9th Lord of the Math

Solution Booklet

Saint Stephen's High School

December 3, 2016

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Team Finals

TF1 Problem The Russian Cyrillic alphabet has 33 letters, 21 of which are consonants and 10 are vowels. The remaining two letters do not fit in either category, and are called "signs". The two signs cannot appear at the beginning of a word and can only follow a consonant. How many three-letter Russian "words" (strings of letters) with at least one vowel satisfy the above condition?

Answer 21 370

Solution Denote by C a consonant, V a vowel, and S a sign. Then only strings with length three that are of the following types satisfy the conditions: VVV, VVC and permutations, VCC and permutations, VCS, and CSV. There are $10^3 = 1000$ strings of the form VVV, $21 \times 10^2 \times 3 = 6300$ strings of the form VVC (including permutations), $21^2 \times 10 \times 3 = 13\,230$ strings of the form VCC (including permutations), and $21 \times 10 \times 2 \times 2 = 840$ strings of the form VCS or CSV, giving a total of 21 370 possible strings.

TF2 Problem *Taxicab Geometry* is one kind of non-Euclidean geometry where all points are in the xy-plane and the distance function is defined as the sum of the positive differences of their corresponding

x-, and y-coordinates. For example, the distance between $(1,2)$ and $(.5,-6)$ is 8.5. Find the area of the circle in a taxicab geometry centered at the origin with radius 50 units. A circle is defined as the set of points that are equidistant from its center.

Answer 5000 square units

Solution It can be easily verified that this circle has the shape of a square, with vertices $(\pm 50, 0)$ and $(0, \pm 50)$. Therefore, the area is $50^2 \cdot 2 = 5000$ square units.

TF3 Problem ISBNs are numbers that are used to identify most published books. It consists of ten digits, where the first nine digits a_1 to a_9 range from 0 to 9, and the tenth digit a_{10} from 0 to 10. The first nine digits give information about the book for identification, and the tenth digit is a "check digit" to check if the first nine digits might be wrongly encoded. The check digit a_{10} is chosen given the first nine digits such that

$$\sum_{i=1}^{10} i a_i \equiv 0 \pmod{11}.$$

If the first nine digits are $0 - 825417 - 39$, find the check digit.

Answer 8

Solution Expanding the given condition, we have

$$0(1) + 8(2) + 2(3) + 5(4) + 1(5) + 4(6) + 7(7) \\ + 8(3) + 9(9) + 10\left(a_{10}\right) = 217 + 10a_{10} \equiv 0 \pmod{11}.$$

Since $217 \equiv 8 \pmod{11}$, then $a_{10} = 8$ because $80 + 8 \equiv 0 \pmod{11}$.

TF4 Problem For a positive integer n, let r_n be a positive integer from 1 to 9 inclusive such that $n \equiv r_n \pmod 9$. Define

$$\mathcal{A} = \left\{n \leq 1000 \mid n \equiv 0 \pmod{r_n}\right\}.$$

Find the cardinality of $\mathcal{A}$.

Answer 527

Solution It is easy to see that for all positive integers $x \leq 1000$ such that $r_x = 1, 3,$ or 9, then $x \in \mathcal{A}$. There are 112 numbers that are $1 \pmod 9$, 111 that are $3 \pmod 9$, and 111 that are $9 \pmod 9$.

If $x \equiv 2 \pmod 9$, then $x = 18a + 2$, a a nonnegative integer. Since $x \leq 1000$, $a \leq 55$. Thus there are $55 + 1 = 56$ numbers here. (Note that $a = 0$ is a valid case.)

If $x \equiv 4 \pmod 9$, then $x = 36a + 4$. The possible values for a here are $\{0, 1, \ldots, 27\}$, giving 28 numbers.

If $x \equiv 5 \pmod 9$, then $x = 45a + 5$, and $a \leq 22$. Thus there are 23 numbers for this case.

If $x \equiv 6 \pmod 9$, then $x = 18a + 6$, and like in the case where $x \equiv 2 \pmod 9$, there are 56 possible numbers.

If $x \equiv 7 \pmod 9$, then $x = 63a + 7$. The largest value of a such that $x \leq 1000$ is $a = 15$, so there are 16 numbers for this case.

Finally, if $x \equiv 8 \pmod 9$, then $x = 72a + 8$, and there are 14 possible numbers for x.

In total, the cardinality of $\mathcal{A}$ is $112 + 111 + 111 + 56 + 28 + 23 + 56 + 16 + 14 = 527$.

TF5 **Problem** The two front tires of a new four-wheeled car will wear out after 38 400 km, whereas the two rear tires will wear out after 51 600 km. Also, suppose that five identical tires, including one spare tire, come with the car. If you can easily change the tires whenever you want, what is the maximum distance that can be driven?

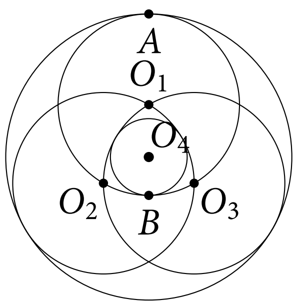
Answer 55 040 km

Solution If we assume that tires wear out at a constant rate, then the total wear (i.e., if the wear on a tire is 1 then the tire is unusable) on the four tires for every kilometer is $\frac{2}{38\,400}+\frac{2}{51\,600}=\frac{1}{11\,008}$. Thus the maximum number of kilometers obviously happens if the wear is spread evenly among the five tires. Thus the total distance that can be traveled is $5 \times 11\,008 = 55\,040$ km.

