Exponents of Jacobians of Graphs and Regular Matroids

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Divisor theory and the Jacobian

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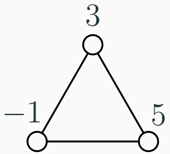

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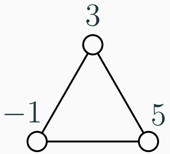

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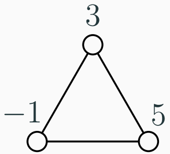

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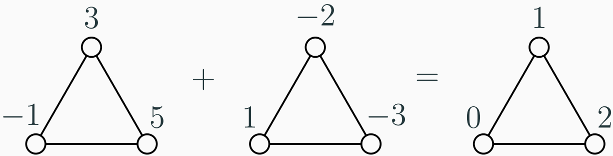

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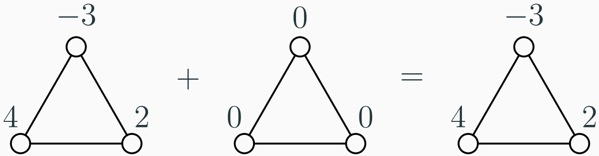

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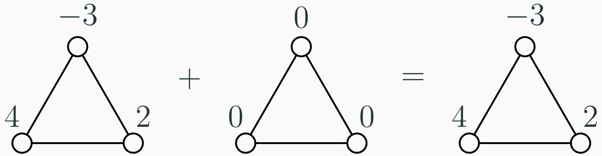

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THE DIVISOR GROUP

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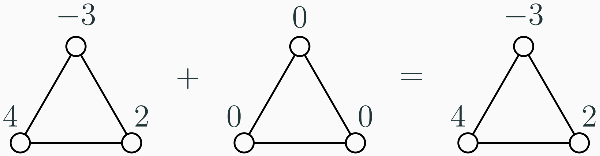

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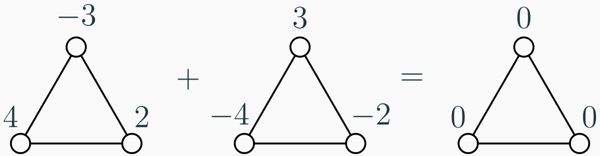

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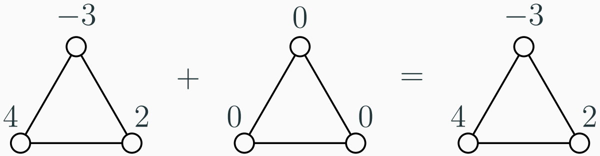

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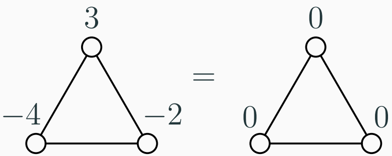

These form the divisor group. For a graph G, we call this $\mathrm{Div}(G)$.

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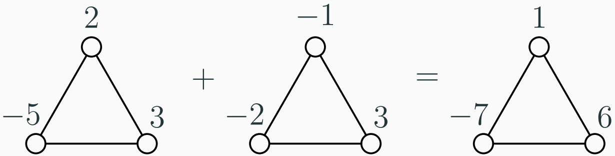

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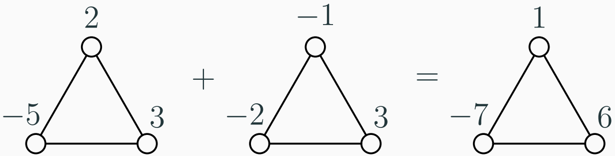

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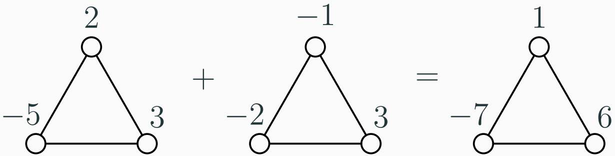

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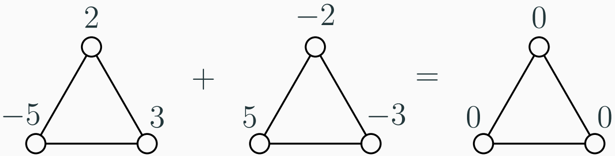

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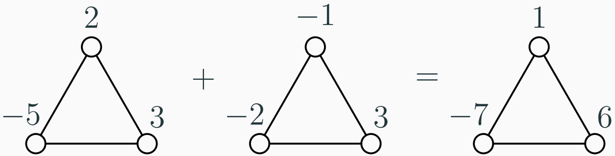

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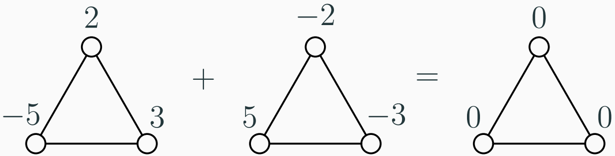

So these form a subgroup of $\mathrm{Div}(G)$, which we call $\mathrm{Div}^0(G)$.

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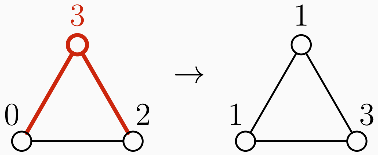

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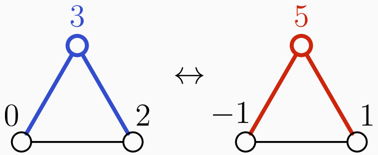

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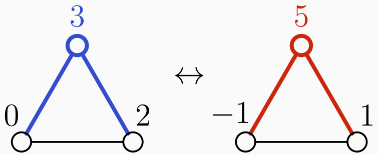

So toppling is reversible. Note that toppling a divisor with degree zero also gives a divisor with degree zero.

