GMATIC 2017

27th Grace Mathineers Interschool Competition

October 18, 2017

Some questions are in literature: four written questions are from SUMO Algebra 2014, one is probably from AMC, the Euler conjecture question is from AIME 1989.

Part I is one point each and had five choices, with choice E being NOTA, except I do not recall the choices. Part II is five points each with solutions.

Oral round mechanics are explained in another document. Answers are based on memory, so they probably have mistakes, and are incomplete.

If you have any corrections or additions, please contact me at `cj@cjquines.com`, or through my Facebook account, Carl Joshua Quines.

Part I

1. Christian and Grace are painting a house together. If they work together for the whole job, it takes them 20 hours to finish. If they both work together until the house is half-painted, and then Grace works alone for the rest of the job, it takes 30 hours in all. How many hours would it take Christian to paint the house alone? [40]

2. What is the value of x? [diagram: A, B, C collinear, D, E, C collinear, $\angle ABD = 80^\circ$, $\angle BDC = 2x^\circ$, $\angle BED = 125^\circ + x^\circ$, $\angle EBC = x^\circ$, $\angle ACD = 125^\circ$.] []

3. What is the maximum number of Mondays in 272727 consecutive days? [38961]

4. The average of seven consecutive integer is 11, what is the largest? [14]

5. Three vertices of a parallelogram are $(0, 1), (1, 2), (2, 1)$, what is its area? [2]

6. If $10 \leq x \leq 20$ and $40 \leq y \leq 60$, what is the maximum value of $\dfrac{x^2 + 4x + 4y}{2y}$? [8]

7. Find the area of a circle with chord AB of length 5, and chord AC of length 7, and the length of arc AB is half the length of arc AC. $\left[\dfrac{625}{51}\pi\right]$

8. Find the value of $9^6 + (69)(9^4) + (195)(9^2) + (6)(9)$. [999999]

9. Find the smallest p such that for all $q > p$ the polynomial $x^3 + x^2 + qx + 9$ has exactly one real root. $\left[-\dfrac{39}{4}\right]$

10. Polynomials P and Q satisfy $P(P(x)) = P(x)^{16} + x^{48} + Q(x)$. What is the smallest possible degree of Q? [35]

11. The mean of five numbers is 69, their median is 83, the range is 70 and the mode is 85. What is the second smallest number among them? [77]

12. The twenty-four permutations of 1234 are sorted in ascending order. What position is 3142 in the list? [14th]

13. The points $A_1, A_2, \ldots, A_{10}$ are equally spaced around a circle with center C. What is $\angle A_1A_5C$? $[18°]$

14. If $x^2yz^3 = 7^3$ and $xy^2 = 7^9$, what is xyz? $\left[7^4\right]$

15. If the sum of the first 50 positive odd numbers is 50^2, what is the sum of the first even numbers? [sic] $\left[50^2 + 50\right]$

16. The positive integers a, b, c satisfy $\dfrac{\frac{a}{b} + \frac{a}{c} + 1}{\frac{b}{c} + \frac{b}{a} + 1} = 11$. How many ordered tuples (a, b, c) also satisfy $a + 2b + c \leq 40$? [42]

17. In triangle ABC, points D, E and F are on sides AB, BC, and CA respectively, such that $AD : DB = p : q$, E is the midpoint of BC, and $3AF = FC$. The square of the area of $\triangle DFB$ is equal to the product of the areas of $\triangle AFE$ and $\triangle CED$. What is $p : q$? $\left[\dfrac{\sqrt{105} - 3}{6}\right]$

18. Given three numbers, we take one and add to it the average of the other two. The three possible results are $65, 69$ and 76. What is their average? [35]

19. The point P is on the line $y = 5x + 3$ and let $Q = (3, -2)$. Let M be the midpoint of PQ. What is the locus of M? $[y = 5x - 7]$

20. The rectangle in the diagram has height 4. What is its length? [diagram: three circles C_1, C_2 and C_3 are inside a rectangle such that C_1 is tangent to three sides, C_2 and C_3 are each tangent to two sides, all three circles are pairwise externally tangent.] $\left[3 + \sqrt{8}\right]$

Part II

1. Find all positive integer solutions to $16xy(x^2 + 4y^2) = 81z^4$.[1] $[(6k, 3k, 4k), k \in \mathbb{N}]$

2. Three people each have a three-faced coin, with faces labeled heads, tails, and sides. They each flip the coin once, and everyone who got heads or tails flipped the coin again. What is the probability all three coins are now sides? $\left[\dfrac{125}{729}\right]$

3. Cyclic quadrilateral $ABCD$ has center O. The circle with diameter AB intersects AC and AD again at M, N respectively. Prove that line MN passes through the midpoint of BD.