TF6 **Problem** Three circles with equal radii Γ_1, Γ_2, Γ_3 are on a plane such that each of the three circles passes through the other two circles' centers. A smaller circle Γ_4 is internally tangent to all three circles, and the three circles are all internally tangent to a larger circle Γ_5. If the product of the lengths of the radii of the five circles is equal to 162, find the radius of Γ_1.

Answer 3

Solution



Let O_n be the center of circle Γ_n. (In this case $O_3 = O_4$.) We assign $O_2(0,0)$, $O_3(r,0)$, such that r is the radius of Γ_1. Then, since

$O_1O_2O_3$ is equilateral, $O_1\left(\frac{r}{2}, \frac{\sqrt{3}}{2}r\right)$. Since O_4 lies on the altitudes of $\triangle O_1O_2O_3$, it is the orthocenter and thus the centroid, as the triangle is equilateral. Then $O_4\left(\frac{r}{2}, \frac{\sqrt{3}}{6}r\right)$.

Note that A, O_1, O_4 and B are collinear and the line that contains them is perpendicular to the segment O_2O_3. Since AO_1 is a radius, then A is r units above O_1, i.e., $A\left(\frac{r}{2}, \frac{\sqrt{3}+2}{2}r\right)$; similarly BO_1 is a radius thus $B\left(\frac{r}{2}, \frac{\sqrt{3}-2}{2}r\right)$. A radius of Γ_4 is O_4B, which has length $\frac{\sqrt{3}}{6}r - \frac{\sqrt{3}-2}{2}r = \frac{3-\sqrt{3}}{3}r$. On the other hand, a radius of Γ_5 is O_4A, which has length $\frac{\sqrt{3}+2}{2}r - \frac{\sqrt{3}}{6}r = \frac{3+\sqrt{3}}{3}r$.

Therefore the product of the lengths of the radii is $r^3 \cdot \frac{3-\sqrt{3}}{3}r \cdot \frac{3+\sqrt{3}}{3}r = 162$, or $\frac{2}{3}r^5 = 162$. Solving for r, we have $r = 3$.

TF7 Problem The volume of a right circular cylinder is $6\sqrt{3}\,\text{cm}^3$. What is its minimum possible total surface area?

Answer $18\sqrt[3]{\pi}\,\text{cm}^2$

Solution Let r and h be the radius and the height of the cylinder, respectively. Also let TSA be the total surface area, and V the volume. Then from AM-GM, $TSA = 2\pi r^2 + 2\pi rh \geq 3\sqrt[3]{2\pi r^2 (\pi rh)^2} = 3\sqrt[3]{2\pi (\pi r^2 h)^2} = 3\sqrt[3]{2\pi V^2} = 18\sqrt[3]{\pi}\,\text{cm}^2$.

TF8 **Problem** Find $\cot\left(\cot^{-1} 2 + \cot^{-1} 3 + \cot^{-1} 5 + \cot^{-1} 7\right)$.

Answer $\dfrac{11}{23}$

Solution Note that $\operatorname{Arg}(2+i) = \cot^{-1} 2$, $\operatorname{Arg}(3+i) = \cot^{-1} 3$, $\operatorname{Arg}(5+i) = \cot^{-1} 5$, and $\operatorname{Arg}(7+i) = \cot^{-1} 7$. Then, by de Moivre's formula, we have

$$\begin{aligned}
&\cot^{-1} 2 + \cot^{-1} 3 + \cot^{-1} 5 + \cot^{-1} 7 \\
&= \operatorname{Arg}(2+i) + \operatorname{Arg}(3+i) + \operatorname{Arg}(5+i) + \operatorname{Arg}(7+i) \\
&= \operatorname{Arg}\left((2+i)(3+i)(5+i)(7+i)\right) \\
&= \operatorname{Arg}(110 + 230i) = \tan^{-1}\frac{23}{11} = \cot^{-1}\frac{11}{23}
\end{aligned}$$

TF9 **Problem** $\overline{x34y73}$ is divisible by 7. Find the sum of all possible values of $y - x$.

Answer 3

Solution Since $34\,073 \equiv 4 \pmod 7$, $100\,000 \equiv 5 \pmod 7$, and $100 \equiv 2 \pmod 7$ then x and y satisfy the equivalence $(5x+2y) \equiv 3 \pmod 7$. But $(5x + 2y) \equiv (-2x + 2y) \equiv 10 \pmod 7$, so $(y - x) \equiv 5 \pmod 7$.

Therefore either $y - x$ is -2 or 5. The sum is $5 - 2 = 3$.

TF10 **Problem** The sum of the first ten terms of an arithmetic sequence is 155 and the sum of the first two terms of a geometric sequence is 9. Find all possible ordered pairs of the common difference and the common ratio (d, r) if the common difference is the first term of the geometric sequence and the common ratio is the first term of the arithmetic sequence.

