10th Lord of the Math

Solution Booklet

Saint Stephen's High School

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JIF-1 **Problem** What is the coefficient of the term containing x^4y^3 in the expansion $(x+y)^7$?

Answer 35

Solution Using the binomial theorem, the coefficient is $\binom{7}{3} = 35$.

JIF-2 **Problem** Triangle ABC has lengths $AB = 6$, $BC = 7$, and $AC = 8$. Find the length of the angle bisector from A to BC.

Answer 6

Solution Let AD ($D \in BC$) be the angle bisector. Let $BD = x$, which implies that $DC = 7 - x$.

Then, from the angle bisector theorem, we have $8x = 6(7-x) \Rightarrow 8x = 42 - 6x \Rightarrow x = 3$. Therefore, $BD = 3$ and $DC = 4$.

The length of AD can now be determined using Stewart's theorem:

$$AC^2 \cdot BD + AB^2 \cdot DC = BC \cdot (BD \cdot DC + AD^2)$$

$$64 \cdot 3 + 36 \cdot 4 = 7(3 \cdot 4 + AD^2)$$

$$192 + 144 = 7(12 + AD^2)$$

$$AD = 6$$

JIF-3 **Problem** Convert the octal number 2017_8 to base ten.

Answer 1039

Solution In base ten $2017_8 = 2 \times 512 + 1 \times 8 + 7 = 1039$.

JIF-4 **Problem** If real numbers x and y satisfy $(x+5)^2+(y-12)^2 = 14^2$, find the minimum value of $x^2 + y^2$.

Answer 1

Solution The line connecting the centers of circles $C_1 : (x + 5)^2 + (y - 12)^2 = 14^2$ and $C_2 : x^2 + y^2 = c^2$ has equation $12x + 5y = 0$. The line intersects C_1 at $\left(\frac{5}{13}, -\frac{12}{13}\right)$. If c^2 is chosen to minimize $x^2 + y^2$ subject to C_1, then C_1, C_2 and the line connecting the centers of the two circles should concur at the point of tangency of C_1 and C_2, i.e., $\left(\frac{5}{13}, -\frac{12}{13}\right) \in C_2$. Therefore, $c^2 = \left(\frac{5}{13}\right)^2 + \left(-\frac{12}{13}\right)^2 = 1$.

JIF-5 **Problem** The mean of three numbers is 7 and the mode is 5. Find the range.

Answer 6

Solution Since 5 is the mode and the mean is greater than 5, the two smaller numbers are 5. Then the third number is $7 \times 3 - 5 - 5 = 11$. The range is $11 - 5 = 6$.

JIF-6 **Problem** Evaluate $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7}$.

Answer $\frac{1}{2}$

Solution From De Moivre's theorem, $x = \operatorname{cis}\frac{2\pi}{7}$ is a seventh root of unity. Thus $x^7 - 1 = 0$. Since $x \neq 1$, $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

Substituting $x = \operatorname{cis}\frac{2\pi}{7}$ into the RHS gives $\sum_{n=0}^{6} \operatorname{cis}\frac{2n\pi}{7} = 0$. The real parts of equal complex numbers are also equal; thus $\sum_{n=0}^{6} \cos \frac{2n\pi}{7} = 0$. However, from trigonometric identities, $\cos \frac{2\pi}{7} = \cos \frac{12\pi}{7}$, $\cos \frac{4\pi}{7} = -\cos \frac{3\pi}{7} = \cos \frac{10\pi}{7}$ and $\cos \frac{6\pi}{7} = -\cos \frac{\pi}{7} = \cos \frac{8\pi}{7}$.

Substituting these results in, we have $-2\cos \frac{3\pi}{7} + 2\cos \frac{2\pi}{7} - 2\cos \frac{\pi}{7} + 1 = 0 \Rightarrow \cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} = \frac{1}{2}$.

JIF-7 **Problem** For how many integers x less than or equal to 2017 is the expression $\sqrt{x\sqrt{x\sqrt{x}}}$ an integer?

Answer 3

Solution The expression is equivalent to $x^{\frac{7}{8}} = \left(\sqrt[8]{x}\right)^7$. Since this is an integer, $\sqrt[8]{x}$ also has to be an integer. Since $3^8 = 6561 > 2017$, and $x \geq 0$, we have three possible values for x: 0^8, 1^8, and $2^8 = 256$.

JIF-8 **Problem** For odd integers n, define the *double factorial* to be $n!! = 1 \cdot 3 \cdot 5 \cdots (n-2) \cdot n$. Find the last three digits of 55!!.

Answer 625

Solution It is easy to see that $5^3 \mid 55!!$, and that 55!! is odd. Thus, it only suffices to find 55!! mod 8 to get 55!! mod 1000 due to the Chinese remainder theorem, since 8 and 125 are two coprime numbers that multiply to 1000.

Note that the product of any four consecutive odd integers is congruent to $(-3) \cdot$

$(-1) \cdot 1 \cdot 3 \equiv 9 \equiv 1 \pmod 8$. Thus

$$55!! \equiv \prod_{i=1}^{28}(2i-1) \equiv 1^7 \equiv 1 \pmod 8.$$

Since $55!! \equiv 1 \pmod 8$ and $55!! \equiv 0 \pmod{125}$, then $55!! \equiv 625 \pmod{1000}$.

JIF-9 **Problem** Real numbers a and b satisfy $3^a = 7$ and $7^b = 27$. Find the value of ab.

Answer 3

Solution By the properties of exponents, $3^{ab} = 7^b = 27 = 3^3$. Therefore, $ab = 3$.

JIF-10 **Problem** Real numbers m and n satisfy $m^2 - 2m - 5 = 0$ and $n^2 - 2n - 5 = 0$. Find all possible values of $m^3 - n^3$.

