P be a point on AC such that $BCDP$ is cyclic. Prove that BD is tangent to (ADP).

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In triangle ABC, $AC = BC$. Point M is the midpoint of AB. Point D lies on line CM. Let K and L be the feet of the perpendiculars from D and C onto BC and AD, respectively. Prove that K, L, and M are collinear.

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Rather than proving ℓ, m, and n are concurrent, it's more common to let P as ℓ and m's intersection and then prove that P lies on n.

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2. Prove that $ABCP$ is cyclic. You might want to use the fact that $\triangle ABE \cong \triangle CBF$. Then, D lies on this circle because $ABCD$ is a square, so in fact $ABCPD$ is cyclic.

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2. Prove that $ABCP$ is cyclic. You might want to use the fact that $\triangle ABE \cong \triangle CBF$. Then, D lies on this circle because $ABCD$ is a square, so in fact $ABCPD$ is cyclic.
3. Similarly, prove $EPGF$ is cyclic, and prove $EPGFB$ is cyclic.
4. Finally, $\angle DPB = 90^\circ$ by $(ABCPD)$. Also $\angle BPG = 90^\circ$, why? Why do these two show that D, P, and G are collinear.

Example

In parallelogram $ABCD$, $AC > BD$. Let P be a point on AC such that $BCDP$ is cyclic. Prove that BD is tangent to (ADP).

Recall the tangency criteria. Let ABC be inscribed in a circle, and P a point outside the circle. If P and C lie on opposite sides of AB, then PA is tangent to (ABC) if and only if $\angle PAB = \angle ACB$.

1. For this problem, it's sufficient to prove $\angle PDB = \angle DAP$. Prove this.

Example

In triangle ABC, $AC = BC$. Point M is the midpoint of AB. Point D lies on line CM. Let K and L be the feet of the perpendiculars from D and C onto BC and AD, respectively. Prove that K, L, and M are collinear.

Right angles make lots of cyclic quadrilaterals. That's because two right angles facing each other can form opposite angles of a cyclic quadrilateral, and two right angles facing in the same direction can form adjacent vertices of a cyclic quadrilateral.

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2. Similarly, prove that $(CDLK)$.
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4. Now draw another diagram where D still lies on the line CM, but *outside* the triangle. Does the logic of the previous case still work?

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You may notice our angle chase doesn't work completely now. We'll fix this using something called directed angles. But first, some exercises!

Let's angle chase!

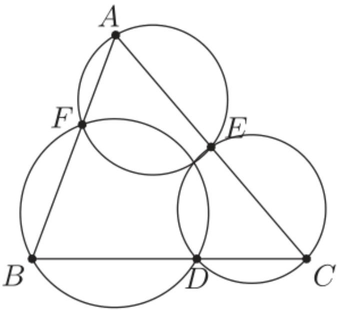
1. Let $ABCDE$ be a convex pentagon such that $BCDE$ is a square with center O and $\angle A = 90^\circ$. Prove that AO bisects $\angle BAE$.
2. Two parallel lines are tangent to a circle with center O. A third line, also tangent to the circle, meets the two parallel lines at A and B. Prove that AO is perpendicular to OB.
3. Let ABC be a triangle, O be the circumcenter, and let AO meet BC at D. Point K is selected so that KA is tangent to (ABC) and $\angle KCB = 90^\circ$. Prove that KD is parallel to AB.
4. Let $ABCD$ be a cyclic quadrilateral. Let M be the midpoint of arc BC not containing A or D. Let E and F be the intersections of AM and DM with BC, respectively. Show that A, E, F, D lie on the same circle.

Sometimes, angle chasing problems have lots of different configurations.

Example

(Miquel) Points D, E, and F are chosen on lines BC, CA, and AB of a triangle ABC. Prove that (AEF), (BFD), and (CDE) concur.

This is a classic example of a problem which is hard with normal angles, because there are so many different cases!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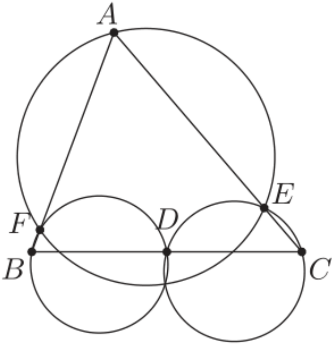
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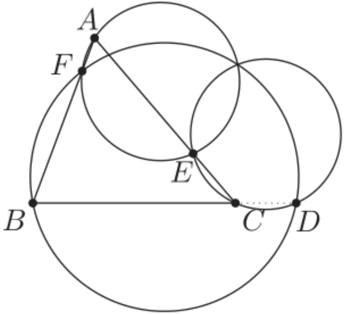
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The trick is to use **directed angles**. Directed angles are taken counterclockwise, and then taken modulo 180°.