Answer $(3, 2)$ and $\left(\frac{2}{3}, \frac{25}{2}\right)$

Solution As from the problem let the common difference, and the first term of the geometric sequence, be d; and let the common ratio, and the first term of the arithmetic sequence, be r. Then we know that $r + (r + d) + (r + 2d) + \cdots + (r + 9d) = 10r + 45d = 155$, and $d + dr = 9$. Solving for r in the second equation gives $r = \frac{9}{d} - 1$ (since $d \neq 0$). Then, $\frac{90}{d} - 10 + 45d = 155$. This simplifies to $90 + 45d^2 = 165d$, or $3d^2 - 11d + 6 = 0 \Rightarrow (d-3)(3d-2) = 0$. Therefore $d = 3$ or $d = \frac{2}{3}$. If $d = 3$ then $r = 2$; if $d = \frac{2}{3}$ then $r = \frac{25}{2}$. Thus the ordered pairs are $(3, 2)$ and $\left(\frac{2}{3}, \frac{25}{2}\right)$.

TF11 **Problem** Suppose you have a circular pizza divided into six equal slices, and you have to choose one flavor for each slice. If there are three flavors to choose from, and adjacent slices have to have different flavors, how many ways are there to flavor the pizza?

Answer 14

Solution Three cases have to be considered: one where there are three slices of one flavor and three of another; one where there are two slices for each of the three flavors; and another where there is one slice of the first flavor, two of the second, and three of the third. One cannot have 4 or more slices for one flavor as it will require some adjacent slices to have the same flavor.

Case 1. Three slices for each of two flavors. If the flavors are A and B then one can only have $ABABAB$ as the order of the flavors. Note that $ABABAB = BABABA$ since this is just rotation by one slice. Therefore there are $\binom{3}{2} = 3$ ways in this case.

Case 2. Two slices for each of three flavors. The only possible permutations are $ACBCAB$, $ABCBAC$, $ABCABC$, $ACBABC$, and $ACBACB$. All other permutations are rotations of any of the above. Thus there are five ways here.

Case 3. One slice for the first flavor, two for the second, three for the third. If we first denote the flavor with three slices as A, the one with two as B, and the last one C our only permutation is $ABABAC$. Now there are three choices for A, two for B and one for C. Thus we have 6 ways.

Therefore we have a total of $3 + 5 + 6 = 14$ ways.

TF12 **Problem** In a diving competition, 5 judges score each dive on a scale from 1 to 10. The point value of the dive is obtained by dropping the highest and lowest scores and multiplying the sum of the remaining scores by the degree of difficulty. If a dive with a degree of difficulty of 3.2 received scores of 7.5, 8.0, 9.0, 6.0 and 8.5, what was the point value of the dive?

Answer 76.8

Solution The sum of the scores excluding the highest and lowest scores is 24. Multiplying by 3.2 we get 76.8.

TF13 **Problem** For what real values of k will the function $f(x) = (k-2)x + 3k - 4$, $x \in \mathbb{R}$ be even, and for what values of k will the function be odd?

Answer Even: $k = 2$; Odd: $k = \frac{4}{3}$

Solution An even function f satisfies $f(-x) = f(x)$. Therefore we have $-(k-2)x+3k-4=(k-2)x+3k-4$ for all real x, thus $k-2=0$. Thus $k=2$ will make the function even.

An odd function, meanwhile, satisfies $f(-x)=-f(x)$. Therefore we have $-(k-2)x+3k-4=-((k-2)x+3k-4)$ for all real x, which implies $3k-4=0$. Thus $k=\frac{4}{3}$ will make the function odd.

TF14 Problem Lewis has five cards. Each card has one black and one white face. He shuffles the five cards and puts them in a row. If Lewis can flip consecutive cards with the same face to the other face, what is the expected value of the minimum number of flips needed to make all the cards black face up?

Answer $\frac{3}{2}$

Solution We work by cases. For brevity a "white card" means a card whose white face is face up.

Case 1. There are exactly 0 white cards. The probability of this happening is $\frac{1}{32}$ and 0 flips are needed.

Case 2. There is exactly 1 white card. The probability of this happening is $\frac{5}{32}$ and 1 flip is needed.

Case 3. There are exactly 2 white cards. The probability of the two cards being adjacent is $\frac{4}{32}$ and 1 flip is needed. The probability that the two are not adjacent is $\frac{6}{32}$ and 2 flips are needed.

Case 4. There are exactly 3 white cards. The probability that the three cards are adjacent is $\frac{3}{32}$ and 1 flip is needed. The probability that exactly two of the three cards are adjacent is $\frac{6}{32}$ and 2 flips are needed. The probability that none are adjacent is $\frac{1}{32}$ and 3 flips are needed.

Case 5. There are exactly 4 white cards. The probability that the four cards are adjacent is $\frac{2}{32}$ and 1 flip is needed. The probability that exactly three of the four are adjacent is $\frac{3}{32}$ and 2 flips are needed.

Case 6. There are exactly 5 white cards. The probability that this happens is $\frac{1}{32}$ and 1 flip is needed.

Therefore, the expected value is $\frac{1}{32} \cdot 0 + \frac{5}{32} \cdot 1 + \frac{4}{32} \cdot 1 + \frac{6}{32} \cdot 2 + \frac{3}{32} \cdot 1 + \frac{6}{32} \cdot 2 + \frac{1}{32} \cdot 3 = \frac{2}{32} \cdot 1 + \frac{3}{32} \cdot 2 + \frac{1}{32} \cdot 1 = \frac{48}{32} = \frac{3}{2}$.

TF15 Problem In the cryptarithm $\overline{APAT} + \overline{APAT} = \overline{WALO}$, each letter consistently represents one digit from 0 to 9. Two letters cannot represent the same digit, and a number cannot start with the digit zero.