Answer $0, \pm 18\sqrt{6}$

Solution A quadratic polynomial has at most two roots. We consider two cases.

Case 1. $m = n$. Then $m^3 - n^3 = 0$.

Case 2. $m \neq n$. Then from Vieta's formulas, $m + n = 2$ and $mn = -5$. Now $m^3 - n^3 = (m-n)(m^2 + mn + n^2) = \pm\sqrt{(m+n)^2 - 4mn} \cdot ((m+n)^2 - mn) = \pm\sqrt{4 - (-5)} \cdot (2^2 - 4(-5)) = \pm 2\sqrt{6} \cdot 9 = \pm 18\sqrt{6}$. Note that the sign of $m^3 - n^3$ varies if $m > n$ or $m < n$.

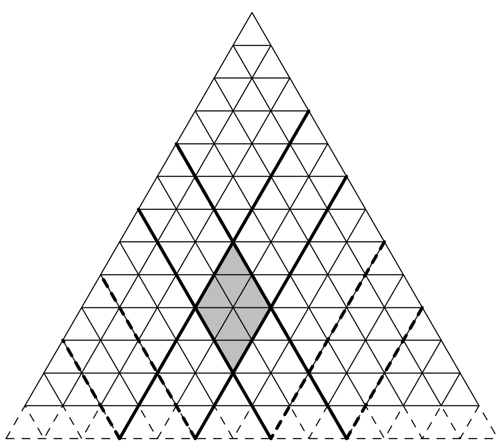
Thus there are three possible values for $m^3 - n^3$: $0, \pm 18\sqrt{6}$.

JIF-11 **Problem** An equilateral triangle with side length 12 is divided into 144 congruent non-overlapping equilateral triangles with side length 1. How many parallelograms are bounded by the segments of the triangles?

Answer 3003

Solution Consider first the parallelograms whose sides are not parallel to the bottom side of the triangle. To find the number of such parallelograms, we construct a bijection. First draw one extra row at the bottom of the triangle, as shown. If we extend the four segments to intersect this extra row, four points are obtained. It is easy to see that these four points uniquely determine a parallelogram in this case. Thus the number of parallelograms here is $\binom{14}{4} = 1001$, as there are fourteen points on the extra row.

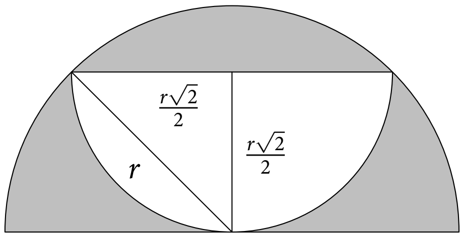
Similarly we have 1001 parallelograms whose sides are not parallel with the left side and 1001 whose sides are not parallel with the right side of the triangle. Thus there are 3003 parallelograms.



JIF-12 **Problem** An infinite sequence of inscribed semicircles with decreasing radii is drawn such that their straight edges are all parallel. Find the area of all the regions inside an odd number of semicircles, if the radius of the largest semicircle is 100 units?

Answer $\dfrac{10\,000\pi}{3}$

Solution Consider two semicircles constructed in this fashion. If the larger semicircle has radius r, then by the Pythagorean theorem, the smaller one has radius $\dfrac{r\sqrt{2}}{2}$.



Therefore, the area of the shaded region is

$$\frac{10\,000\pi}{2} - \frac{5000\pi}{2} + \frac{2500\pi}{2} - \frac{1250\pi}{2} + \cdots = \frac{\frac{10\,000\pi}{2}}{1 - \left(-\frac{1}{2}\right)} = \frac{10\,000\pi}{2} \cdot \frac{2}{3} = \frac{10\,000\pi}{3}$$

JIF-13 **Problem** How many rectangles can be formed by a 4×101 grid of squares such that the rectangles contain the square on the second column, second row?

Answer 1200

Solution Each rectangle can be determined by a pair of horizontal lines and a pair of vertical lines. A rectangle that contains the square on the second column, second row would have one line to the left, one to the right, one above, and one below the said square. Thus the number of rectangles is $2 \times 100 \times 2 \times 3 = 1200$.

JIF-14 **Problem** An ordered list of 100 real numbers is such that the sum of any 9 consecutive numbers on the list are all equal, i.e., if a_n is the nth term, then $\sum_{i=1}^{9} a_i = \cdots = \sum_{i=92}^{100} a_i$, and that the sum of any 11 consecutive numbers are also all equal. Find the largest possible value of the range of the numbers.

Answer 0

Solution From the given it is implied that if $x \equiv y \pmod 9$ or $x \equiv y \pmod{11}$, then $a_x = a_y$. But the list has more than 9×11 elements, implying that all the numbers must be equal. Thus the range is zero.

JIF-15 **Problem** The sum of n consecutive integers is 192. Find the greatest possible value for n.

Answer 384

Solution Note that $(-191)+(-190)+\cdots+191+192 = 192$, which has 384 consecutive integers. It is clear that a longer string off consecutive integers cannot sum up to 192. Let x be the smallest number among the consecutive integers. Then $x + (x + 1) + \cdots + (x + n - 1) = 192 \Rightarrow nx + \frac{n(n-1)}{2} = 192 \Rightarrow n(n + x - 1) = 384$. Since both n and $n + x - 1$ have to be integers, $n \leq 384$. We have shown that $n = 384$ is possible; thus it is the maximum value.

JTF-1 **Problem** Evaluate $\sin(55°)\sin(65°)\sin(175°)+\sin(125°)\sin(130°)\sin(240°)+\sin(35°)\sin(95°)\sin(115°)+\sin(50°)\sin(145°)\sin(150°)$.

