Constructions

Carl Joshua Quines

November 5, 2018

Disclaimer: This is an *experimental* problem set. I make no guarantees of usefulness. Take all of this with a grain of salt.

All of the problems in this set have solutions that make use of a construction that makes it easier. The idea is "certain parts of a problem should signal trying a construction". It may not solve the problem, but is often a key step. The problems are categorized under type of construction, and what feature the problems have that triggers the construction.

Hints are provided after the problems. References used in preparing this problem set appear at the end.

1 Problems

1.1 Unrolling

If you have a condition like $AB + CD = EF$, you *almost always* want to "unroll" or "open the gates".[1] These conditions have a point in common on the same side or both sides. Here, we distinguish between two kinds, based on the construction:

Same point on the same side

The first is "constructive": turning the sum of two segments to a segment. If $AX + AY$ appears on one side, construct a point Y' on ray AX such that $XY' = AY$. Then $AX + AY = AY'$, and AY' hopefully makes an isosceles triangle, parallelogram, isosceles trapezoid, or cyclic quadrilateral.

Note that you can try constructing on ray AY instead. Trying to construct on the opposite direction seems to help sometimes.

1. (IGO 2017/E3) In regular pentagon $ABCDE$, point F lies on segment AB such that $\angle FCD = 90^\circ$. Prove that $AE + AF = BE$.

2. (Brazil 2006/1) In triangle ABC, the internal bisector of $\angle B$ meets AC at P. Let I be the incenter of triangle ABC. Prove that if $AP + AB = CB$, then API is an isosceles triangle.

3. (IMO 1997/2) Angle A is the smallest in triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet line AU at V and W, respectively. Lines BV and CW intersect at T. Show that $AU = TB + TC$.

4. (van Schooten's theorem) Let P be a point on minor arc BC of equilateral triangle ABC. Then $PA = PB + PC$.

5. (IMO 2001/5) Let ABC be a triangle with $\angle BAC = 60^\circ$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB + BP = AQ + QB$, what are the angles of the triangle?

[1] Problem 5 in section Grab Bag is one of the few exceptions I know.

Same point on opposite sides

The other is "destructive": turning the difference of two segments to a segment. If AX appears on one side and AY on the other, choose a point X' on ray AY such that $AX' = AX$. This makes AXX' an isosceles triangle, and $X'Y$ the difference, where $X'Y$ hopefully makes an isosceles triangle, parallelogram, isosceles trapezoid, or cyclic quadrilateral.

Note that you can try constructing on ray AX instead. Trying to construct on the opposite direction seems to help sometimes.

A specific form of this looks like $AX + BY = XY$. In this case, there's a point P on XY that forms isosceles triangles AXP and BYP. This may or may not help: sometimes the point on rays AX or BY helps more.

1. (IGO 2016/E5) Let $ABCD$ be a convex quadrilateral such that $\angle ADC = 135°$ and $\angle ADB - \angle ABD = 2\angle DAB = 4\angle CBD$. If $BC = CD\sqrt{2}$, prove that $AB = BC + AD$.

2. (IMO 1985/1) A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

3. (JBMO SL 2014/G1) Let ABC be a triangle with $\angle ABC = \angle BCA = 40°$. The angle bisector of $\angle ABC$ meets side AC at D. Prove that $BD + DA = BC$.

4. (Bosnia and Herzegovina TST 2011/1) Let ABC be a triangle such that $AB + AC = 2BC$. Show that the midpoints M of AB, N of AC, its incenter I, and A lie on the same circle.

5. (Polish JMO Finals 2018/2) Let $ABCD$ be a trapezoid with AB parallel to CD and $AB+CD = AD$. Diagonals AC and BD intersect at E. A line passing through E parallel to AB cuts AD at F. Prove that $\angle BFC = 90°$.

