

# GMATIC 2016

Oral Round

October 19, 2016

If you have any corrections or additions, please contact me at [cj@cjquines.com](mailto:cj@cjquines.com), or through my Facebook account, Carl Joshua Quines.

- E1. Compute the unique positive integer that, when squared, is equal to six more than five times itself. [6]
- E2. David flipped a fair coin five times. Compute the probability that the fourth coin flip is the first to land heads.  $\left[\frac{1}{16}\right]$
- E3. The analog clock in front of you displays the time 3:40. Compute the number of degrees of the smaller angle by the minute and the hour hand of the clock.  $[130^\circ]$
- E4. Isosceles trapezoid  $ABCD$  has  $AB = 10$ ,  $CD = 20$ ,  $BC = DA$ , and the area of  $ABCD$  is 180. Compute the length of  $BC$ . [13]
- E5. The coordinates of three vertices of a parallelogram are  $A(1, 1)$ ,  $B(2, 4)$ ,  $C(-5, 1)$ . Compute the area of the parallelogram. [18]
- E6. An infinite geometric sequence has a first term of 12, and all terms in the sequence sum to 9. Compute the common ratio between consecutive terms of the sequence.  $\left[-\frac{1}{3}\right]$
- A1. James writes down three integers. Alex picks some two of these integers, takes the average of them, and adds the result to the third integer. If the possible final results Alex could get are 42, 13, 37, what are the three integers James originally chose?  $[-20, 28, 38]$
- A2. Consider a  $4 \times 4$  grid of squares, each of which are originally colored red. Every minute, Piet can jump on one of the squares, changing the color of it and any adjacent squares to blue. What is the minimum number of minutes it will take Piet to change the entire grid to blue? [4]
- A3. For any positive integer  $x \geq 2$ , define  $f(x)$  to be the product of the distinct prime factors of  $x$ . For example,  $f(12) = 2 \times 3 = 6$ . Compute the number of integers  $2 \leq x \leq 100$  such that  $f(x) < 10$ . [23]
- A4. Kelvin the Frog is trying to hop across a river. The river has 10 lily pads on it, and he must hop on them in a specific order (the order is unknown to Kelvin). If Kelvin hops to the wrong lily pad at any point, he will be thrown back to the wrong side of the river and will have to start over. Assuming Kelvin is infinitely intelligent, what is the minimum number of hops he will need to guarantee reaching the other side? [176]
- A5. Let  $ABCD$  be a convex quadrilateral whose diagonals  $AC$  and  $BD$  meet at  $P$ . Let the area of triangle  $APB$  be 24 and let the area of triangle  $CPD$  be 25. What is the minimum possible area of quadrilateral  $ABCD$ ?  $[49 + 20\sqrt{6}]$
- A6. For how many triples  $(x, y, z)$  of integers between  $-10$  and  $10$  inclusive do there exist reals  $a, b, c$  that satisfy  $ab = x$ ,  $ac = y$ ,  $bc = z$ ? [4061]

- D1. Find any quadruple of positive integers  $(a, b, c, d)$  satisfying  $a^3 + b^4 + c^5 = d^{11}$  and  $abc < 10^5$ .  
[(128, 32, 16, 4) or (160, 16, 8, 4)]
- D2. A graph consists of 6 vertices. For each pair of vertices, a coin is flipped, and an edge connecting the two vertices is drawn if and only if the coin shows heads. Such a graph is good, if, starting from any vertex  $V$  connected to at least one other vertex, it is possible to draw a path starting and ending at  $V$  that traverses each edge exactly once. What is the probability that the graph is good?  
 $\left[ \frac{507}{16384} \right]$
- D3. A number  $n$  is bad if there exists some integer  $c$  for which  $x^x \equiv c \pmod{n}$  has no integer solutions for  $x$ . Find the number of bad integers between 2 and 42 inclusive.  
[25]
- D4. For how many pairs of nonzero integers  $(c, d)$  with  $-2015 \leq c, d \leq 2016$  do the equations  $cx = d$  and  $dx = c$  both have an integer solution?  
[8061]
- D5. Let  $ABC$  be a triangle that satisfies  $AB = 13, BC = 14, AC = 15$ . Given a point  $P$  in the plane, let  $P_A, P_B, P_C$  be the reflections of  $A, B, C$  across  $P$ . Call  $P$  good if the circumcircle of  $P_A P_B P_C$  intersects the circumcircle of  $ABC$  at exactly 1 point. The locus of good points  $P$  encloses a region  $S$ . Find the area of  $S$ .  
 $\left[ \frac{4225}{64} \pi \right]$
- D6. Find the sum of all positive integers  $n < 2015$  that can be expressed in the form  $\left\lfloor \frac{x}{2} \right\rfloor + y + xy$ , where  $x$  and  $y$  are positive integers.  
[2029906]
- C1. Compute the smallest positive integer  $x > 100$  such that every permutation of the digits of  $x$  is prime.  
[113]
- C2. We say that a number is arithmetically sequenced if the digits, in order, form an arithmetic sequence. Compute the number of 4-digit positive integers which are arithmetically sequenced.  
[30]
- C3. Let  $ABC$  be a triangle with  $AB = 12, BC = 5, AC = 13$ . Let  $D$  and  $E$  be the feet of the internal and external angle bisectors from  $B$ , respectively. Let  $\omega$  be the circumcircle of  $\triangle BDE$ , extend  $AB$  so that it intersects  $\omega$  again at  $F$ . Extend  $FC$  to meet  $\omega$  again at  $X$ , and extend  $AX$  to meet  $\omega$  again at  $G$ . Find  $FG$ .  
 $\left[ \frac{1560}{119} \right]$
- DoD. Alice and Bob are playing a game in which Alice has a  $1/3$  probability of winning, a  $1/2$  probability of tying, and a  $1/6$  probability of losing. Given that Alice and Bob played a game which did not end in a tie, compute the probability that Alice won.  
 $\left[ \frac{2}{3} \right]$