I don't have enough time to write a full explanation here (it's much easier to explain face-to-face). The quick explanation is to write a "regular" proof, and then simply rewrite all the angles so that they're facing "counterclockwise".

What do I mean by "facing counterclockwise?" I mean that if you write $\measuredangle ABC$, you're referring to the angle when you take AB, and turn it counterclockwise until it aligns with BC.

Sometimes this is a reflex angle. Make sure you see the difference between $\measuredangle ABC$ and $\measuredangle CBA$. One of them is a reflex angle, one of them is not.

Don't! Forget! To! Direct! Your! Angles!

Example

In triangle ABC, $AC = BC$. Point M is the midpoint of AB. Point D lies on line CM. Let K and L be the feet of the perpendiculars from D and C onto BC and AD, respectively. Prove that K, L, and M are collinear.

Here's a segment of our old proof: By $(CAML)$, $\angle ALM = \angle ACM$. We know that $\angle CLA = 90^\circ$. By $(CDLK)$, $\angle KLC = \angle KDC$.

1. Draw a diagram where the points of ABC are labeled counterclockwise. Make sure D lies on the *segment* CM. To direct $\angle ALM$, we write it as $\measuredangle ALM$: you start with segment AL, turn it counterclockwise to get LM.
2. Similarly, $\measuredangle CLA = 90^\circ$ (right angles are always directed), and $\measuredangle KLC = \measuredangle KDC$. Note that it's **not** true that $\measuredangle CLK = \measuredangle KDC$, do you see why?
3. Now draw a different diagram; you should see that the angle chase still works.

Let's use directed angles!

1. (Reim) Suppose that the circles ω_1 and ω_2 intersect at distinct points A and B. Let CD be any chord on ω_1, and let E and F be the second intersections of the lines CA and BD, respectively, with ω_2. Prove EF is parallel to DC.
2. (Tritangent) Let ABC be an acute triangle with altitudes BE and CF. Let M be the midpoint of BC. Prove that ME, MF, and the line through A parallel to BC are all tangents to (AEF).
3. In cyclic quadrilateral $ABCD$, let I_1 and I_2 denote the incenters of $\triangle ABC$ and $\triangle DBC$, respectively. Prove that $I_1 I_2 BC$ is cyclic.
4. (Simson) Let ABC be a triangle and P be a point on (ABC). Let X, Y, and Z be the feet of the perpendiculars from P onto BC, CA, and AB, respectively. Prove that X, Y, and Z are collinear.

This example wasn't actually proven during our lecture. But it really is just angle chasing:

Example

(Incenter–Excenter) Let ABC be a triangle with incenter I, A-excenter I_A, and denote by L the midpoint of the arc BC not containing A. Show that L is the center of a circle through I, I_A, B, C.

You should prove it!

The following problems also weren't discussed during lecture; feel free to come back to them later.

Let's use the incenter–excenter lemma!

1. (CGMO 2012) Let ABC be a triangle. The incircle of $\triangle ABC$ is tangent to AB and AC at D and E, respectively. Let O be the circumcenter of $\triangle BCI$. Prove that $\angle ODB = \angle OEC$.
2. (Nine-point) Let ABC be a triangle with orthocenter H. Let D, E, F be the altitudes from A, B, C to the opposite sides. Show that the midpoint of AH lies on (DEF).
3. (IMO 2006) Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$, with equality if and only if $P = I$.
4. (JBMO 2010) Let AL and BK be angle bisectors of scalene triangle ABC. The perpendicular bisector of BK intersects line AL at M. Point N lies on line BK such that LN is parallel to MK. Prove that $LN = NA$.

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Answer: **Draw stuff in the right order!** First, choose a point O as the circumcenter, and draw a circle. Then choose points A, B, and C on it to form the triangle.

The key is to then use the incenter–excenter lemma. By using the ruler, you can construct the line passing through O perpendicular to BC. This line hits the circumcircle at the midpoint of arc BC, say it's M. Then AM is an angle bisector, no compasses needed!