6. (IZhO 2010/2) In a cyclic quadrilateral $ABCD$ with $AB = AD$ points M and N lie on the sides BC and CD respectively so that $MN = BM + DN$. Lines AM and AN meet the circumcircle of $ABCD$ again at points P and Q respectively. Prove that the orthocenter of the triangle APQ lies on the segment MN.

7. (ISL 2010/G5) Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90°$, prove that $2\angle BDA = \angle CDE$.

1.2 Parallelograms

Existing parallelogram or isogonality

If there is an existing parallelogram, you should try constructing another if it's useful. In particular, if you have parallelogram $ABCD$, and a point E, then the same point F completes parallelograms $ABEF$ and $CDFE$.

Completing this diagram is helpful when you have other information about the angles: here, completing the diagram forms isogonal points or cyclic quadrilaterals because of the additional information.

1. (BMO2 2013/2) The point P lies inside triangle ABC so that $\angle ABP = \angle PCA$. The point Q is such that $PBQC$ is a parallelogram. Prove that $\angle QAB = \angle CAP$.

2. (Canada 1997/4) The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180^\circ$. Prove that $\angle OBC = \angle ODC$.

3. (ISL 2012/G2) Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, and G are concyclic.

4. (Taiwan TST2 2014/6) Let P be a point inside triangle ABC, and suppose lines AP, BP, CP meet the circumcircle again at T, S, R (here $T \neq A$, $S \neq B$, $R \neq C$). Let U be any point in the interior of PT. A line through U parallel to AB meets CR at W, and the line through U parallel to AC meets BS again at V. Finally, the line through B parallel to CP and the line through C parallel to BP intersect at point Q. Given that RS and VW are parallel, prove that $\angle CAP = \angle BAQ$.

5. (EGMO 2016/2) Let $ABCD$ be a cyclic quadrilateral, and let diagonals AC and BD intersect at X. Let C_1, D_1 and M be the midpoints of segments CX, DX and CD, respectively. Lines AD_1 and BC_1 intersect at Y, and line MY intersects diagonals AC and BD at different points E and F, respectively. Prove that line XY is tangent to the circle through E, F and X.

6. (ELMO 2012/5) Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

Equal segments

If you have two segments of the same length but they are far, a general way to take advantage is to construct a parallelogram. Sometimes, this makes an isosceles triangle or isosceles trapezoid.

1. ([1]) Let $ABCDE$ be a convex pentagon with $AB = BC$ and $CD = DE$. If $\angle ABC = 2\angle CDE = 120^\circ$ and $BD = 2$, find the area of $ABCDE$.

2. Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of triangle CGI at C and J. Prove that $IJ = AH$.

3. (IMO 2018/1) Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.[2]

4. (Iran TST3 2017/6) Let O and H be the circumcenter and orthocenter of triangle ABC. Let P be the reflection of A with respect to line OH, and suppose that P is not on the same side of line BC as A. Points E and F lie on AB and AC respectively such that $BE = PC$ and $CF = PB$. Let K be the intersection of lines AP and OH. Prove that $\angle EKF = 90^\circ$.

[2] For me, this feels conceptually closer to Problem 5 in section Grab Bag. Compare the solution posted on `http://aops.com/community/c6h1670580p10626602`.

Midpoints

If you have a midpoint of a segment, try completing a parallelogram with that segment as a diagonal. Sometimes useful to convert to angle conditions, sometimes forms cyclic quadrilaterals and the like.

1. ([1]) Let ABC be a triangle and let M be the midpoint of BC. Squares $ABQP$ and $ACYX$ are constructed externally. Show that $PX = 2AM$.

2. (USAMO 2003/4) Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

3. (NIMO 8/8) The diagonals of convex quadrilateral $BSCT$ meet at the midpoint M of $\overline{ST}$. Lines BT and SC meet at A, and $AB = 91$, $BC = 98$, $CA = 105$. Given that $\overline{AM} \perp \overline{BC}$, find the positive difference between the areas of $\triangle SMC$ and $\triangle BMT$.