Answer $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

Solution

$$\begin{aligned}
&\sin(55°)\sin(65°)\sin(175°)+\sin(125°)\sin(130°)\sin(240°)\\
&+\sin(35°)\sin(95°)\sin(115°)+\sin(50°)\sin(145°)\sin(150°)\\
=&\cos(35°)\cos(25°)\sin(5°)-\cos(35°)\cos(40°)\cos(30°)\\
&+\sin(35°)\cos(5°)\cos(25°)+\cos(40°)\sin(35°)\sin(30°)\\
=&\cos(25°)(\cos(35°)\sin(5°)+\cos(5°)\sin(35°))\\
&-\cos(40°)(\cos(35°)\cos(30°)-\sin(35°)\sin(30°))\\
=&\cos(25°)\sin(40°)-\cos(40°)\cos(65°)\\
&=\cos(25°)\sin(40°)-\cos(40°)\sin(25°)\\
=&\sin(15°)=\frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

JTF-2 **Problem** Rugged Corp is making a new tablet that can survive falls from heights (but not from 2018 floors high). To test this the company gives John, a quality assurance tester, two units of the said tablet on a 2017-floor skyscraper. John will try to determine the highest floor from which a dropped tablet will survive the fall. The company will not give John extra tablets but John can reuse a tablet that did not break. What is the minimum number of drops John needs to guarantee that he finds the highest "safe" floor?

Answer 64

Solution We denote $a + \cdots + b$ as the sum of all integers between a and b, inclusive. First, note that 64 is the smallest integer n that satisfies $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \geq 2017$, as $\frac{63(64)}{2} = 2016$. This is how the drops will work:

We drop the first tablet on the following floors, until it breaks: floors 64, 64 + 63, $64 + 63 + 62, \ldots, 64 + \cdots + 12 = 2014$, 2015, and 2016. (We know the tablet will break on the top floor.) There are 55 floors in this sequence.

If the first tablet breaks on the 64th floor, then the maximum number of drops needed to know the highest floor is 63 for the second tablet, as John would need to drop his second (and last) tablet from the first floor going up, as there are no replacements. Thus at most 64 drops are needed.

Now, if the first tablet breaks on floor $64 + \cdots + n$, $12 \leq n \leq 63$, then we know that the floor we want is between floors $64 + \cdots + (n+1)$ and $64 + \cdots + n$. To get to floor $64 + \cdots + n$ the first tablet would have been dropped $65 - n$ times; the second tablet will require a maximum of $n - 1$ drops from $(64 + \cdots + (n+1)) + 1$ to $64 + \cdots + n$. We still need a maximum of 64 drops.

If the first tablet doesn't break from the 2014th floor, we proceed with the 2015th and 2016th floors, requiring a maximum of 55 drops.

Thus John needs to do a minimum of $\max(64, 64, 55) = 64$ drops.

Note that if the initial floor is not 64, it will not be optimal. If it is < 64, then if the tablet will only break at the 2017th floor, more than 64 drops are needed. On the other hand, if it is > 64, then if the tablet breaks at the 64th floor, more than 64 drops are needed.

JTF-3 **Problem** How many permutations of $STEPHENIAN$ are there such that the vowels are in alphabetical order?

Answer 75 600

Solution There are a total of $\dfrac{10!}{2!2!} = 907\,200$ ways to rearrange the letters without any restrictions, since there are two E's and two N's.

Consider the rearrangements of the vowels only. There are $\dfrac{4!}{2!} = 12$ ways to arrange the vowels. This means that the 907 200 ways can be grouped into groups of 12, such that within each group S, T, P, H and the two N's are in the same positions. Only one in each group have the vowels in alphabetical order.

Therefore, the number of permutations whose vowels are in alphabetical order is $\dfrac{907\,200}{12} = 75\,600$.

JTF-4 **Problem** What is the largest factor of $11!$ that is one more than a multiple of 6?

Answer 385

Solution The prime factorization of $11!$ is $2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$. Since the factor we are finding is not divisible by 6, then it can be expressed as $5^{a_1}7^{a_2}11^{a_3}$, where $a_1 \in \{0, 1, 2\}$, and $a_2, a_3 \in \{0, 1\}$. Since $5 \equiv 11 \equiv (-1) \pmod 6$ and $7 \equiv 1 \pmod 6$, the largest factor that is congruent to $1 \pmod 6$ is $5 \times 7 \times 11 = 385$.

JTF-5 **Problem** How many integers x satisfy $(x-20)^{17}(x-17)^{20}(x-2017)^{2017} \leq 0$?

Answer 1999

Solution Obviously, $x = 17, 20, 2017$ are solutions. Note that $(x-17)^{20}$ is always non-negative. We consider two cases:

Case 1. $(x-20)^{17} < 0$ and $(x-2017)^{2017} > 0$. Then $x < 20$ and $x > 2017$, which is impossible.

Case 2. $(x-20)^{17} > 0$ and $(x-2017)^{2017} < 0$. Then $20 < x < 2017$. There are $2016 - 21 + 1 = 1996$ integers that satisfy this.

Adding $x = 17, 20, 2017$, there are a total of 1999 integers.

JTF-6 **Problem** Let a, b, c, d be four positive integers. If the LCM of a and b is 60, the LCM of a, b, and c is 120, and the LCM of b, c, and d is 24. What is the least possible value of the LCM of c and d?

Answer 8

Solution The LCM of c and d is at least the bigger of the two, so minimizing c and d is a good tactic if c and d on a large scale. From the given at least one of a and b is divisible by 4 and both are not divisible by 8. Since the LCM of a, b, and c is 120, then c is divisible by 8. This means that the LCM is at least 8.

Say $c = 8$. Since the LCM of b, c and d is 24, then we can choose b as a factor of 3 and $d = 1$, so that the LCM is equal to 8. Since this is possible with $a = 20$, $b = 3$, $c = 8$, $d = 1$, the minimum LCM of c and d is 8.