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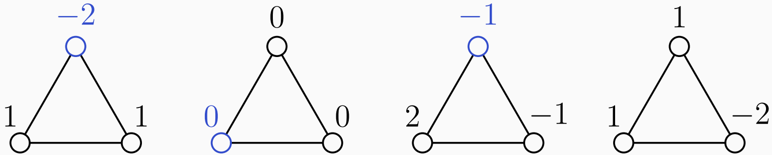

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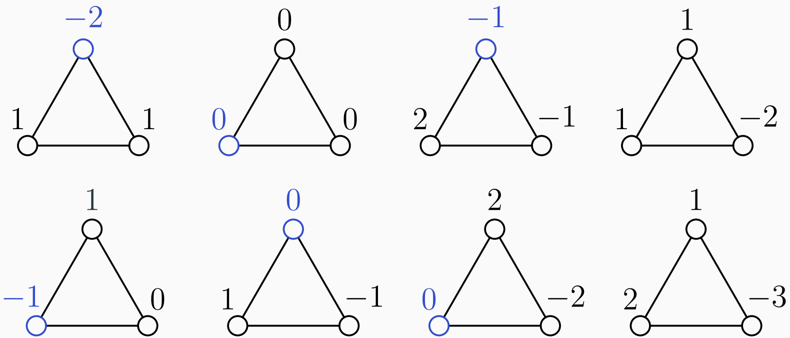

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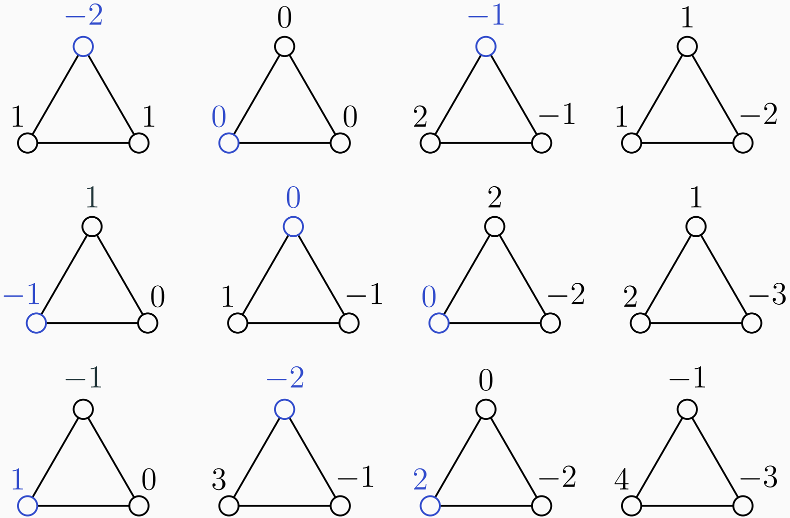
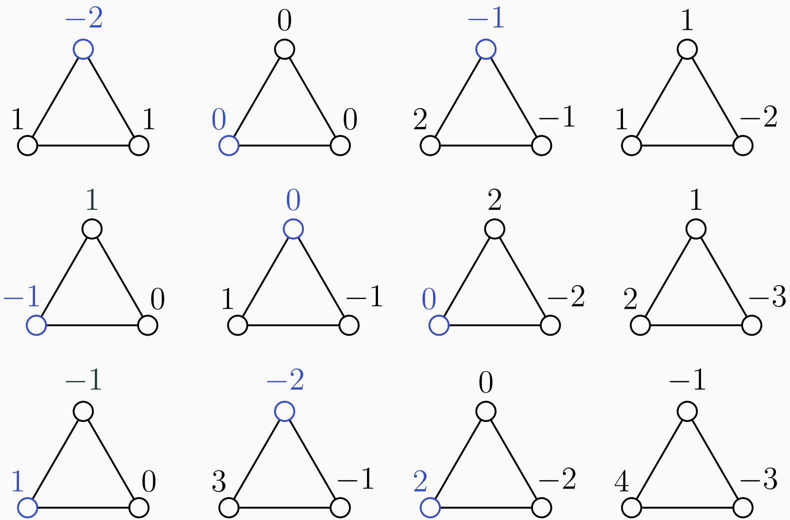
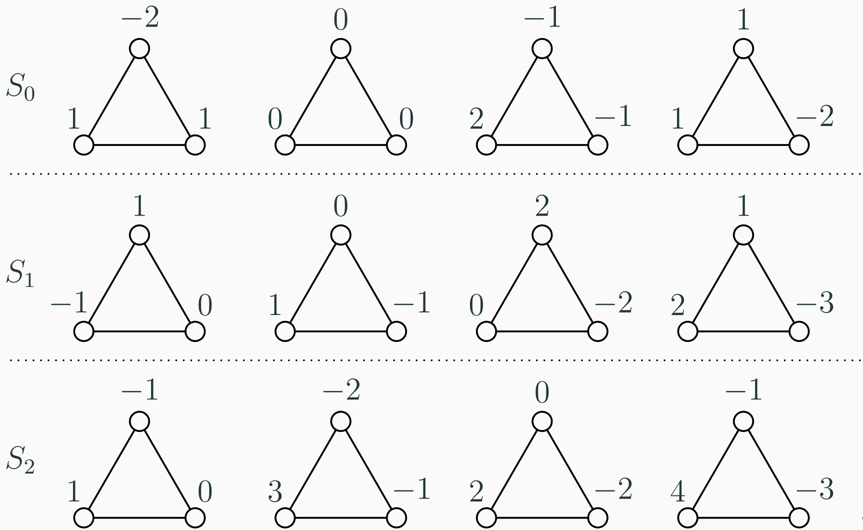