What is the sum of all possible values of the 4-digit number $\overline{WALO}$? The letter O does not necessarily represent the digit zero.

Answer 40 226

Solution First, A can only be 1, 2, 3 or 4. Also, either W is one more than L or one less. In fact, A cannot be 1 or 3 because there are no integer solutions for $2P = 1$, $2P = 11$, $2P = 3$ or $2P = 13$.

Case 1. $A = 2$. First consider the case where $W = 4$ and $L = 5$: $\overline{2P2T} + \overline{2P2T} = \overline{425O}$. Here, $P = 1$ as there is no carry-over to the thousands place. On the other hand, there is a carry-over to the tens place; thus $T \geq 5$. T cannot be 5 as it has already been used. If $T = 6$, then $O = 2$; but $A = 2$. If $T = 7$, $O = 4$; if $T = 8$, $O = 6$; and if $T = 9$, $O = 8$. These are the three solutions.

$$\begin{array}{r} 2\ 1\ 2\ \cancel{5} \\ +\ 2\ 1\ 2\ \cancel{5} \\ \hline 4\ 2\ 5\ 0 \end{array} \qquad \begin{array}{r} 2\ 1\ 2\ 6 \\ +\ 2\ 1\ 2\ 6 \\ \hline 4\ 2\ 5\ \cancel{2} \end{array} \qquad \begin{array}{r} 2\ 1\ 2\ 7 \\ +\ 2\ 1\ 2\ 7 \\ \hline \mathbf{4\ 2\ 5\ 4} \end{array}$$

$$\begin{array}{r} 2\ 1\ 2\ 8 \\ +\ 2\ 1\ 2\ 8 \\ \hline \mathbf{4\ 2\ 5\ 6} \end{array} \qquad \begin{array}{r} 2\ 1\ 2\ 9 \\ +\ 2\ 1\ 2\ 9 \\ \hline \mathbf{4\ 2\ 5\ 8} \end{array}$$

If $W = 5$ and $L = 4$, then $P = 6$ and $T \leq 4$ since there is a carry-over to the thousands place and none to the tens place. If $T = 0$, $O = 0$, contradicting the fact that all letters must represent different digits. If $T = 1$, $O = 2$; but $A = 2$. Similarly $T \neq 2$. If $T = 3$, $O = 6$, but $P = 6$. $T \neq 4$ since $L = 4$. Therefore there are no solutions for this case.

$$
\begin{array}{ccccc}
 & 2 & 6 & 2 & \cancel{0} \\
+ & 2 & 6 & 2 & \cancel{0} \\
\hline
 & 5 & 2 & 4 & \cancel{0}
\end{array}
\quad
\begin{array}{ccccc}
 & 2 & 6 & 2 & 1 \\
+ & 2 & 6 & 2 & 1 \\
\hline
 & 5 & 2 & 4 & \cancel{2}
\end{array}
\quad
\begin{array}{ccccc}
 & 2 & 6 & 2 & \cancel{2} \\
+ & 2 & 6 & 2 & \cancel{2} \\
\hline
 & 5 & 2 & 4 & \cancel{4}
\end{array}
$$

$$
\begin{array}{ccccc}
 & 2 & 6 & 2 & 3 \\
+ & 2 & 6 & 2 & 3 \\
\hline
 & 5 & 2 & 4 & \cancel{6}
\end{array}
\quad
\begin{array}{ccccc}
 & 2 & 6 & 2 & \cancel{4} \\
+ & 2 & 6 & 2 & \cancel{4} \\
\hline
 & 5 & 2 & 4 & 8
\end{array}
$$

Case 2. $A = 4$. First consider the case where $W = 8$ and $L = 9$. Then $2P = 4$ or $P = 2$. Since there is a carry-over to the tens digit, $T \geq 5$. If $T = 6$, $O = 2$ but already $P = 3$. If $T = 7$, $O = 4$ but $L = 4$. $T \neq 8$ as $W = 8$ and $T \neq 9$ as $L = 9$. Thus, T can only be 5, and $O = 0$.

$$
\begin{array}{ccccc}
 & 4 & 2 & 4 & 5 \\
+ & 4 & 2 & 4 & 5 \\
\hline
 & \mathbf{8} & \mathbf{4} & \mathbf{9} & \mathbf{0}
\end{array}
\quad
\begin{array}{ccccc}
 & 4 & 2 & 4 & 6 \\
+ & 4 & 2 & 4 & 6 \\
\hline
 & 8 & 4 & 9 & \cancel{2}
\end{array}
\quad
\begin{array}{ccccc}
 & 4 & 2 & 4 & 7 \\
+ & 4 & 2 & 4 & 7 \\
\hline
 & 8 & 4 & 9 & \cancel{4}
\end{array}
$$

$$\begin{array}{r} 4\ 2\ 4\ \cancel{8} \\ +\ 4\ 2\ 4\ \cancel{8} \\ \hline 8\ 4\ 9\ 6 \end{array} \qquad \begin{array}{r} 4\ 2\ 4\ \cancel{9} \\ +\ 4\ 2\ 4\ \cancel{9} \\ \hline 8\ 4\ 9\ \cancel{8} \end{array}$$

If $W = 9$ and $L = 8$, then $P = 7$. Now $T \leq 4$ as there is no carry-over to the tens digit. If $T = 0$, $O = 0$ which cannot be; if $T = 2$, $O = 4$ but already $A = 4$. Similarly $T \neq 4$. If $T = 1$, $O = 2$; if $T = 3$; $O = 6$. These are the two solutions.