JTF-7 **Problem** How many integers evenly divide 999 999?

Answer 128

Solution $999\,999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$. Therefore we have $(3+1)(1+1)^4 = 64$ positive factors. If x evenly divides 999 999, $-x$ does too; thus the total number of integers that evenly divide 999 999 is 128.

JTF-8 **Problem** Find the sum of the digits of the decimal expansion of $\frac{10^{27}+2}{6}$.

Answer 158

Solution Note that $\frac{10^{27}+2}{6} = \frac{1}{2} \cdot \left(\frac{10^{27}-1}{3} + 1\right) = \frac{1}{2} \cdot \underbrace{3 \cdots 3}_{26\ 3\text{'s}}4 = 1\underbrace{6 \cdots 6}_{25\ 6\text{'s}}7$. The sum of the digits is $1 + 6 \times 25 + 7 = 158$.

JTF-9 **Problem** The answers to each of the five statements below is either true or false.

1. The answers of Statement 3 and Statement 4 are different.

2. Statement 1 is true.

3. There are more False answers than True answers.

4. There are more False answers than True answers in the above statements.

5. There is an unequal amount of True and False answers in the above statements.

What are the answers to each of the five statements?

Answer False, False, True, True, False

Solution If #2 is true, then #1 is true. This means that #4 is false, in turn implying that #3 is true. This contradicts #1 being true.

Therefore, #2 is false. Then, #1 is false, #4 is true, #3 is true, and #5 is false.

JTF-10 **Problem** There are 1007 points in the interior of a convex pentagon such that no three of 1012 points, including the vertices of the pentagon, are collinear. The pentagon is partitioned into several triangles. Each vertex of each of these triangles is either a vertex of the pentagon or one of the 1007 points. How many triangles result?

Answer 2017

Solution Consider the interior angles formed in the triangles. If x triangles are formed, then the sum of the interior angles of all the triangles, in degrees, is $180x$.

However, this sum is also equal to the sum of the internal angles of the pentagon and $360°$ for each point, since the sum of the internal angles that have a specific point as vertex is $360°$. This is equal to $540 + 360 \cdot 1007$ in degrees.

Therefore, $180x = 540 + 360 \cdot 1007 \Rightarrow x = 3 + 2 \cdot 1007 = 2017$.

JTF-11 **Problem** How many pounds of H_2O must be evaporated from 50 pounds of a 3% salt solution so that the remaining solution will be 5% salt?

Answer 20

Solution There are $50(0.03) = 1.5$ pounds of salt in the original solution. If x is the number of pounds of water that must be evaporated, then the remaining solution contains $50 - x$ pounds. Since the amount of salt is unchanged, then $(50 - x)(0.05) = 1.5 \Rightarrow 50 - x = 30 \Rightarrow x = 20$.

JTF-12 **Problem** Find the remainder when $\binom{4034}{2017}$ is divided by 2017^2.

Answer 2

Solution From the Vandermonde identity, we have

$$\binom{4034}{2017} = \sum_{i=0}^{2017} \binom{2017}{i}\binom{2017}{2017-i}.$$

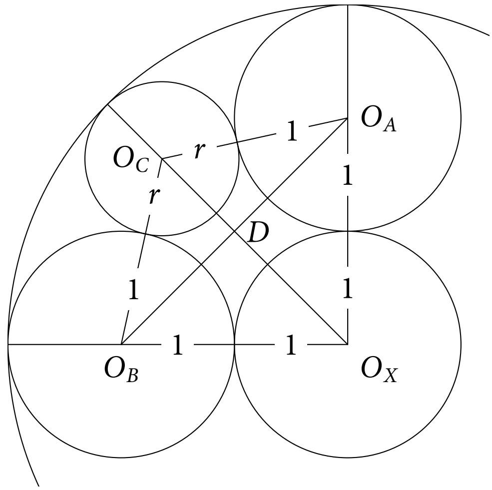
Note that the first and last terms in the expansion of the right-hand side, $\binom{2017}{2017} \cdot \binom{2017}{0}$, each equal 1. Since 2017 is a prime, then it divides $\binom{2017}{i}$ for all integers i from 1 to 2016. Thus for all integers i from 1 to 2016, $\binom{2017}{i}\binom{2017}{2017-i}$ is divisible by 2017^2.

Thus, the remainder when $\binom{4034}{2017}$ is divided by 2017^2 is 2.

JTF-13 **Problem** Circle X with radius 3 contains a concentric circle Y with radius 1. Two other circles, A and B, are congruent to Y and are tangent to both X and Y such that the centers of circles A, B and Y form a right triangle. A smaller circle C is tangent to circles A, B, and X. Find the radius of circle C.

Answer $\dfrac{9-3\sqrt{2}}{7}$

Solution Let O_Z be the center of circle Z, and r be the radius of circle C.



Note that quadrilateral $O_AO_CO_BO_X$ is a kite, as $O_XO_A = O_XO_B = 2$ and $O_BO_C = O_CO_A = r+1$. Furthermore, $O_AO_B = 2\sqrt{2}$ and $O_XO_C = 3-r$. Therefore,

$$[O_AO_CO_BO_X] = \frac{2\sqrt{2}(3-r)}{2} = 3\sqrt{2} - \sqrt{2}r,$$

and $[O_AO_XO_B] = 2$.

Let D be the midpoint of O_AO_B. Since $\triangle O_AO_BO_C$ is isosceles, we know that $O_CO_X \perp O_AO_B$. From the Pythagorean theorem, $O_CD = \sqrt{(O_CO_B)^2 - (O_BD)^2} = \sqrt{(r+1)^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{r^2+2r-1}$.