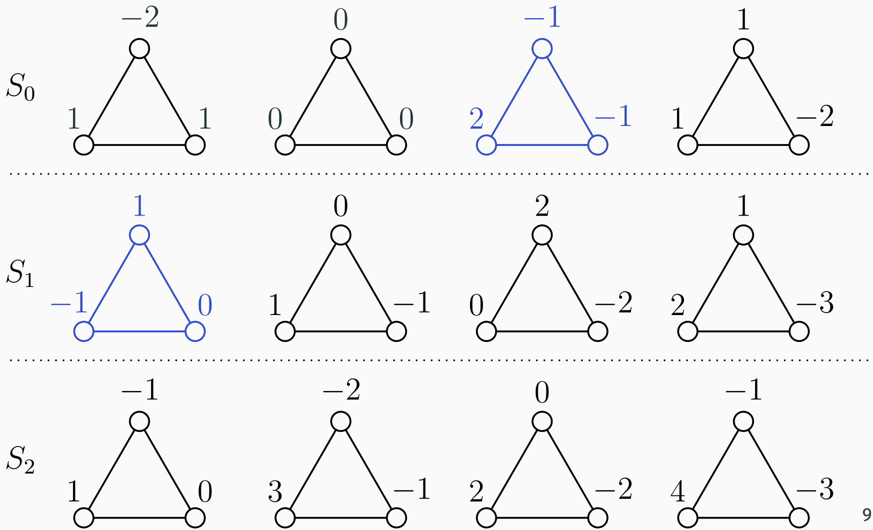
These sets partition the divisors with degree zero of this graph. All divisors with degree zero are in one of these equivalence sets.



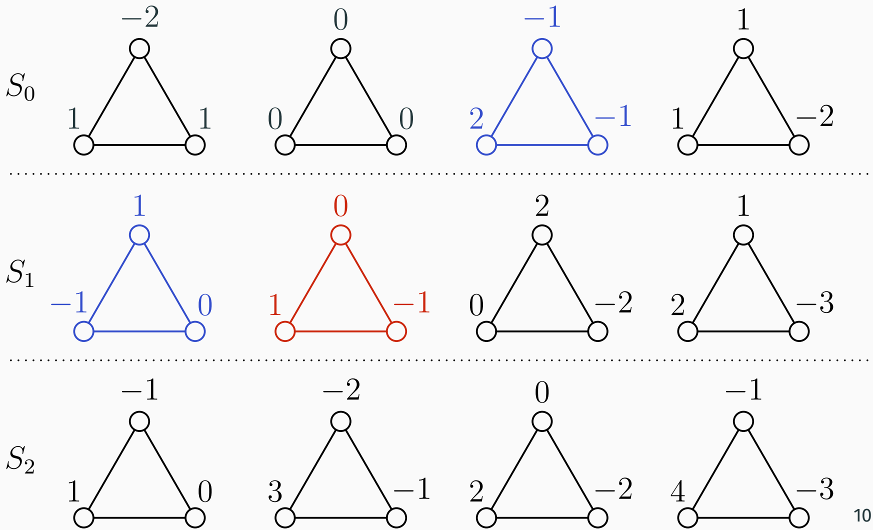
Let’s label the equivalence sets like so.



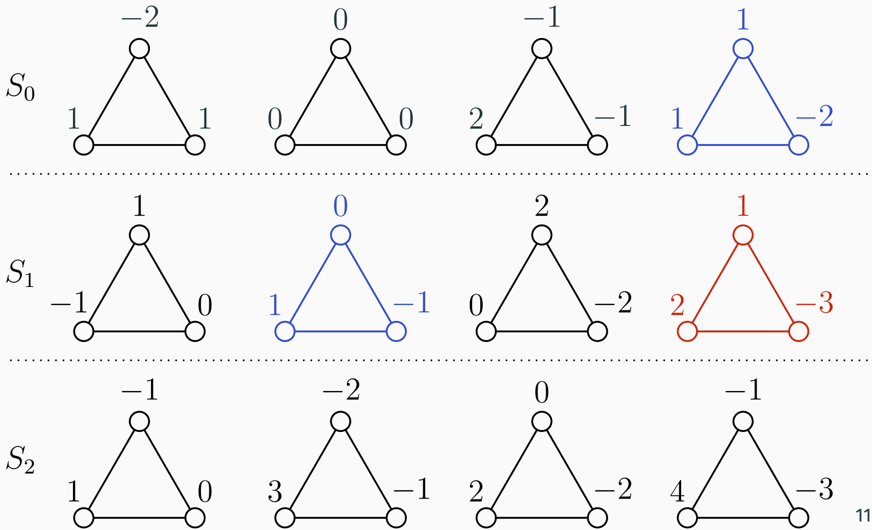
What is the sum of these two divisors?



