Compass and ruler

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With a compass and a ruler, here are two “classical” constructions. Two of these are from Sharygin, because Sharygin almost always has a compass-and-ruler construction problem.

1. Construct a triangle given three segments: the first has the same length of one of its sides, the second has the same length as the altitude to that side, and the third has the same length as its circumradius.

2. (Sharygin) Construct points X and Y on sides AB and BC, respectively, of triangle ABC so that $AX = BY$ and $XY \parallel AC$.

3. (Sharygin) Let AP and BQ be altitudes of acute-angled triangle ABC. Using a compass and a ruler, construct a point M on side AB such that $\angle AQM = \angle BPM$.

Still with a compass and ruler, here are some slightly different kinds of constructions:

4. Given a segment of length $7\sqrt{3}$, construct a segment of length $\sqrt{7}$.

5. Given three concurrent lines, construct a triangle with those lines as angle bisectors.

6. Disect a given 19° angle into 19 equal parts.

7. Construct a heptagon given the midpoints of its sides.

8. Construct an equilateral triangle whose vertices lie on three given parallel lines.

With slightly different tools, we can construct other things too. We assume that no matter what tools we have, we can always:

- Mark the intersections of two objects drawn, and
- Mark an arbitrary point on a given object or on the plane.

Then:

9. With an unmarked, infinite two-sided ruler, construct the midpoint of a given segment.

10. Given a regular unit hexagon and a ruler, construct a segment of length (a) 2019, (b) $\dfrac{1}{2019}$, (c) $\sqrt{2019}$.

    Extension: For which n can we construct segments of the length $\sqrt{n}$? These are exactly the integers n where we can write $n = a^2 - ab + b^2$ for some integers a, b. These are closely related to the Eisenstein integers, numbers of the form $a + b\omega$, where ω satisfies $\omega^3 = 1$ and $\omega \neq 1$.

11. Construct a segment joining two points with only a ruler shorter than their distance.

12. (Prasolov) An angle is drawn on a sheet of paper. Using only the corner of a rectangular metal sheet, draw its bisector.

13. (Half of ELMO 2018/3) Let A be a point in the plane, and ℓ a line not passing through A. Evan has a special compass which has the ability to draw a circle through three distinct noncollinear points. Construct the reflection of A over ℓ.

14. (Prasolov) Take a coin, trace its circumference, and choose a point on it. Using the coin, we can construct the point's antipode.

Hints

1. To use the altitude, draw a line ℓ perpendicular to the side m. Draw a circle centered on $\ell \cap m$ with radius the altitude. Let it intersect ℓ at point P. Then draw a line through P perpendicular to ℓ.

2. Draw angle bisector BZ, then line ZY parallel to AB.

3. Choose M such that (QPM) is tangent to AB.

4. Lots of different ways to do this; one way is power of a point.

5. Pick a point A on the first line. Reflect it over the other two lines.

6. Modulo $360°$.

7. All you have to do is construct a parallelogram!

8. Pick a point A on the first line. Rotate the whole configuration $60°$ about A, and overlay it on the original diagram.

9. In triangle ABC, let E and F be on AC and AB such that $EF \parallel BC$. Let $X = BE \cap CF$. What is $AX \cap BC$?

10. Construct the triangular grid.

11. Draw two rays through the first point A such that point B is contained within the two rays, and the rays form a sufficiently small angle. Pick points P, P' on the first ray, Q, Q' on the second ray, such that $PQ' \cap P'Q = B$. Use $PQ \cap P'Q'$.

12. First figure out how to reflect a point over another point. Then use Thales's theorem.

13. Invert about the circle.

14. Let ω_1 be the initial circle, and A_1 be the given point. Draw a circle ω_2 through A_1, and let $A_2 = \omega_1 \cap \omega_2$. Draw a circle ω_3 through A_2, and let $A_3 = \omega_2 \cap \omega_3$. Draw ω_4 through A_3, and let $B_1, A_4 = \omega_1 \cap \omega_3$. Finally, draw ω_5 through both B_1 and A_4. The desired point is $\omega_1 \cap \omega_5$.

References

Yaglom's Geometric Transformations I is the source for problems 5, 7, and 8. Problems 1, 6, 9, 11, 12, 14 are taken from Prasolov's Problems in Plane and Solid Geometry. Pretty much all the problems here are folklore anyway.