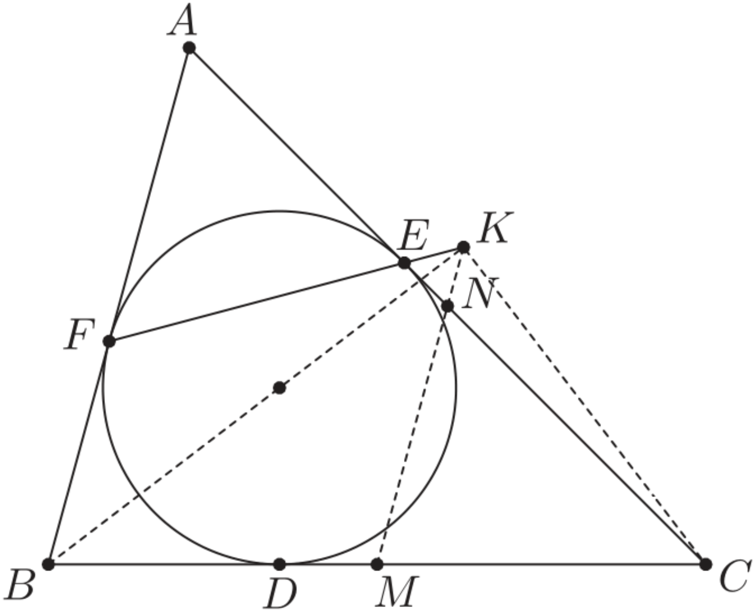
For more details, I strongly recommend reading Some Notes on Constructing Diagrams.

Let's play a game of "identify the cyclic quadrilaterals"! Lots of geometry problems involve finding a cyclic quadrilateral and proving it's cyclic. An important skill is identifying which quadrilaterals are cyclic based on just the diagram.

Example

(Right Angle on Incircle Chord) The incircle of $\triangle ABC$ is tangent to BC, CA, AB at D, E, F, respectively. Let M and N be the midpoints of BC and AC, respectiely. Ray BI meets line EF at K. Show that $BK \perp CK$. Then show that K lies on line MN.

Diagram next slide.


A
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4. If you pick two adjacent vertices, you could try to show the subtended angles are equal. If we pick E and K, you need to show $\measuredangle IEC = \measuredangle IKC$ (nope). Or K and C, and show $\measuredangle EKI = \measuredangle ECI$ (nope). Or C and I, and show $\measuredangle KCE = \measuredangle KIE$ (also nope).

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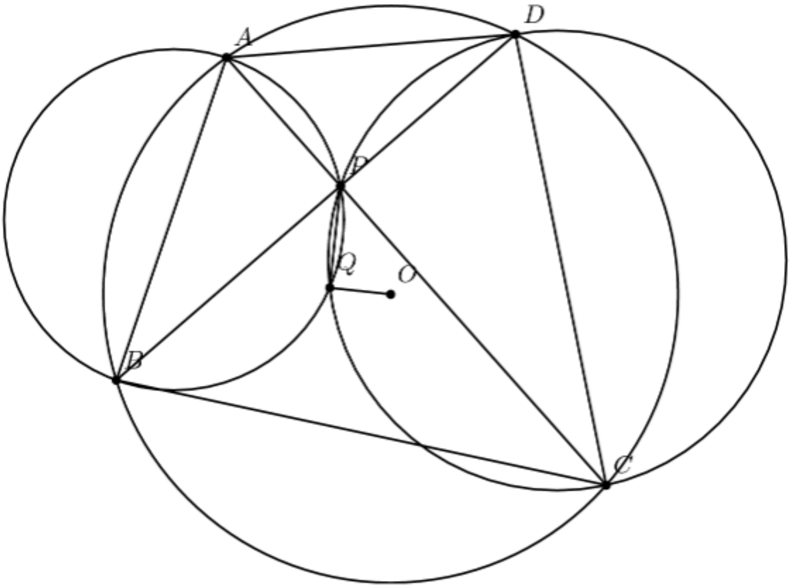
5. If we pick I and E, we can try to show $\measuredangle KIC = \measuredangle KEC$. This is promising, because $\measuredangle KIC$ is nice, it's also $\measuredangle BIC$. And $\measuredangle KEC = \measuredangle FEA$ due to vertical angles. Can you finish the proof from here?