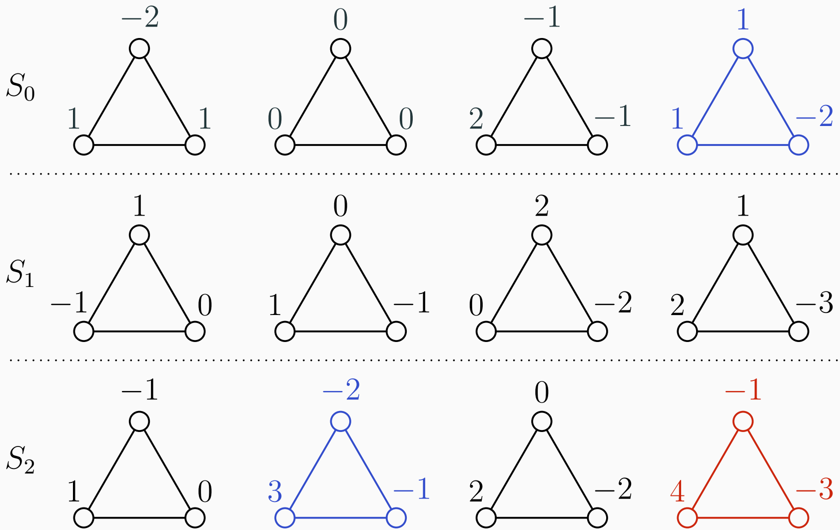
This divisor in S_0 and this one in S_1 add to one in S_1.



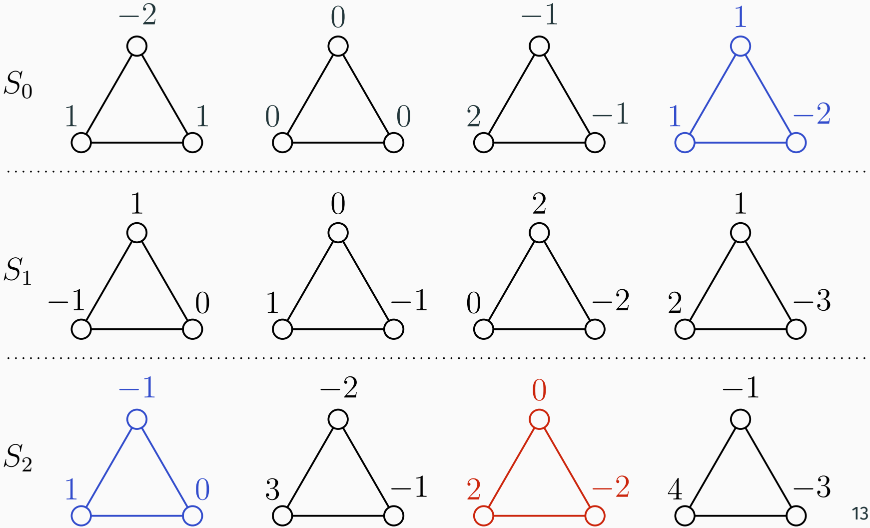
Any choice of divisors works! So we can say $S_0 + S_1 = S_1$.



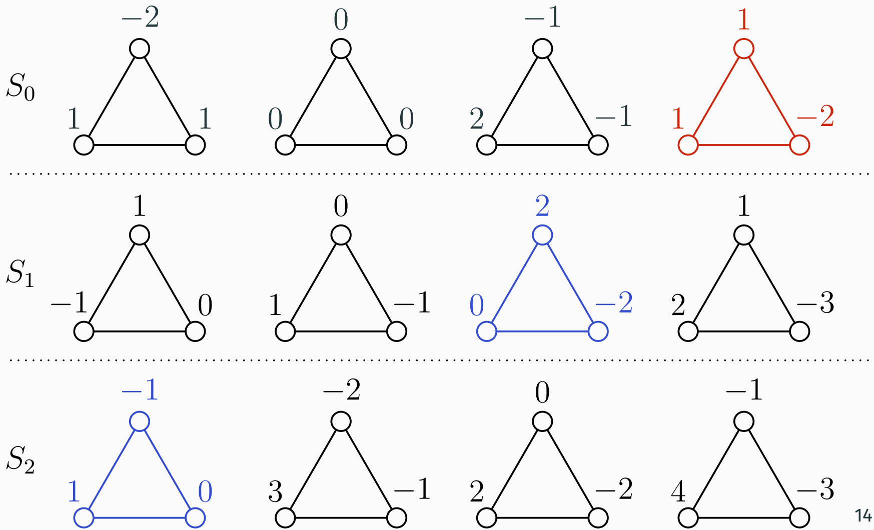
Similarly, the sum of the two divisors in blue is the divisor in red.



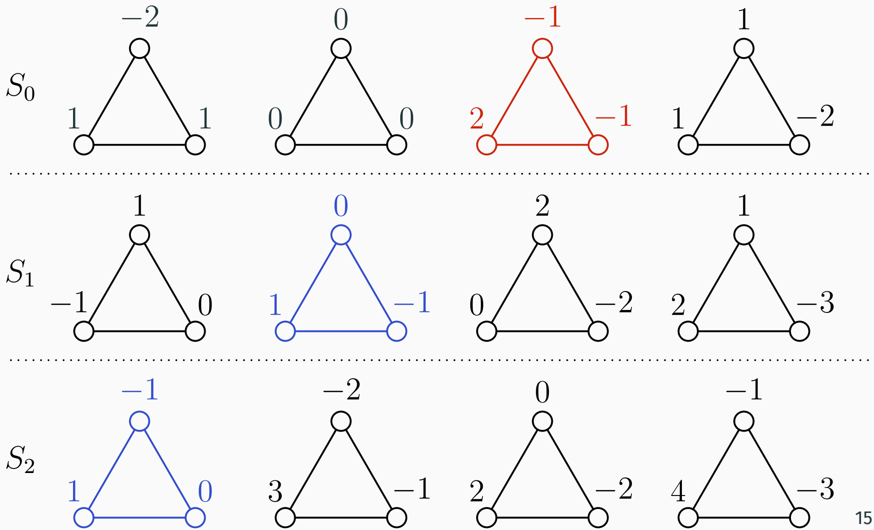
Here's another example showing $S_0 + S_2 = S_2$.