$$\begin{array}{r} 4\ 7\ 4\ \cancel{0} \\ +\ 4\ 7\ 4\ \cancel{0} \\ \hline 9\ 4\ 8\ \cancel{0} \end{array} \qquad \begin{array}{r} 4\ 7\ 4\ 1 \\ +\ 4\ 7\ 4\ 1 \\ \hline \mathbf{9\ 4\ 8\ 2} \end{array} \qquad \begin{array}{r} 4\ 7\ 4\ 2 \\ +\ 4\ 7\ 4\ 2 \\ \hline 9\ 4\ 8\ \cancel{4} \end{array}$$

$$\begin{array}{r} 4\ 7\ 4\ 3 \\ +\ 4\ 7\ 4\ 3 \\ \hline \mathbf{9\ 4\ 8\ 6} \end{array} \qquad \begin{array}{r} 4\ 7\ 4\ \cancel{4} \\ +\ 4\ 7\ 4\ \cancel{4} \\ \hline 9\ 4\ 8\ \cancel{8} \end{array}$$

Our solutions are 4254, 4256, 4258, 8490, 9482, 9486. Their sum is 40 226.

Individual Semifinals

Easy Round

IS-E1 **Problem** Bulbasaur, Charmander, and Squirtle have some berries to eat. All three are in a generous mood, so Bulbasaur gives Charmander as many berries as Charmander has and Squirtle as many Squirtle has. Then, Charmander does the same, giving Bulbasaur and Squirtle as many berries as they each have. Finally, Squirtle gives Bulbasaur and Charmander as many berries as each have. If after this each has 16 berries, how many berries did Bulbasaur have at first?

Answer 26

Solution We work backwards. Before Squirtle shared its berries we know that Bulbasaur and Charmander each have $\frac{16}{2} = 8$ berries. Since there is a total of 48 berries, Squirtle has $48-8-8 = 32$ berries. Before Charmander shared, Bulbasaur has $\frac{8}{2} = 4$ berries and Squirtle, $\frac{32}{2} = 16$ berries, leaving Charmander with $48 - 4 - 16 = 28$ berries. Thus, at the beginning of the game, before Bulbasaur shared, Charmander has $\frac{28}{2} = 14$ berries and Squirtle, $\frac{16}{2} = 8$ berries. Thus, Bulbasaur has $48 - 14 - 8 = 26$ berries.

IS-E2 Problem Find the last digit of $7^{11} + 8^{10} - 9^{12}$.

Answer 4

Solution Note that $7^{11} < 9^{11}$ and $8^{10} < 9^{11}$. Thus $7^{11}+8^{10} < 9^{11}+9^{11} < 9^{12}$, and the given expression is negative. From modulo arithmetic it is known that the last digit of 7^{11} is 3, the last digit of 8^{10} is 4, and the last digit of 9^{12} is 1. Therefore the last digit of $7^{11} + 8^{10} - 9^{12}$ is not $3 + 4 - 1 = 6$, but $10 - 6 = 4$.

IS-E3 Problem A convex polyhedron consists solely of hexagonal and quadrilateral faces. If for all vertices three faces meet at a vertex, how many quadrilateral faces are there?

Answer 6

Solution Let there be m hexagonal faces and n quadrilateral faces. We have $m + n$ faces, $\frac{6m + 4n}{3}$ vertices, and $\frac{6m + 4n}{2}$ edges, since two adjacent faces of convex polyhedra always meet at an edge. By Euler's polyhedron formula we have $\frac{6m + 4n}{3} + (m+n) = \frac{6m + 4n}{2} + 2$. Simplifying the equation, the m's cancel out, and $n = 6$.

IS-E4 Problem At how many points do the graphs of $y = 2\log x$ and $y = \log(2x)$ intersect in the xy-plane?

Answer 1

Solution $2\log x = \log(2x)$ implies that $x^2 = 2x$, since $2\log x = \log x^2$. This gives us $x = 0$ or $x = 2$. If $x = 0$, there is no value for y as $\log 0$ is undefined; if $x = 2$, $y = 2\log 2$. Therefore there is only one intersection.

IS-E5 **Problem** How many permutations of five distinct letters are there if one letter has to be in front of another?

Answer 60

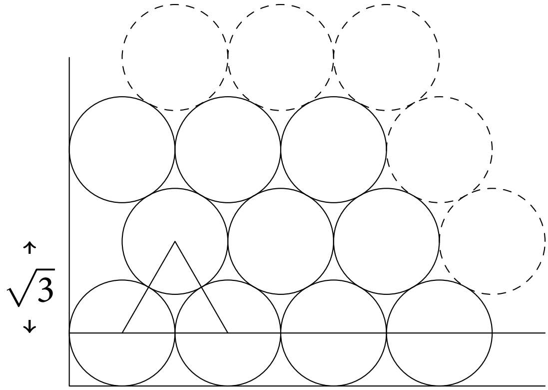
Solution Without the restriction, there are $5! = 120$ permutations. Now there is a bijection between permutations where one letter, say A, is always in front of another, say B, and permutations where B is ahead of A. Thus exactly half of the permutations satisfy the given condition, or $\dfrac{120}{2} = 60$ permutations.

Average Round

IS-A1 **Problem** At most how many circles of radius 1 cm can fit inside a square with side 20 cm such that the circles do not overlap?