4. (Euler's quadrilateral theorem) In quadrilateral $ABCD$, let M and N be the midpoints of diagonals AC and BD, respectively. Then $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$.

5. (ISL 2009/G4) Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

Tangents

These two problems have the same configuration and are conceptually similar in terms of the parallelogram to complete.

1. (IMO 2017/4) Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

2. (ELMO 2015/3) Let ω be a circle and C a point outside it; distinct points A and B are selected on ω so that $\overline{CA}$ and $\overline{CB}$ are tangent to ω. Let X be the reflection of A across the point B, and denote by γ the circumcircle of triangle BXC. Suppose γ and ω meet at $D \neq B$ and line CD intersects ω at $E \neq D$. Prove that line EX is tangent to the circle γ.

1.3 Halving segments and angles

If you have one segment twice another, like $AB = 2XY$, construct the midpoint of XY. If you have one angle twice another, like $\angle ABC = 2\angle XYZ$, construct the angle bisector of $\angle XYZ$. This gives you three equal segments or three equal angles to work with. Sometimes helpful.

1. (IGO 2015/E2) Let ABC be a triangle with $\angle A = 60^\circ$. The points M, N, and K lie on BC, AC, and AB respectively such that $BK = KM = MN = NC$. If $AN = 2AK$, determine the angles of triangle ABC.

2. (IGO 2018/I2) In convex quadrilateral $ABCD$, the diagonals AC and BD meet at P. Suppose $\angle DAC = 90^\circ$ and $2\angle ADB = \angle ACB$. If we have $\angle DBC + 2\angle ADC = 180^\circ$, prove that $2AP = BP$.

3. ([5]) In isosceles triangle ABC, $AB = AC$. Let D be on side BC such that $BD = 2DC$. Point P lies on segment AD such that $\angle ABP = \angle PAC$. Prove that $\angle BAC = 2\angle DPC$.

4. (USA TST 2001/5) In triangle ABC, $\angle B = 2\angle C$. Let P and Q be points on the perpendicular bisector of segment BC such that rays AP and AQ trisect $\angle A$. Prove that $PQ < AB$ if and only if $\angle B$ is obtuse.

1.4 Incenters and excenters

There doesn't seem to be any similarities between these problems, other than the fact they can be solved by constructing incenters or excenters. Sometimes the center already exists as a point, and identifying it makes the problem easier.

1. In triangle ABC, $\angle BAC = 120°$. The bisectors of the angles $\angle BAC, \angle ABC$, and $\angle BCA$ intersect the opposite sides at the points D, E and F, respectively. Prove that the circle with diameter EF passes through D.

2. (USAMTS 2012/3/3) In quadrilateral $ABCD$, $\angle DAB = \angle ABC = 110°$, $\angle BCD = 35°$, $\angle CDA = 105°$, and AC bisects $\angle DAB$. Find $\angle ABD$.

3. (All-Russian 2011/10/4) Triangle ABC has perimeter 4. Points X and Y lie on rays AB and AC respectively such that $AX = AY = 1$. Segments BC and XY intersect at point M. Prove that the perimeter of either triangle ABM or triangle ACM is 2.[3]

4. (ISL 2009/G3) Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

5. (USAMO 1999/6) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

6. (ISL 2017/G1) Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

7. (Geolympiad Summer 2015/5) Let ABC be a triangle and P be in its interior. Let Q be the isogonal conjugate of P. Show that $BCPQ$ is cyclic if and only if $AP = AQ$.

8. (IMO 2004/5) In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies $\angle PBC = \angle DBA$ and $\angle PDC = \angle BDA$. Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

1.5 Supplementary angles make two similar triangles

An oddly specific construction, which is kind of hard to tell when needed, involves a cyclic quadrilateral $ABCD$. We construct the point P on the segment AC such that BP and BD are isogonal. Then triangles ABP and DBC are similar, and triangles PBC and ABD are similar as well. Sometimes can be applied without cyclic quadrilaterals.