Similarly, $S_1 + S_2 = S_0$.



We see $S_0 + S_1 = S_1$, and $S_0 + S_2 = S_2$, and $S_1 + S_2 = S_0$.



When we complete the addition table:

$+$	S_0	S_1	S_2
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S_1	S_1	S_2	S_0
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Note that the identity here is S_0, which is the equivalence set containing the zero divisor.

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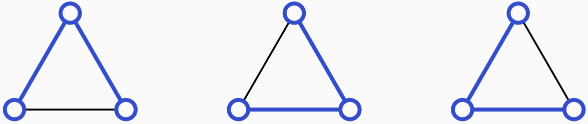
The Jacobian provides structure to the divisors, which helps us study them.

A subgraph with $n - 1$ edges that connects every vertex is called a spanning tree.



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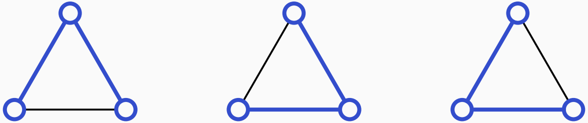
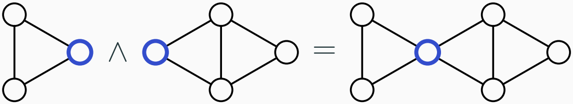


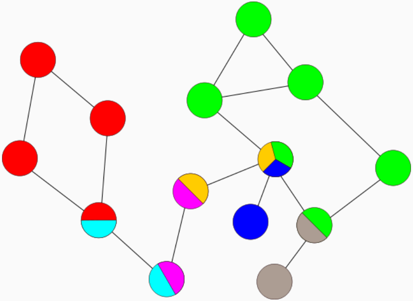
Surprisingly, the number of elements in $\mathrm{Jac}(G)$ is the number of spanning trees! Our definition didn't relate to spanning trees at all, but suddenly they're involved. Why?

We take the wedge sum of two graphs by taking two vertices and making them the same:



It is known that $\mathrm{Jac}\,(G_1 \wedge G_2) = \mathrm{Jac}\,(G_1) \oplus \mathrm{Jac}\,(G_2)$.

We can split a graph into parts that we can wedge sum together:



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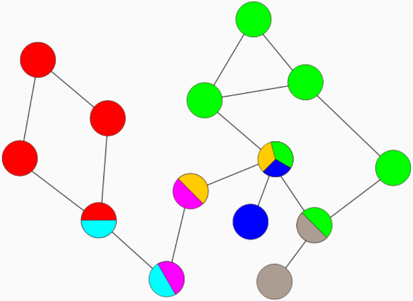

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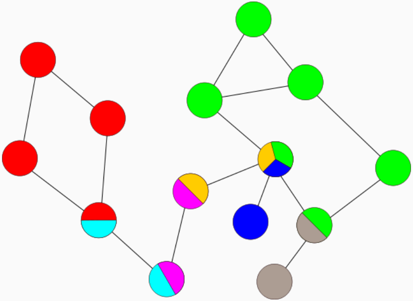

These parts can't be split into any more parts, so they're called biconnected. Therefore, to study the Jacobian, we only need to study the Jacobians of biconnected graphs.

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Similarly, the exponent in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is 2:

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Since $\operatorname{Jac}(G)$ is a finite abelian group, it has an exponent.

Conjecture

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We now show one way of proving this conjecture for $k = 2$ and $k = 3$.

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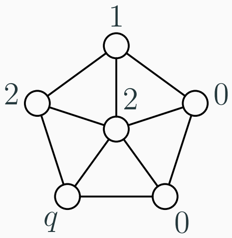

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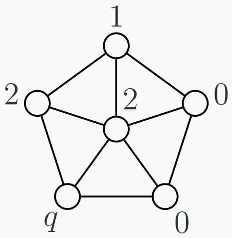
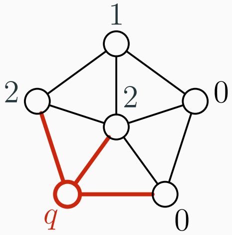


Because it has degree zero, we know what q is.

Think of the label of each vertex as the number of firefighters on that vertex.

We start a fire at vertex q. The fire spreads through the edges.



A vertex is safe as long as the number of firefighters is at least the number of burning edges next to it.

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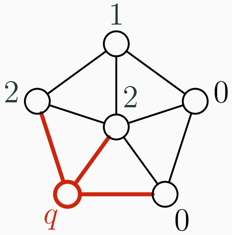
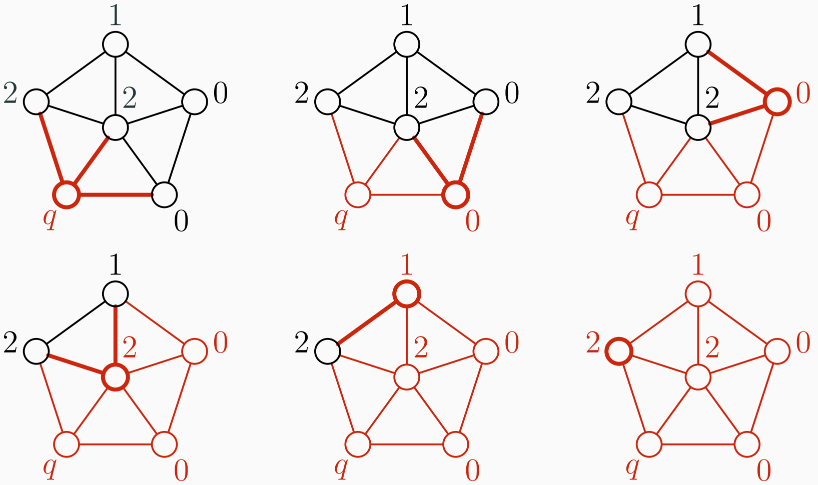


A vertex is safe as long as the number of firefighters is at least the number of burning edges next to it.