Answer 105

Solution The circle have to be packed such that adjacent circles' centers form equilateral triangles. Refer to the following figure.



Since the diameter of each circle is 2 cm, then ten of them can fit exactly on the bottom row. Thus the second bottom row will have one less (9 circles). The rows will alternate having 10 and 9 circles. Now we have to find how many rows there will be.

If there are n rows, then the height will be $2 + (n-1)\sqrt{3}$. Now $2 + (n-1)\sqrt{3} \le 20$, or $n \le 6\sqrt{3} + 1$. But $100 < 108 < 121 \Rightarrow 10 < 6\sqrt{3} < 11 \Rightarrow 11 < 6\sqrt{3}+1 < 12$. Therefore there are $\lfloor 6\sqrt{3} + 1 \rfloor = 11$ rows. There are six rows with ten circles and five rows with nine circles, thus we can fit at most $60 + 45 = 105$ circles.

IS-A2 **Problem** Evaluate the series $\sum_{k=0}^{2016} k\mathrm{i}^k$, where $\mathrm{i} = \sqrt{-1}$.

Answer $1008 - 1008\mathrm{i}$

Solution The sum is equal to $\sum_{k=1}^{2016} k\mathrm{i}^k$. We consider the sum $\sum_{k=4n+1}^{4n+4} k\mathrm{i}^k$, where n is a nonnegative integer. This sum is $(4n+4) - (4n+2) +$

$((4n+1)-(4n+3))\mathrm{i} = 2-2\mathrm{i}$. This happens 504 times, thus the answer is $1008-1008\mathrm{i}$.

IS-A3 Problem An unfair coin is tossed n times, for some positive integer n. If the variance of the distribution of the number of heads is $\sqrt{99}$, find the minimum possible value for n.

Answer 40

Solution The variance of the binomial distribution $\mathrm{Bi}(n,p)$ is $np(1-p)=\sqrt{99}$, where p is the probability of heads. By AM-GM, $p(1-p)<\frac{1}{4}$ (as $p\neq 1-p$), so $\sqrt{99}<\frac{n}{4}$, or $n>4\sqrt{99}=\sqrt{1584}$. Since $39=\sqrt{1521}<\sqrt{1584}<\sqrt{1600}=40$, the minimum possible value of n is 40.

IS-A4 Problem A turtle born on January 1 in the first half of the nineteenth century was x years old in the year x^2. How old is it now, in years?

Answer 210 years old

Solution We see that $42=\sqrt{1764}<\sqrt{1800}<\sqrt{1849}=43<\sqrt{1850}$. Thus $x=43$, and the turtle was born in $43^2-43=1806$. It is now 210 years old.

IS-A5 Problem Find the sum of all values of x that satisfy the equation $2\lfloor x\rfloor = x+2\{x\}$, where $\lfloor x\rfloor$ is defined as the least integer greater than or equal to x, and $\{x\} = x - \lfloor x \rfloor$.

Answer 4

Solution Substituting $x = \lfloor x \rfloor + \{x\}$ into the original equation gives us $2\lfloor x \rfloor = \lfloor x \rfloor + 3\{x\}$, or $\lfloor x \rfloor = 3\{x\}$. Since $0 \le \{x\} < 1$, $0 \le \lfloor x \rfloor < 3$. Thus $\lfloor x \rfloor$ can only be 0, 1, or 2.

If $\lfloor x \rfloor = 0$, then $\{x\} = 0$, and $x = 0$; if $\lfloor x \rfloor = 1$, then $\{x\} = \frac{1}{3}$ and $x = \frac{4}{3}$; if $\lfloor x \rfloor = 2$, then $\{x\} = \frac{2}{3}$ and $x = \frac{8}{3}$. The sum of all possible values, then, is 4.

Difficult Round

IS-D1 **Problem** A polynomial f with positive integer coefficients satisfies $f(12) = 311 \times 113$. If the sum of its coefficients is 31, find all possible values of $f(10)$.

Answer 57 034

Solution Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then $f(12) = a_n 12^n + a_{n-1}12^{n-1} + \cdots + a_1 \cdot 12 + a_0$. A possible set of values for the coefficients are the consecutive digits of the base-12 representation of $311 \times 113 = 35\,143_{10} = 18\,407_{12}$. Note that the sum of the digits here is 20, and that $20 + 12 - 1 = 31$.

Currently our function is $f(x) = x^4 + 8x^3 + 4x^2 + 0x + 7$, and to make the sum of the coefficients 31, 1 has to be decreased from a coefficient and 12 added to the coefficient of the lower power of x. We can't remove 1 from 0, as all coefficients are positive (having a coefficient as zero just means the term won't exist, and so does not violate the conditions). Thus the possible functions are $f_1(x) = x^4 + 8x^3 + 3x^2 + 12x + 7$, $f_2(x) = x^4 + 7x^3 + 16x^2 + 7$, $f_3(x) = 20x^3 + 4x^2 + 7$. Then $f_1(10) + f_2(10) + f_3(10) = 18\,427 + 18\,607 + 20\,407 = 57\,034$.

IS-D2 Problem Evaluate $\sqrt[3]{2^1\sqrt[3]{2^1\sqrt[3]{2^1\sqrt[3]{2^3\sqrt[3]{2^5\sqrt[3]{2^9\cdots}}}}}}$, where the exponents of 2 are from the successive terms of the sequence $1, 1, 1, 3, 5, 9, \ldots$, where the first three terms are 1 and succeeding terms are generated by getting the sum of the last three terms.

