Angle chasing

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Warm-ups

1. Do twenty push-ups.

2. In quadrilateral $WXYZ$ with perpendicular diagonals, we are given $\angle WZX = 30°$, $\angle XWY = 40°$, and $\angle WYZ = 50°$. Compute $\angle WXY$.

3. (Orthic triangle) In triangle ABC, let D, E, and F denote the feet of the altitudes from A, B, and C, respectively. Let H be the orthocenter of triangle ABC.
   a) Identify six cyclic quadrilaterals with vertices among $\{A, B, C, D, E, F, H\}$.
   b) Show that $\triangle AEF \sim \triangle ABC$. What about other triangles?
   c) Prove that H is the incenter of triangle DEF.
   d) Let X be the reflection of H over BC. Show that X lies on (ABC).
   e) Let Y be the reflection of H over the midpoint of BC. Show that AY is a diameter of (ABC).

Lecture

1. A reminder of cyclic quadrilaterals and tangent circles

2. Directed angles

3. Incenter–excenter lemma

4. Drawing a diagram properly

   Required reading: Some Notes on Constructing Diagrams

5. Geoguessing

Angle chasing

1. Let $ABCDE$ be a convex pentagon such that $BCDE$ is a square with center O and $\angle A = 90°$. Prove that AO bisects $\angle BAE$.

2. Two parallel lines are tangent to a circle with center O. A third line, also tangent to the circle, meets the two parallel lines at A and B. Prove that AO is perpendicular to OB.

3. Let ABC be a triangle and let AO meet BC at D. Point K is selected so that KA is tangent to (ABC) and $\angle KCB = 90°$. Prove that KD is parallel to AB.

4. (BAMO 1999) Points O, A, and B are collinear in that order. Let P be a point on the circle with diameter AB. Point Q lies on line PA such that OQ and OA are perpendicular. Prove that $\angle BQP = \angle BOP$.

5. Let $ABCD$ be a cyclic quadrilateral. Let M be the midpoint of arc BC not containing A or D. Let E and F be the intersections of AM and DM with BC, respectively. Show that A, E, F, D lie on the same circle.

Configuration issues

1. Suppose that the circles ω_1 and ω_2 intersect at distinct points A and B. Let CD be any chord on ω_1, and let E and F be the second intersections of the lines CA and BD, respectively, with ω_2. Prove EF is parallel to DC.

2. (Tritangent) Let ABC be an acute triangle with altitudes BE and CF. Let M be the midpoint of BC. Prove that ME, MF, and the line through A parallel to BC are all tangents to (AEF).

3. In cyclic quadrilateral $ABCD$, let I_1 and I_2 denote the incenters of $\triangle ABC$ and $\triangle DBC$, respectively. Prove that $I_1 I_2 BC$ is cyclic.

4. (Simson) Let ABC be a triangle and P be a point on (ABC). Let X, Y, and Z be the feet of the perpendiculars from P onto BC, CA, and AB, respectively. Prove that X, Y, and Z are collinear.

Incenter–excenter lemma

1. (CGMO 2012) Let ABC be a triangle. The incircle of $\triangle ABC$ is tangent to AB and AC at D and E, respectively. Let O be the circumcenter of $\triangle BCI$. Prove that $\angle ODB = \angle OEC$.

2. (Nine-point) Let ABC be a triangle with orthocenter H. Let D, E, F be the altitudes from A, B, C to the opposite sides. Show that the midpoint of AH lies on (DEF).

3. (IMO 2006) Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$, with equality if and only if $P = I$.

4. (JBMO 2010) Let AL and BK be angle bisectors of scalene triangle ABC. The perpendicular bisector of BK intersects line AL at M. Point N lies on line BK such that LN is parallel to MK. Prove that $LN = NA$.

5. (IMO 2002) Let BC be a diameter of circle Ω center at O. Let A be a point of Ω such that $\angle AOB < 120°$. Let D be the midpoint of arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets Ω at E and at F. Prove that I is the incenter of triangle CEF.

Geoguessing

1. (IMO 2004) Let ABC be an acute scalene triangle. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of the side BC. The bisectors of $\angle BAC$ and $\angle MON$ intersect at R. Prove that (BMR) and (CNR) intersect at a point on BC.

2. (Russia 1996) Points E and F are given on the side BC of a convex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle CAF = \angle EDB$.

3. (Iran 2004) Let $ABCD$ be a cyclic quadrilateral. The perpendiculars to AD and BC at A and C respectively meet at M, and the perpendiculars to AD and BC at D and B meet at N. If the lines AD and BC meet at E, prove that $\angle DEN = \angle CEM$.

4. (IMO 2013) Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. ω_1 is the circumcircle of triangle BWN and X is a point such that WX is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that WY is a diameter of ω_2. Show that the points X, Y, and H are collinear.

5. (NIMO 2013) Let ABC be a triangle with orthocenter H and let M be the midpoint of BC. Denote by ω_B the circle passing through B, H, and M, and denote by ω_C the circle passing through C, H and M. Lines AB and AC meet ω_B and ω_C again at P and Q, respectively. Rays PH and QH meet ω_C and ω_B again at R and S, respectively. Prove that M, R, S are collinear.

6. (EGMO 2012) Let ABC be an acute-angled triangle with circumcenter Γ and orthocenter H. Let K be a point on Γ on the other side of BC from A. Let L be the reflection of K in the line AB, and let M be the reflection of K in the line BC. Let E be the second point of intersection of Γ with (BLM). Show that the lines KH, EM, and BC concur.