[3] Fun fact: this is often wrongly cited as Russia 2010.

1. (Ptolemy's theorem) In cyclic quadrilateral $ABCD$, $AC \cdot BD = AB \cdot CD + BC \cdot DA$.
2. (Generalization, Philippines 2019) In triangle ABC, D and E are points on sides AB and AC respectively. Point Y is in triangle ABC such that $\angle DYB$ and $\angle EYC$ are supplementary. Let X be a point inside the triangle such that $\angle XBC = \angle EBA$ and $\angle XCB = \angle DCA$. Prove that $\angle BAC$ and $\angle EXD$ are complementary.
3. (IOM 2018/6) The incircle of a triangle ABC touches the sides BC and AC at points D and E, respectively. Suppose P is the point on the shorter arc DE of the incircle such that $\angle APE = \angle DPB$. The segments AP and BP meet the segment DE at points K and L, respectively. Prove that $2KL = DE$.

1.6 Grab bag

These problems all involve a construction as a key step in their solution.

1. (Canada 2000/4) Let $ABCD$ be a convex quadrilateral with $\angle CBD = 2\angle ADB$, $\angle ABD = 2\angle CDB$ and $AB = CB$. Prove that $AD = CD$.
2. (EGMO 2013/1) The side BC of triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then triangle ABC is right angled.
3. (Italy TST 2001/1) The diagonals AC and BD of a convex quadrilateral $ABCD$ intersect at point M. The bisector of $\angle ACD$ meets the ray BA at K. Given that $MA \cdot MC + MA \cdot CD = MB \cdot MD$, prove that $\angle BKC = \angle CDB$.
4. (USA TST 2000/2) Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.
5. On arc BC of the circumcircle of triangle ABC, two points X and Y are chosen such that they both lie on the side of BC that is opposite of A, and $\angle BAX = \angle CAY$. Let M be the midpoint of chord AX. Show that $BM + CM \geq AY$.[4]
6. In triangle ABC, $\angle BCA = \angle CAB + 90°$. Point D is on ray BC such that $AC = AD$. Let point E be such that $\angle EBC = \angle CAB$ and $2\angle EDC = \angle CAB$. Prove that $\angle CED = \angle ABC$.
7. In triangle ABC, point D lies on line BC such that C is between B and D. Suppose there exists a unique point X on line AD such that $AX/BX = CX/DX$. Prove that triangle ABC is isosceles.
8. (Iran TST3 2017/2) Let P be a point in the interior of quadrilateral $ABCD$ such that: $\angle BPC = 2\angle BAC$, $\angle PCA = \angle PAD$, and $\angle PDA = \angle PAC$. Prove that $\angle PBD = |\angle BCA - \angle PCA|$.
9. (IMO 2018/6) A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle BXA + \angle DXC = 180°$.

[4] Compare with IMO 2018/1; in particular this solution on AoPS.

Adventitious quadrangles

Classical; don't take these too seriously. A general method for solving these problems exist, see [3].

1. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20^\circ$. Points D and E are on AC and AB respectively such that $\angle CBD = 40^\circ$ and $\angle BCE = 50^\circ$. Determine $\angle CED$.

2. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20^\circ$. Points D and E are on AC and AB respectively such that $\angle CBD = 50^\circ$ and $\angle BCE = 60^\circ$. Determine $\angle CED$.

3. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20^\circ$. Points D and E are on AC and AB respectively such that $\angle CBD = 60^\circ$ and $\angle BCE = 70^\circ$. Determine $\angle CED$.

4. In convex quadrilateral $ABCD$, $AB = BC = CD$, $\angle ABC = 70^\circ$, and $\angle BCD = 170^\circ$. Determine $\angle DAB$.

5. In convex quadrilateral $ABCD$, $\angle ABD = 12^\circ$, $\angle ACD = 24^\circ$, $\angle DBC = 36^\circ$, and $\angle BCA = 48^\circ$. Determine $\angle ADC$.