Then $[O_AO_CO_B] = \dfrac{2\sqrt{2}\sqrt{r^2+2r-1}}{2} = \sqrt{2r^2+4r-2}$. This means that

$$[O_AO_CO_BO_X] = [O_AO_XO_B] + [O_AO_CO_B] = 3\sqrt{2} - \sqrt{2}r = 2 + \sqrt{2r^2+4r-2}$$

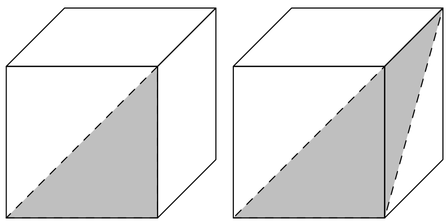
. This equation simplifies to $7r^2 - 18r + 9 = 0$, with roots $r = \dfrac{9 \pm 3\sqrt{2}}{7}$.

Note that from the question circle C with radius r is smaller than the circles with radius 1. Since $\frac{9-3\sqrt{2}}{7} < 1 < \frac{9+3\sqrt{2}}{7}$, we have $r = \frac{9-3\sqrt{2}}{7}$.

JTF-14 **Problem** How many right triangles can be formed using the vertices of a cube?

Answer 48

Solution There are two kinds of right triangles that can be formed: one isosceles with sides $1, 1, \sqrt{2}$ and the other with sides $1, \sqrt{2}, \sqrt{3}$, as shown. There are 24 for each, so there are 48 right triangles in all.



JTF-15 **Problem** AB is a diameter of circle O. P is a point on $\overrightarrow{AB}$ outside the circle and C is a point on the circle such that CP is a tangent. If $AP = 8$ and $CP = 4$, find the length of AB.

Answer 6

Solution Let the radius of the circle be x. Since CP is a tangent, OCP is a right triangle with OP as hypotenuse. We have $CP = 4$, $OC = x$ and $OP = 8 - x$. From the Pythagorean theorem, we have $x^2 + 16 = x^2 - 16x + 64$ or $x = 3$. Therefore the diameter AB has length 6.

SE1-E1 **Problem** Suppose a statistician wants to test if two standard dice are loaded or not, using the sum of the numbers on top as a one-tailed test statistic. If he rolls the dice and both turn up five, what is the p-value?

Answer $\frac{1}{6}$

Solution Since the test statistic X is considered to be one-tailed, the p-value is equal to $P(X = 10) + P(X = 11) + P(X = 12)$. But $P(X = 10) = \frac{3}{36}$, $P(X = 11) = \frac{2}{36}$, and $P(X = 12) = \frac{1}{36}$. Therefore, the p-value is $\frac{1}{6}$.

SE1-E2 **Problem** A 100×100 grid (with ten thousand cells) is drawn. The positive integers from 1 to 10 000 are written, one each, in each cell such that the sum of the numbers on each row, column, or main diagonal are all equal to a certain constant. Find this constant.

Answer 500 050

Solution The sum of all positive integers less than or equal to 10 000 is $\frac{1}{2} \cdot 10\,000 \cdot 10\,001 = 50\,005\,000$. Since the sum of the numbers on each row should be equal to k, then $100k = 50\,005\,000$. Therefore, $k = 500\,050$.

SE1-E3 **Problem** Suppose f is a real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. Find $f(21)$.

Answer 64

Solution Substitute $x = 1$: $f(5) = f(1 + f(1)) = 4f(1) = 16$.

Substitute $x = 5$: $f(21) = f(5 + f(5)) = 4f(5) = 64$.

SE1-E4 **Problem** Find the probability of drawing, without replacement, two aces from a deck of 52 playing cards.

Answer $\frac{1}{221}$

Solution We multiply the probabilities for each of the two draws: $\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{13 \times 17} = \frac{1}{221}$.

SE1-E5 **Problem** Find the product of all real zeros of $\log x + \log(x + 2) - 3$.

Answer $\sqrt{1001} - 1$

Solution The equation $\log x + \log(x + 2) = 3$ is equivalent to $x^2 + 2x = 10^3$, or $x^2 + 2x - 1000 = 0$. This quadratic equation has roots $\frac{-2 \pm \sqrt{4004}}{2} = -1 \pm \sqrt{1001}$. Since $\log(-1 - \sqrt{1001})$ is undefined, the only real zero is $-1 + \sqrt{1001}$.

SE1-E6 **Problem** Let $f(n)$ denote the sum of the distinct prime factors of positive integer n. Let $g(k)$ be the kth smallest integer mn $(m, n \in \mathbb{Z}^+)$, such that $f(m + n) = f(mn)$. What is $g(1) + g(2) + g(3)$?

Answer 15

Solution We check for smaller numbers.

$$mn = 2: \quad 2 = f(2) = f(2 \cdot 1) \neq f(2 + 1) = f(3) = 3$$

$$mn = 3: \quad 3 = f(3) = f(3 \cdot 1) \neq f(3 + 1) = f(4) = 2$$

$$mn = 4: \quad 4 = f(4) = f(2 \cdot 2) = f(2 + 2) = f(4) = 4$$

$$mn = 5: \quad 5 = f(5) = f(5 \cdot 1) = f(5+1) = f(6) = 5$$

$$mn = 6: \quad 5 = f(6) = f(2 \cdot 3) = f(2+3) = f(5) = 5$$

Therefore, $g(1) = 4$, $g(2) = 5$, $g(3) = 6$; $g(1) + g(2) + g(3) = 4 + 5 + 6 = 15$.

SE1-E7 **Problem** If $\sin\theta = \frac{7}{25}$, find $\sin 2\theta$.