The top four vertices are protected, but the lower-right vertex is not protected. It will burn the next turn.

In this graph, everything eventually burns:



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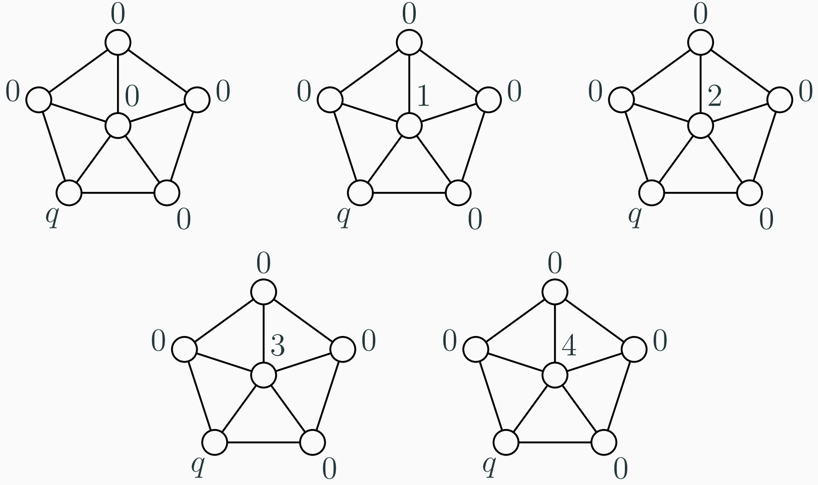
So every q-reduced divisor corresponds to a different element in $\mathrm{Jac}(G)$.

Let Δ be the maximum degree. Here's Δ q-reduced divisors:



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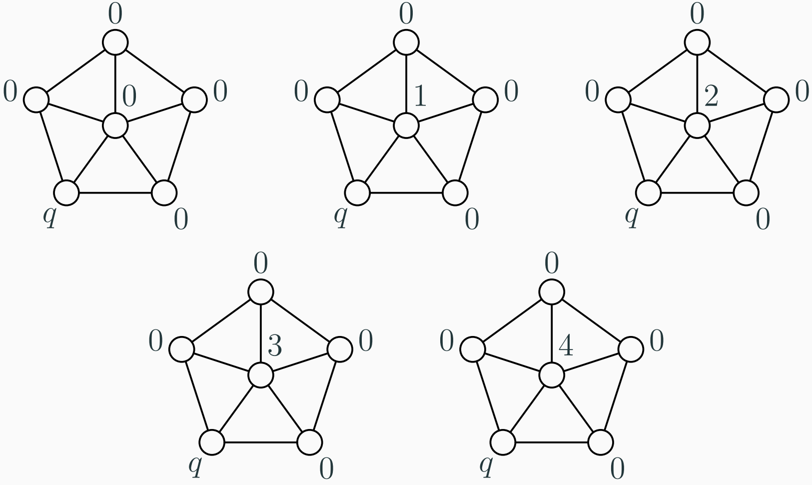
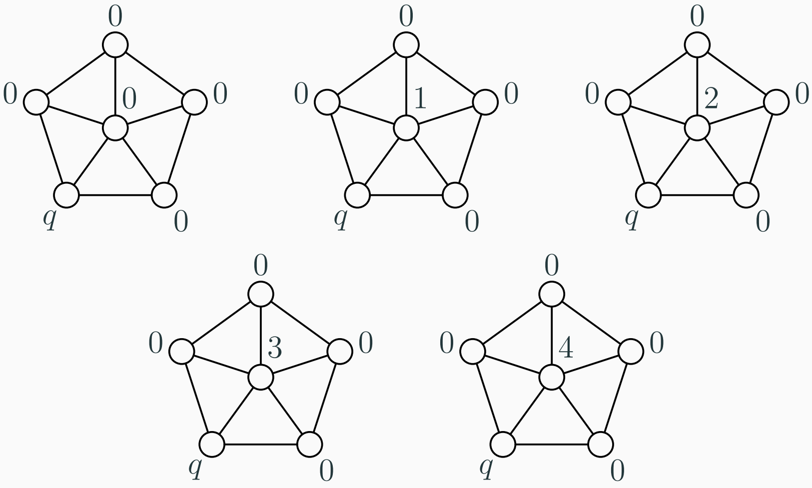


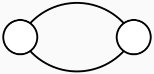
If D is the top center divisor, then $2D$, $3D$, $4D$ are not zero.

Let Δ be the maximum degree. Here's Δ q-reduced divisors:



So D has order $\geq \Delta$, so the exponent of $\mathrm{Jac}(G) \geq \Delta$.