Example

(China) A convex quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC and BD intersect at P. The circumcircles of triangles ABP and CDP intersect again at Q. If O, P, and Q are three distinct points, prove that OQ is perpendicular to PQ.

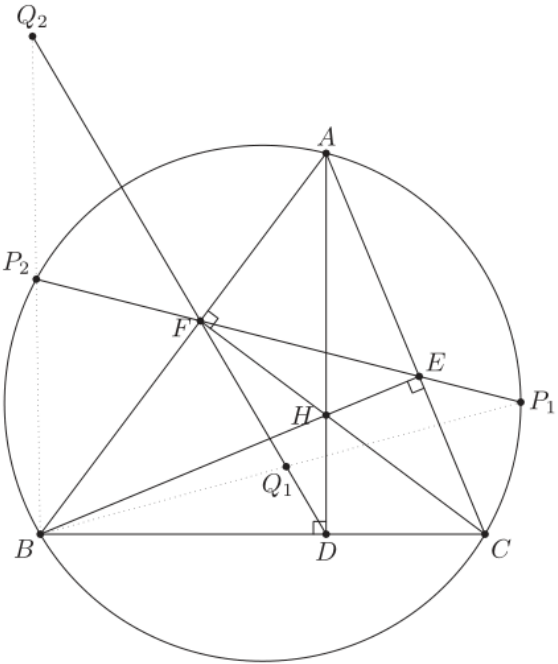

D
A
P
Q
O
B
C

The key cyclic quadrilateral here is $BQOC$ or $AQOD$.

Verify this with some different diagrams.

Example

(ISL 2010) Let ABC be an acute triangle with D, E, and F the feet of the altitudes lying on BC, CA, and AB, respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.


Q_2
A
P_2
F
E
P_1
H
Q_1
B
D
C

Note that there are two possible choices for P (and thus Q), indicated with P_1 and P_2 (and thus Q_1 and Q_2). In both cases, you have $APQF$ as cyclic.

Verify this with some different diagrams.

The following exercises weren't discussed in class, and all involve "identifying a cyclic quadrilateral and proving it's cyclic."

Let's do some exercises!

1. Do twenty push-ups.
2. (IMO 2004) Let ABC be acute and scalene. The circle with diameter BC meets AB and AC at M and N. Let O be the midpoint of BC. The bisectors of $\angle BAC$ and $\angle MON$ meet at R. Show (BMR), (CNR), and BC concur.
3. (Russia 1996) Let E and F be on side BC of convex quad. $ABCD$ (with E closer than F to B). Given $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$, show $\angle CAF = \angle EDB$.
4. (NIMO 2013) Let ABC have orthocenter H. Let M be the midpoint of BC. Lines AB and AC meet (BHM) and (CHM) again at P and Q. Rays PH and QH meet (CHM) and (BHM) again at R and S. Show M, R, S are collinear.
5. (Iran 2004) Let $ABCD$ be cyclic. Perpendiculars to AD and BC at A and C meet at M, and perpendiculars to AD and BC at D and B meet at N. Lines AD and BC meet at E. Show $\angle DEN = \angle CEM$.

Let's get even more exercise!

1. (BAMO 1999) Points O, A, and B are collinear in that order. P is a point on the circle with diameter AB. Q lies on line PA such that $OQ \perp OA$. Show $\angle BQP = \angle BOP$.
2. (IMO 2013) Let ABC be acute with orthocenter H. W is on BC, between B and C. M and N are the feet of the altitudes drawn from B and C. X is such that WX is a diameter of (BWN). Y is such that WY is a diameter of (CWM). Show X, Y, H are collinear.
3. In ABC, D and E are on BC so AD is an altitude and AE is an angle bisector. M is on AE such that $BM \perp AE$ and N is on AC such that $EN \perp AC$. Show D, M, and N are collinear.
4. (IMO 2002) BC is a diameter of Ω with center O. A is on Ω such that $\angle AOB < 120°$. D is the midpoint of arc AB (not containing C). The line through O parallel to DA meets line AC at I. The perpendicular bisector of OA meets Ω at E and F. Show I is the incenter of CEF.

References

Much of the content, including some diagrams, are copied from Chapter 1 in Euclidean Geometry in Mathematical Olympiads. The first three chapters can be previewed online on Google Books. Diagrams are being reused here under fair use. (Please don't sue me Evan.)