4. Positive integers $x_1, x_2, \ldots, x_{100}$ are such that k is a divisor of $\gcd(x_{k-1}, x_k)$ for each integer $2 \leq k \leq 100$. Find the minimum value of $x_1 + x_2 + \cdots + x_{100}$. [343400]

Oral round

E1. A speaker talked for an hour to a group of students. 10% of the students heard the entire talk, and 30% slept through the entire talk. Half the remainder heard one-third of the talk and the other half heard two-thirds of the talk. What was the average number of minutes of the talk heard by the students? [24]

[1] Sketches: (1) Fermat's little theorem, (2) Probability for one person is $5/9$, (3) Simson's and draw a rectangle, (4) $k(k+1)|x_k \implies x_k \geq k(k+1)$.

E2. Lianne and Christen went fishing. Lianne has a $\frac{1}{3}$ probability of catching more fish, half the chance of them catching the same number of fish, and $\frac{1}{6}$ probability of catching less fish. Given that Lianne and Christen did not catch the same number of fishes [sic]. What is the chance that Lianne caught more fish? $\left[\frac{2}{3}\right]$

E3. What is the greatest possible area for a rhombus constructed on a 3 in. by 9 in. sheet of paper? [15 sq. in.]

E4. The coordinates of the three vertices of a parallelogram are $A(1,1), B(2,4)$, and $C(-5,1)$. Compute for the area of the parallelogram [18]

E5. Bryce flips a fair coin five times. What is the probability that the fourth coin flip is the first coin flip that lands heads? $\left[\frac{1}{16}\right]$

E6. Grace's Honda Jazz gets 15 km per liter of gasoline. How many km can she drive with Php 1347 worth of gasoline that she bought at Php 44.90 per liter? [390]

E7. Find the last 2 digits of the expansion of 67^{20172} [27]

E8. The hands of a clock display 3:40. How many degrees is the smaller angle formed by the hands of the clock? [130°]

E9. If $\sin\frac{\pi}{6} + \tan\frac{\pi}{4} + \cos\frac{\pi}{3} = \sec G$ and $0 \le G < 2$ [sic]. Find all the values of G. $\left[\frac{\pi}{3}\right]$

A1. Lance has 2 red socks, 2 blue socks, and 2 green socks. He grabs three different socks at random. What is the chance that all are different colors? $\left[\frac{2}{5}\right]$

A2. Upon entering a tall building, you are confronted with a problem – stairs or elevator? If you decide to take the stairs, it will take you 20 seconds to walk up each flight of stairs. If you decide to take the elevator, you will have to wait for 3 minutes for the elevator to arrive, after which it will take 3 seconds to move up each floor. What is the minimum number of floors for which taking the elevator takes less time than taking the stairs?[3] [11]

A3. Shaq eats ice cream in a right circular cone with an opening of radius 5 and a height of 10. If Shaq's ice cream scoops are always perfectly spherical, compute the radius of the largest scoop he can get such that at least half of the scoop is contained within the cone. $[2\sqrt{5}]$

A4. Isosceles trapezoid $ABCD$ has $AB = 10, CD = 20, BC = AD$, and an area of 180. Compute the length of BC. [13]

A5. Find the sum of all positive divisors of 104 060 041. [105101005]

A6. Two drivers, Edwards and Domeng, were 225 km apart. They traveled towards each other at the same constant speed of x kph with Edwards having a head start of 30 minutes. Upon meeting, each continued to the other's starting point at a constant speed of $(x-10)$ kph. If Edwards completed the entire trip in 5 hours, find x. [50]

[2] This was repealed several questions later during the average round and replaced (orally, I might add) with another question that was so confusing no one remembers what it was.

[3] Rescinded several questions later after accusations of ambiguity. Replaced with question "Eight points are equally spaced on a unit circle. What is the product of the distances from one point to each other point?"

A7. Find the number of positive integers that are divisors of at least one $10^{10}, 15^7, 18^{11}$ [sic] [435]

D1. In 1769, Euler conjectured that if the sum of n kth powers is itself a kth power, then n must be at least as large as k. However, this was disproven in 1996 with the equation $x^5 = 133^5 + 110^5 + 84^5 + 27^5$. Find the value of x. [144]

D2. A middle-aged man, noting that his present age was a prime number, observed that the next occurrence of this kind for him was exactly as far away as was the most recent one of this kind. The last time he could have made such an observation was when he was five years old. How old is the man? [53]

D3. Find the numerical value of $\sin 40° \sin 80° + \sin 80° \sin 160° + \sin 160° \sin 320°$. $\left[\frac{3}{4}\right]$

D4. Point C lies on a circle with diameter AB. The bisector of angle CAB intersects BC at D and intersects the circle at E. If $BD = 25$ and $CD = 7$, find BE. [20]

D5. Find all positive integers n such that there exists integers x, y, z satisfying $n = x^2 + y^2 + z^2 + x$. [all of them]