Suppose the exponent of $\text{Jac}(G)$ is 2. As the exponent is at least Δ, $2 \geq \Delta$. The graph is biconnected, so it must be a cycle. It's known $\text{Jac}(C_n) = \mathbb{Z}_n$. So the only graph with exponent 2 is:



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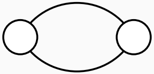
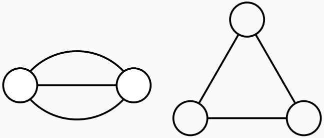


Suppose the exponent is 3, then $3 \geq \Delta$. It's known if equality holds, then the graph has 2 vertices and Δ edges. Otherwise, $\Delta = 2$, so it must be C_3. So the graphs with exponent 3 are:



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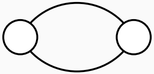

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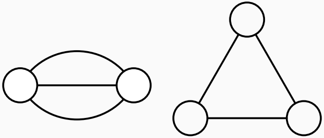
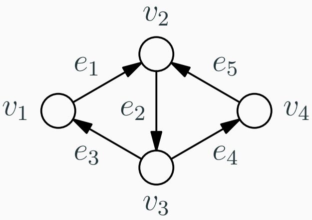
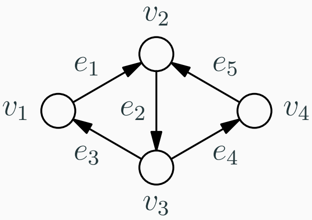


We now explain a different way to prove the conjecture for $k = 2$.

Cycle and cut spaces



The incidence matrix describes which vertices are connected by which edges.



The incidence matrix describes which vertices are connected by which edges. Here's an example of an incidence matrix.

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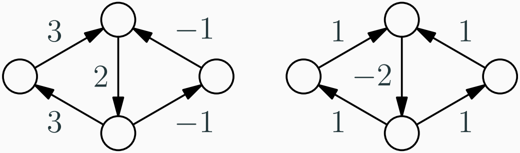

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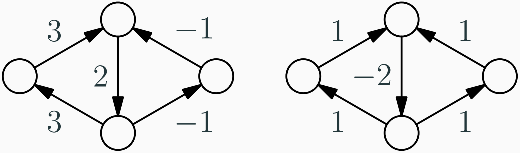

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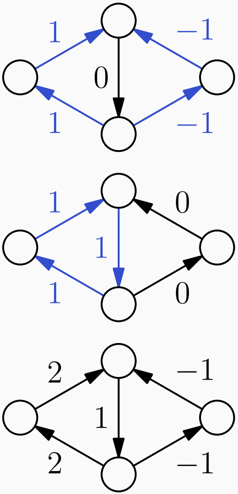
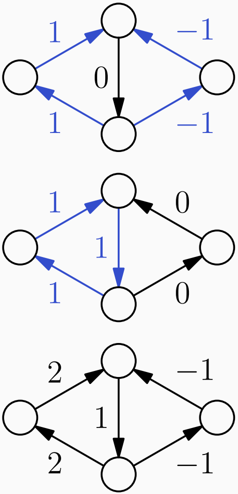
So these two vectors are orthogonal as their inner product is 0.



Let Q be a cycle. The vector $z_Q \in C^1$ is:

$$z_Q(e) = \begin{cases} 0 & \text{if } e \text{ is not in } Q, \\ 1 & \text{if } e \text{ is aligned with } Q, \\ -1 & \text{if } e \text{ goes opposite as } Q. \end{cases}$$



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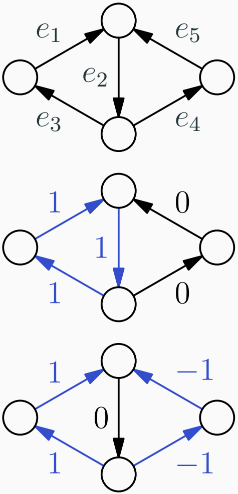

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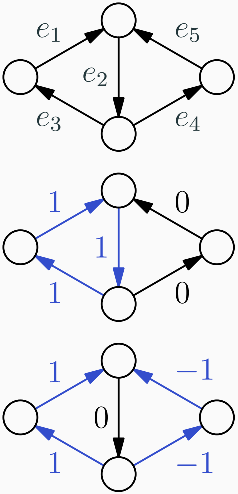

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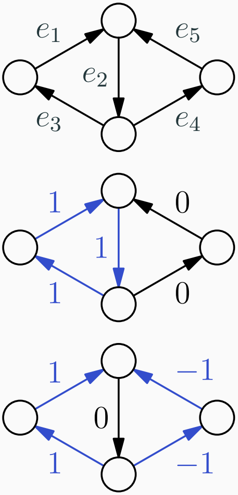
For the first z_Q,
$D(z_Q) = D(e_1) + D(e_2) + D(e_3) = 0.$



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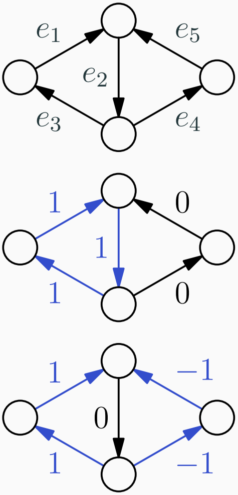

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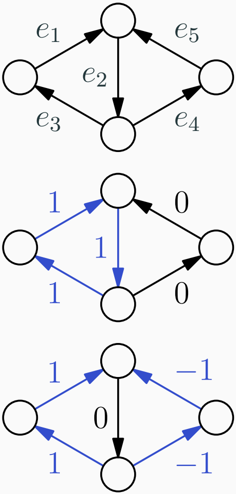

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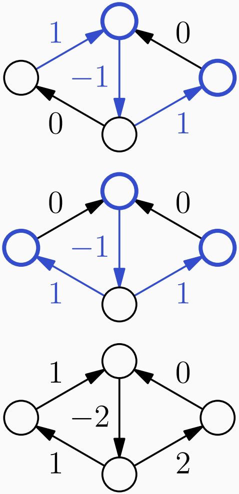
It is known that every vector that satisfies $D(v) = 0$ is in Z. So $\ker D = Z$.