6. In convex quadrilateral $ABCD$, $\angle ABD = 38^\circ$, $\angle ACD = 48^\circ$, $\angle DBC = 46^\circ$, and $\angle BCA = 22^\circ$. Determine $\angle ADC$.

2 Hints

Same point on the same side

1. Construct on ray AE.
2. Construct on ray AB.
3. Construct on ray BT.
4. Find a clever way to prove lengths of equal segments.
5. Construct on rays AQ and AB.

Same point on opposite sides

1. Construct on ray AD.
2. Complete two isosceles triangles on AB.
3. Construct on ray BC.
4. Complete two isosceles triangles on BC.
5. Complete two isosceles triangles on BC.
6. Complete two isosceles triangles on MN.
7. Construct on ray AE.

Existing parallelogram or isogonality

1. Existing parallelogram $PBQC$, additional point A.
2. Existing parallelogram $ABCD$, additional point O.
3. Existing parallelogram $CEDG$, additional point F.
4. Existing parallelogram $PBQC$, additional point A.
5. Existing parallelogram C_1MD_1X, additional point Y.
6. Complete $BPCQ$. Use as existing parallelogram, additional point A.

Equal segments

1. Segment AB is a side. Rearrange to make area easier to calculate.
2. Segment JI is a side; H is a vertex.
3. Segment FD is a side; A is a vertex. Similarly, segment GE is a side; A is a vertex.
4. Segment PC is a side; B is a vertex.

Midpoints

1. Segment BC is a diagonal; A is a vertex.
2. Segment CF is a diagonal; D is a vertex.
3. Segment ST is a diagonal; A is a vertex.
4. Segment AC is a diagonal; B is a vertex. Also, segment AC is a diagonal; D is a vertex.
5. Segment CD is a diagonal; F is a vertex. Similarly, segment AB is a diagonal; F is a vertex.

Tangents

1. Complete parallelogram $RATP$.
2. Complete parallelogram $AYXC$.

Incenters and excenters

1. Construct A-excircle.
2. Observe C is A-excenter of ABD.
3. Construct A-excircle.
4. Construct A-excircle.
5. Observe F is A-excenter of ACD.
6. Construct incenter.
7. Construct incenter.
8. Construct B- and D-excenters of BPD.

Supplementary angles make two similar triangles

1. Construct K on diagonal AC such that BK and BD are isogonal.
2. Construct Z on side BC such that $\angle XZB = \angle DYB$.
3. Construct X on diagonal DE such that PX and PF are isogonal.

Grab bag

1. Intersect opposite sides of $ABCD$.
2. Observe A is centroid of EBD.
3. Draw circle with center C passing through D. Intersect with line AC.
4. Let AC and BD meet at P. Construct the midpoints of AD, BC, DP, and AP.
5. Construct an isosceles trapezoid with base AX and one vertex at B.
6. Let M be the midpoint of CD. Intersect AM with DE and BE.
7. Intersect AD with circumcircle of ABC.
8. Let T be the point such that BPT and APC are similar. Intersect AC and BT.
9. Construct the isogonal conjugate of X.

No hints will be provided for *Halving segments and angles* and *Adventitious quadrangles*.

References

[1] Evan Chen, All you have to do is construct a parallelogram!

[2] Evan Chen, Euclidean Geometry in Mathematical Olympiads.

[3] Hiroshi Saito, Completion of finding proofs for generalized Langley's problems in elementary geometry.

[4] Carlos Shine, Angle Chasing, MOP 2010.

[5] Carlos Shine, Some Useful Constructions, MOP 2012.

Thanks Andrew Wu, Sharvil Kesarwani, Vincent Huang, Ankan Bhattacharya, Michael Diao, Sean Ty, and Albert Patupat for suggestions. If you have an addition or correction, please contact me at `cj@cjquines.com`.