Answer $\sqrt[7]{16}$

Solution Denote by a the given expression. Then

$$a = 2^{\frac{1}{3}}2^{\frac{1}{9}}2^{\frac{1}{27}}2^{\frac{3}{81}}2^{\frac{5}{243}}2^{\frac{9}{729}}\cdots = 2^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{3}{81}+\frac{5}{243}+\frac{9}{729}+\cdots}.$$

Let the exponent of a be x. Then

$$x = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{3}{81} + \frac{5}{243} + \frac{9}{729} + \cdots \qquad (1)$$

$$3x = 1 + \frac{1}{3} + \frac{1}{9} + \frac{3}{27} + \frac{5}{81} + \frac{9}{243} + \cdots \qquad (2)$$

$$9x = 3 + 1 + \frac{1}{3} + \frac{3}{9} + \frac{5}{27} + \frac{9}{81} + \cdots \qquad (3)$$

$$27x = 9 + 3 + 1 + \frac{3}{3} + \frac{5}{9} + \frac{9}{27} + \cdots \qquad (4)$$

(4) − (3) − (2) − (1) gives us $(27 - 9 - 3 - 1)x = 8$ or $x = \frac{8}{14} = \frac{4}{7}$. Therefore $a = 2^{\frac{4}{7}} = \sqrt[7]{2^4} = \sqrt[7]{16}$.

IS-D3 **Problem** How many 2016-digit positive integers that only have digits 1, 2, 3, and 4 are there such that the number has an even number of 2's?

Answer $2^{4031} + 2^{2015}$

Solution Let a_n be the number of n-digit integers that satisfy the above condition. Also, let b_n be the number of n-digit integers with only 1, 2, 3, and 4 but with an odd number of 2's. Then $a_n = 3a_{n-1} + b_{n-1}$, because a_n consists of all numbers with either (a) 1, 3, or 4 as the first digit and with an even number of 2's among the succeeding $(n-1)$ digits, or (b) 2 as the first digit and with an odd number of 2's among the succeeding $(n-1)$ digits. Similarly

$b_n = 3b_{n-1} + a_{n-1}$. Also, the initial values are $a_1 = 3$, $b_1 = 1$.

Adding the two equations together gives us $a_n + b_n = 4(a_{n-1} + b_{n-1})$, $a_1 + b_1 = 4$. Therefore, $a_n + b_n = 4^n$.

Similarly, subtracting one from the other results in $a_n - b_n = 2(a_{n-1} - b_{n-1}$, $a_1 - b_1 = 2$. Therefore, $a_n - b_n = 2^n$. This means that $a_n = \dfrac{4^n + 2^n}{2}$ and $b_n = \dfrac{4^n - 2^n}{2}$.

The question is asking for a_{2016}, which is $\dfrac{4^{2016} + 2^{2016}}{2} = 2^{4031} + 2^{2015}$.

IS-D4 Problem For each of the about 7 billion people in the world, compute the product of the number of fingers in his/her right hand, left hand, right foot and left foot. Suppose the about 7 billion products are also multiplied together. Give a reasonable estimate, within 5% of the exact answer, of this value. You may leave your answer in exponential form.

Answer 0

Solution An amputee will have zero as the product, and thus the product of all the numbers is zero. Only zero is accepted since $0 \pm 5\% = 0$.

IS-D5 Problem The angle of elevation of a building is observed from a point on the horizontal plane on which it stands. At a point x feet nearer the angle of elevation is the complement of the original angle observed. At another point y feet nearer (from the second point) the angle of elevation is now double the first. Express the height of the building in terms of x and y.

Answer $\sqrt{(x+y)^2 - \left(\frac{x}{2}\right)^2}$

Solution Let the original angle of elevation observed have measure A, the height of the building be h, and the distance between the building and the third point be z. Then from the given $\tan A = \dfrac{h}{x+y+z} = \dfrac{y+z}{h}$, and $\tan 2A = \dfrac{h}{z}$. Now from the double-angle formula $\dfrac{h}{z} = \dfrac{2\left(\frac{y+z}{h}\right)}{1-\left(\frac{y+z}{h}\right)^2} = \dfrac{2(y+z)h}{h^2(y+z)^2}$, or $2z(y+z) = h^2 - (y+z)^2 \Rightarrow h^2 = (y+z)(y+3z)$.

But $\dfrac{h}{x+y+z} = \dfrac{y+z}{h} \Rightarrow h^2 = (y+z)(x+y+z)$. Equating the two implies $x+y+z = y+3z$ as $y+z \neq 0$. Therefore, $z = \dfrac{x}{2}$. Now $h^2 = (y+z)(x+y+z) = y^2 + 2xy + \dfrac{3}{4}x^2 = (x+y)^2 - \left(\dfrac{x}{2}\right)^2$, or $h = \sqrt{(x+y)^2 - \left(\dfrac{x}{2}\right)^2}$.

E **Problem** What is the value of $\varphi^{12} + \varphi^8 + \varphi^5 + 160\varphi$, where $\varphi = \dfrac{-1 + \sqrt{5}}{2}$.

Answer 99

Solution Note that $\varphi^2 + \varphi - 1 = 0$.