Let U be a subset of vertices. Then the vector $b_U \in C^1$ is:

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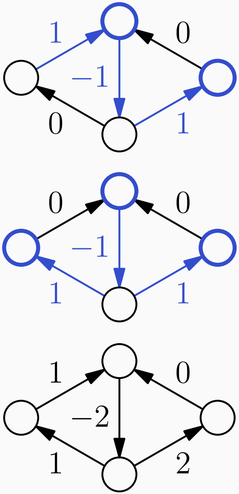

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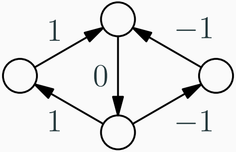

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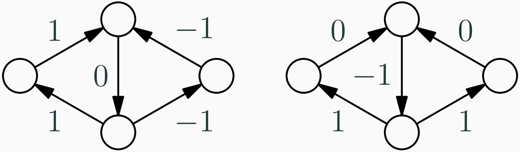

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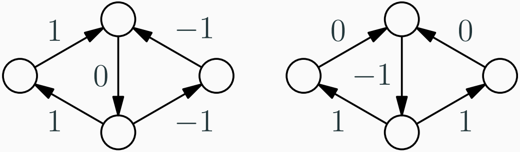

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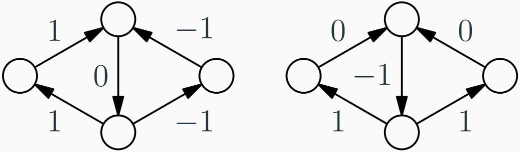

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By writing out elements as linear combinations of cycles and cuts, we can prove Z and B are orthogonal spaces.

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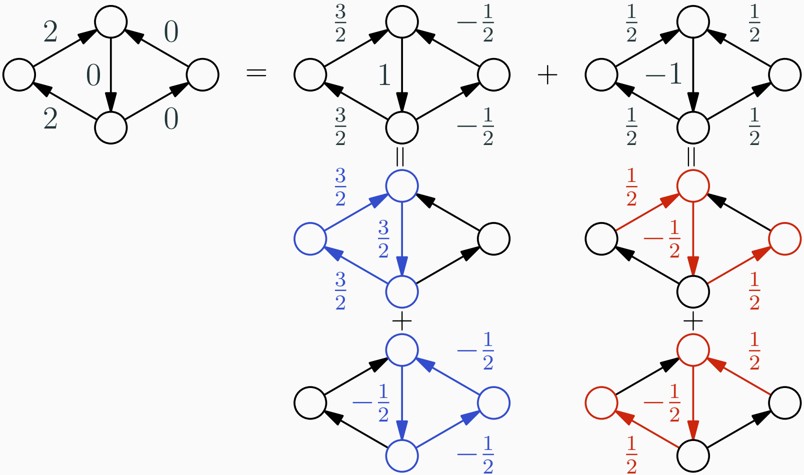

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It's known that $C^1 = Z \oplus B$, but the previous example shows $C_l \neq Z_l \oplus B_l$. Note that $Z_l \oplus B_l$ is contained in C_l, but isn't all of C_l.

For any $c \in C^1$, there exist $c_z \in Z$ and $c_b \in B$ so that $c = c_z + c_b$. Let P be the projection matrix taking c to its cut component c_b. In particular, P is a function $C^1 \to B$ that takes a vector to its cut component.

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It is known that P maps the edge lattice C_l to the dual lattice $B_l^{\#}$ of B_l, which is defined as

$$B_l^{\#} = \{x \in B : \langle x, b \rangle \in \mathbb{Z} \text{ for all } b \in B_l\}.$$

B_l is contained in $B_l^{\#}$, but $B_l^{\#}$ is larger than B_l.

The cut and cycle lattices give us an alternate definition of $\mathrm{Jac}\,(G)$:

$$\mathrm{Jac}\,(G) = C_I/(Z_I \oplus B_I) \cong B_I^{\#}/B_I.$$

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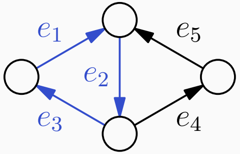
This definition allows us to deal with the Jacobian using matrices.

Regular matroids

Given a graph G, let E be the edge set and D the incidence matrix. We know that the cycles in G are linearly dependent, because they can be summed to zero:



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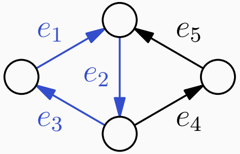

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So we can state the cycles of a graph in terms of sets of column vectors which can be combined to zero:

$$\{e_1, e_2, e_3\}, \{e_1, e_3, e_4, e_5\}, \{e_2, e_4, e_5\}.$$

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Think of a regular matroid as a generalization of a graph.

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For every positive integer k, there are finitely many biconnected graphs G such that the exponent of $\operatorname{Jac}(G)$ is k.

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Conjecture

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We prove this conjecture for $k = 2$ using the projection matrix P.

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Sketch of proof

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But P can be shown to be symmetric, so the diagonal entries must be positive. Furthermore, $P^2 = P$ implies that there are exactly two nonzero entries in each row and column.

We prove this conjecture for $k = 2$ using the projection matrix P.

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WLOG there exist $i < j$ with $P_{i,j} = -\frac{1}{2}$. Then $P(e_i + e_j) = 0$ by symmetry of P, so Pe_i and Pe_j form a circuit. But Pe_i cannot be part of any other circuit. Connectedness of M implies the result.

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