Answer $\pm\frac{336}{625}$

Solution From the Pythagorean identity, $\cos\theta = \pm\frac{24}{25}$, the sign dependent on the quadrant where θ belongs. Thus $\sin 2\theta = 2\sin\theta\cos\theta = \pm 2 \cdot \frac{7}{25} \cdot \frac{24}{25} = \pm\frac{336}{625}$.

SE1-E8 **Problem** There are one thousand lockers and one thousand students in a school. The principal asks the first student to go to every locker and open it. Then he has the second student go to every second locker and close it. The third goes to every third locker and, if it is closed, he opens it, and if it is open, he closes it. The fourth student does this to every fourth locker, and so on. After the process is completed with the thousandth student, how many lockers are closed?

Answer 969

Solution A locker is closed if it is handled by an even number of students. We know that only perfect squares have an odd number of factors. Since there are 31 perfect squares from 1 to 1000, there are $1000 - 31 = 969$ closed lockers.

SE1-E9 **Problem** Find the amplitude of the graph of the function $f(x) = 3\sin x + \cos x$.

Answer $\sqrt{10}$

Solution Divide by $\sqrt{10}$ on both sides to get $\frac{f(x)}{\sqrt{10}} = \frac{3}{\sqrt{10}} \sin x + \frac{1}{\sqrt{10}} \cos x$. Now $\cos \sin^{-1} \frac{1}{\sqrt{10}} = \frac{3}{\sqrt{10}}$, so we can rewrite this as $\frac{f(x)}{\sqrt{10}} = \sin\left(x + \sin^{-1} \frac{1}{\sqrt{10}}\right)$. Therefore the amplitude of the function is $\sqrt{10}$.

SE1-E10 **Problem** Positive integers a, b, and c satisfy $a + \frac{1}{b + \frac{1}{c}} = \frac{15}{2}$. Find $(a-2b-4c)^{2017}$.

Answer 1

Solution Since $\frac{15}{2} = 7 + \frac{1}{2} = 7 + \frac{1}{1 + \frac{1}{1}}$, $a = 7$ and $b = c = 1$. Therefore, $(a - 2b - 4c)^{2017} = 1^{2017} = 1$.

SE1-A1 **Problem** Evaluate the sum $\sum_{k=1}^{99} \frac{k+1}{(k-1)! + k! + (k+1)!}$

Answer $1 - \frac{1}{100!}$

Solution Simply the summand first:

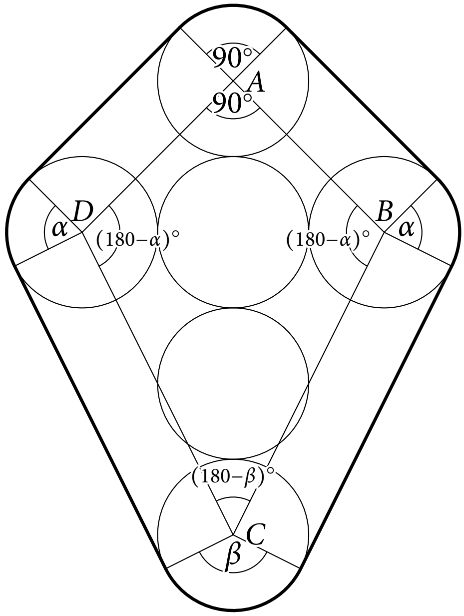
$$\begin{aligned}
\frac{k+1}{(k-1)! + k! + (k+1)!} &= \frac{k+1}{(k-1)!(1 + k + k(k+1))} \\
&= \frac{k+1}{(k-1)!(k+1)^2} = \frac{1}{(k-1)!(k+1)} \\
&= \frac{k}{(k+1)!} = \frac{(k+1)-1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}
\end{aligned}$$

Thus, the sum telescopes to $\frac{1}{1!} - \frac{1}{100!} = 1 - \frac{1}{100!}$

SE1-A2 **Problem** Six circles of radius 1 unit are centered at the points $(0,0)$, $(0,2)$, $(0,-2)$, $(0,-4)$, $(2,0)$, $(-2,0)$. If a rubber band is wrapped around this figure, find the area of the region inside the rubber band.

Answer $12 + 4\sqrt{2} + 4\sqrt{5} + \pi$

Solution



Note that $90 + 2(180 - \alpha) + (180 - \beta) = 360 \Rightarrow 2\alpha + \beta = 270$. Thus, the area inside the rubber band is equal to the sum of the areas of kite $ABCD$, the four rectangles (which share a side with kite $ABCD$), and a circle with radius 1, since the central angles $90°$, α, α and β add up to $360°$.

The kite has diagonals 4 and 6; thus it has area 12. Also, from the Pythagorean theorem, $AB = AD = 2\sqrt{2}$, and $CD = CB = 2\sqrt{5}$. Therefore, the four rectangles have a total area of $4\sqrt{2} + 4\sqrt{5}$. Finally, a circle with radius 1 has area π.

Thus, the area of the interior is $12 + 4\sqrt{2} + 4\sqrt{5} + \pi$.

SE1-A3 **Problem** If for some acute angle θ, $\tan\theta + \cot\theta = 4$, what is $\sin\theta$?

Answer $\dfrac{\sqrt{6} \pm \sqrt{2}}{4}$

Solution From the original equation multiply both sides by $\sin\theta\cos\theta$ (since $\sin\theta \neq 0$ and $\cos\theta \neq 0$) to get $\sin^2\theta + \cos^2\theta = 4\sin\theta\cos\theta$. But the left-hand side of this equation is 1 and the right hand side is precisely $2\sin 2\theta$. Thus we have $\sin 2\theta = \dfrac{1}{2}$ or $2\theta = 30°, 150°$. If $\theta = 15°$, $\sin\theta = \dfrac{\sqrt{6} - \sqrt{2}}{4}$; if $\theta = 75°$, $\sin\theta = \dfrac{\sqrt{6} + \sqrt{2}}{4}$. Thus the values for $\sin\theta$ are $\dfrac{\sqrt{6} \pm \sqrt{2}}{4}$.