First, we show by induction that for all positive integers n,

$$\varphi^n = (-1)^{n+1} F_n \varphi + (-1)^n F_{n-1},$$

where $F_0 = 0$, $F_1 = 1$, $F_i = F_{i-1} + F_{i-2}$ for integers $i \geq 2$ are the Fibonacci numbers.

Now for $n = 1$, LHS $= (-1)^2 F_1 \varphi + (-1)^1 F_0 = \varphi + 0 = \varphi =$ RHS.

Assume that the statement is true for $n = k$. Then $\varphi^k = (-1)^{k+1} F_k \varphi + (-1)^k F_{k-1}$. Now for $n = k + 1$:

$$\begin{aligned}
\varphi^{k+1} &= \varphi^k \varphi = \left((-1)^{k+1} F_k \varphi + (-1)^k F_{k-1}\right) \varphi \\
&= (-1)^{k+1} F_k \varphi^2 + (-1)^k F_{k-1} \varphi \\
&= (-1)^{k+1} F_k (-\varphi + 1) + (-1)^k F_{k-1} \varphi \\
&= (-1)^k \varphi \left(F_k + F_{k-1}\right) + (-1)^{k+1} F_k \\
&= (-1)^k \varphi F_{k+1} + (-1)^{k+1} F_k \\
&= (-1)^{k+2} \varphi F_{k+1} + (-1)^{k+1} F_k
\end{aligned}$$

Thus the statement is true for $n = k + 1$, and by induction true for all positive integers n. This means that

$$\begin{aligned}
\varphi^{12} &= (-1)^{13} F_{12}\varphi + (-1)^{12} F_{11} = -144\varphi + 89 \\
\varphi^{8} &= (-1)^{9} F_{8}\varphi + (-1)^{8} F_{7} = -21\varphi + 13 \\
\varphi^{5} &= (-1)^{6} F_{5}\varphi + (-1)^{5} F_{4} = 5\varphi - 3
\end{aligned}$$

Adding these three equations gives us $\varphi^{12} + \varphi^8 + \varphi^5 = -160\varphi + 99$, or $\varphi^{12} + \varphi^8 + \varphi^5 + 160\varphi = 99$.

A **Problem** In a class election for class head, 44 students write their choice on a slip of paper. Now the teacher counts the votes in a uniformly random order. If Student A gets 26 votes and Student B gets 18 votes, what is the probability that Student A never trailed during the counting process?

Answer $\dfrac{1}{3}$

Solution We define a list like $ABAABAB\ldots$, which tells us who was selected for the ith slip. Now, since the votes for A and votes for B can be rearranged we have $\dbinom{44}{18}$ lists with 26 A's and 18 B's.

We call a list with 26 A's and 18 B's *good* if it satisfies the given condition and *bad* otherwise. A list is bad when there exists a k such that A has

k votes and B has $k + 1$ votes. The smallest k is the first time A trails. For every bad list, after the first instance A trails, we swap all remaining votes of A to B and vice versa. Therefore after this point A gets $17 - k$ votes and B gets $26 - k$ votes, giving us A with $k + 17 - k = 17$ votes and B with $k + 1 + 26 - k = 27$ votes.

After the manipulation, B will always win. Therefore any list with 17 A's and 27 B's can be reverted to a bad list by selecting the first time B has more votes than A (which is guaranteed) then swapping A's votes with B's votes afterwards. Thus there exists a bijection between bad lists and lists with 17 A's and 27 B's, and there are $\binom{44}{17}$ bad lists.

Thus, the probability is

$$\left(1 - \frac{\text{\# of bad lists}}{\text{\# of lists}}\right) = 1 - \frac{\binom{44}{17}}{\binom{44}{18}} = 1 - \frac{44!26!18!}{27!17!44!} = 1 - \frac{2}{3} = \frac{1}{3}.$$

D **Problem** Find all nonnegative integers $N \leq 2064$ such that $\dfrac{\prod_{k=1}^{2064} k!}{N!}$ is a perfect square.

Answer 1032

Solution First we show that 1032 is a solution.

$$\begin{aligned}\prod_{k=1}^{2064} k! &= \prod_{k=1}^{1032} (2k)!(2k-1)! \\ &= \prod_{k=1}^{1032} (2k)\left((2k-1)!\right)^2 \\ &= \left(\prod_{k=1}^{1032} \left((2k-1)!\right)^2\right)\left(2^{1032}\prod_{k=1}^{1032} k\right) \\ &= \left(2^{516}\prod_{k=1}^{1032}(2k-1)!\right)^2 (1032!)\end{aligned}$$

Therefore $N = 1032$ is a possible solution. Now assume for the sake of contradiction that there exists another nonnegative integer M that satisfies this condition. Then it follows that either $\frac{M!}{1032!}$ (if $M > 1032$) or $\frac{1032!}{M!}$ (if $M < 1032$) is a perfect square, since both $\frac{\prod_{k=1}^{2064} k!}{1032!}$ and $\frac{\prod_{k=1}^{2064} k!}{M!}$ are perfect squares.

Note that if $M > 1032$, then $\frac{M!}{1032!}$ will always contain only one copy of 1033, since 1033 is prime. Thus $\frac{M!}{1032!}$ will never be a perfect square.

Similarly, if $M < 1032$, then $\frac{1032!}{M!}$ will always contain only one copy of 1031, since 1031 is prime. Thus $\frac{1032!}{M!}$ will never be a perfect square.

Therefore only $N = 1032$ satisfies the given condition.