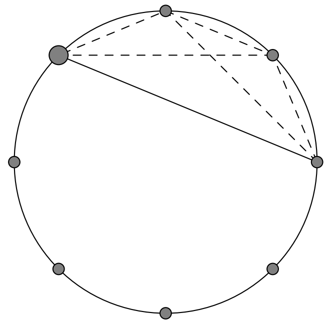
SE1-A4 **Problem** Eight equally-spaced points are on the perimeter of the circle. What is the probability that if three of the points are connected, it forms an acute triangle?

Answer $\dfrac{1}{7}$

Solution We first calculate the number of triangles that can be chosen from these eight points. This is equal to $\dbinom{8}{3} = 56$ triangles.

A right triangle is always formed by choosing two diametrically opposite points and one of the six remaining points. There are four pairs of diametrically opposite points; therefore, there are 24 right triangles that can be formed.

Now we have to calculate the number of obtuse triangles. Refer to the figure below.



First we have to choose a point among the eight. This accounts for 8 ways. Let's say we chose the bigger point shown in the figure. Then, we draw a line to the point 3 points away from the first point, moving clockwise (to avoid double-counting). If we choose any two of the three points, the resulting triangle is always obtuse. Therefore there are $\binom{3}{2}$ ways to choose these two points. In general, there are $8\binom{3}{2} = 24$ obtuse triangles.

This means that there are $56 - 24 - 24 = 8$ acute triangles. The probability, then, is $\frac{8}{56} = \frac{1}{7}$.

SE1-A5 **Problem** Find the greatest common factor of F_{2017} and F_{2016}, where F_n is defined as $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for integers $n > 2$.

Answer 1

Solution For two integers $a > b$, the GCF of a and b is equal to the GCF of $a - b$ and b. Therefore $\text{gcf}(F_{2017}, F_{2016}) = \text{gcf}(F_{2017} - F_{2016}, F_{2016}) = \text{gcf}(F_{2015}, F_{2016}) = \cdots = \text{gcf}(F_2, F_1) = \text{gcf}(1, 1) = 1$.

SE1-A6 **Problem** The value of the function $f(x) = ax + b$ for all integers x is a positive integer that leaves a remainder of 3 when divided by 4. Find all possible values for a.

Answer 0

Solution Note that $f(x) > 0$ for all integers x. The only linear function that satisfies this is the constant function. Thus $a = 0$ and b is any positive integer that leaves a remainder of 3 when divided by 4.

SE1-A7 **Problem** Let $-1 < x < 1$. Find the value of the infinite series $S = 1 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1} + \cdots$.

Answer $\dfrac{1+x}{(1-x)^3}$

Solution Since $n^2 = 2 \cdot \dfrac{n(n+1)}{2} - n$, then

$$S = 2\left(1 + 3x + 6x^2 + \cdots + \frac{n(n+1)}{2}x^{n-1} + \cdots\right) - \left(1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots\right)$$
$$= \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} = \frac{1+x}{(1-x)^3}.$$

SE1-D1 **Problem** There are 432 seats in a cinema and 432 moviegoers with tickets marking their respective assigned seats. The moviegoers line up to enter the cinema. The first moviegoer did not know that the ticket shows an assigned seat, and thus chooses a random seat and sits there. The remaining moviegoers sit on their seat if it is empty — otherwise he/she chooses a random open seat to sit. What is the probability that the 432nd moviegoer sits at his/her assigned seat?

Answer 50%

Solution The last unoccupied seat is either the seat meant for the last moviegoer or the seat of the first moviegoer. If any other seat is vacant when the last moviegoer enters, then it was also vacant when the moviegoer who is supposed to sit there entered the cinema — a contradiction.

But since it is equally likely that the last moviegoer goes to either of the two possible seats (the first moviegoer's seat and the last moviegoer's seat), the probability is $\frac{1}{2}$.

SE1-D2 **Problem** The sum $\sum_{i=1}^{10} \binom{20}{2i-1} 9^{2i-1} 7^{21-2i}$ can be expressed in the form $2^x + 2^y$, where x and y are distinct positive integers. Find $x + y$.

Answer 98

Solution Let S_1 be the above sum. Consider $S_2 = \sum_{i=0}^{10} \binom{20}{2i} 9^{2i} 7^{20-2i}$.

Then, from the binomial theorem,

$$16^{20} = (9+7)^{20} = \sum_{i=0}^{20} \binom{20}{i} 9^i 7^{20-i} = S_1 + S_2$$
$$2^{20} = (9-7)^{20} = \sum_{i=0}^{20} \binom{20}{i} 9^i (-7)^{20-i} = S_1 - S_2,$$

since if i is even then $\binom{20}{i} 9^i 7^{20-i} = \binom{20}{i} 9^i (-7)^{20-i} = S_1 + S_2$.

Adding the two equations together and dividing by two results in $S_1 = \frac{16^{20} + 2^{20}}{2} = \frac{2^{80} + 2^{20}}{2} = 2^{79} + 2^{19}$. The required answer is $79 + 19 = 98$.

SE1-D3 **Problem** Define the *omega function* $\omega(x)$ as the real number that satisfies $x =$

$\omega(x \cdot e^x)$, for real numbers x. Simplify

$$\sqrt[\sqrt[\sqrt[\cdot^{\cdot^{\cdot}}]{2}]{2}]{2}$$

in terms of the omega function.

Answer $e^{\omega(\ln 2)}$

Solution Let x be the given quantity. Then $\sqrt[x]{2} = x \Rightarrow 2^{1/x} = x \Rightarrow x^x = 2 \Rightarrow x \ln x = \ln 2$. Let $a = \ln x$, or $e^x = a$. The equation becomes $e^a \cdot a = \ln 2 \Rightarrow \omega(e^a \cdot a) = \omega(\ln 2) \Rightarrow a = \omega(\ln 2)$. Substituting back the expression for x, we get $\ln x = \omega(\ln 2) \Rightarrow x = e^{\omega(\ln 2)}$.

SE1-D4 **Problem** How many terms on the 2017th row of the Pascal triangle, whose first row has the terms 1 and 1, are not divisible by 3?

Answer 162

Solution This question is equivalent to finding the number of coefficients in the expansion of $(1+x)^{2017}$ that are not divisible by 3. Note that $(1+x)^3 \equiv 1 + x^3 \pmod 3$. Similarly, for k a power of 3, $(1+x)^k \equiv 1 + x^k \pmod 3$.

But $2017 = 729 \times 2 + 243 \times 2 + 27 \times 2 + 9 \times 2 + 1$. Thus

$$\begin{aligned}(1+x)^{2017} &\equiv \left((1+x)^{729}\right)^2 \left((1+x)^{243}\right)^2 \left((1+x)^{27}\right)^2 \cdot \\ &\quad \left((1+x)^9\right)^2 (1+x) \pmod 3 \\ &\equiv (x^{1458} + 2x^{729} + 1)(x^{486} + 2x^{243} + 1) \cdot \\ &\quad (x^{54} + 2x^{27} + 1)(x^{18} + 2x^9 + 1)(x+1) \pmod 3\end{aligned}$$

There are $3^4 \times 2 = 162$ terms in this expansion; it is easy to see that these terms are not like terms.

SE1-D5 **Problem** Find

$$\sum_{i=3}^{\infty} \frac{\sin \frac{i\pi}{(i+1)!}}{\cos \frac{\pi}{i!} \cos \frac{\pi}{(i+1)!}}.$$

Answer $\frac{\sqrt{3}}{3}$

Solution

$$\begin{aligned}
\sum_{i=3}^{\infty} \frac{\sin \frac{i\pi}{(i+1)!}}{\cos \frac{\pi}{i!} \cos \frac{\pi}{(i+1)!}} &= \sum_{i=3}^{\infty} \frac{\sin \left(\frac{\pi}{i!} - \frac{\pi}{(i+1)!} \right)}{\cos \frac{\pi}{i!} \cos \frac{\pi}{(i+1)!}} \\
&= \sum_{i=3}^{\infty} \frac{\sin \frac{\pi}{i!} \cos \frac{\pi}{(i+1)!} - \cos \frac{\pi}{i!} \sin \frac{\pi}{(i+1)!}}{\cos \frac{\pi}{i!} \cos \frac{\pi}{(i+1)!}} \\
&= \sum_{i=3}^{\infty} \left(\tan \frac{\pi}{i!} - \tan \frac{\pi}{(i+1)!} \right)
\end{aligned}$$

Therefore, the series telescopes to $\tan \frac{\pi}{3!} = \frac{\sqrt{3}}{3}$.

SF-1 **Problem** Evaluate $\dfrac{\sin\dfrac{\pi}{12}+\sin\dfrac{\pi}{6}+\sin\dfrac{\pi}{4}}{\cos\dfrac{\pi}{12}+\cos\dfrac{\pi}{6}+\cos\dfrac{\pi}{4}}$.

Answer $\dfrac{\sqrt{3}}{3}$

Solution

$$\begin{aligned}
\frac{\sin\frac{\pi}{12}+\sin\frac{2\pi}{12}+\sin\frac{3\pi}{12}}{\cos\frac{\pi}{12}+\cos\frac{2\pi}{12}+\cos\frac{3\pi}{12}} &= \frac{\sin\frac{2\pi}{12}+2\sin\dfrac{\frac{\pi}{12}+\frac{3\pi}{12}}{2}\cos\dfrac{\frac{\pi}{12}-\frac{3\pi}{12}}{2}}{\cos\frac{2\pi}{12}+2\cos\dfrac{\frac{\pi}{12}+\frac{3\pi}{12}}{2}\cos\dfrac{\frac{\pi}{12}-\frac{3\pi}{12}}{2}} \\
&= \frac{\sin\frac{2\pi}{12}+2\sin\frac{2\pi}{12}\cos\frac{-\pi}{12}}{\cos\frac{2\pi}{12}+2\cos\frac{2\pi}{12}\cos\frac{-\pi}{12}} \\
&= \frac{\sin\frac{2\pi}{12}\left(1+2\cos\frac{-\pi}{12}\right)}{\cos\frac{2\pi}{12}\left(1+2\cos\frac{-\pi}{12}\right)} = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}
\end{aligned}$$

SF-2 **Problem** Find the sum of the reciprocals of all positive integers that can be expressed in the form $3^a \cdot 4^b$, where a and b are positive integers such that $a+b$ is odd.

Answer $\dfrac{7}{120}$

Solution Either a is odd and b is even, or a is even and b is odd. Thus the sum is

$$\begin{aligned}\sum_{a+b \text{ odd}} \frac{1}{3^a \cdot 4^b} &= \sum_{a \text{ even}} \frac{1}{3^a} \cdot \sum_{b \text{ odd}} \frac{1}{4^b} + \sum_{a \text{ odd}} \frac{1}{3^a} \cdot \sum_{b \text{ even}} \frac{1}{4^b} \\ &= \frac{\frac{1}{9}}{1-\frac{1}{9}} \cdot \frac{\frac{1}{4}}{1-\frac{1}{16}} + \frac{\frac{1}{3}}{1-\frac{1}{9}} \cdot \frac{\frac{1}{16}}{1-\frac{1}{16}} \\ &= \frac{1}{8} \cdot \frac{4}{15} + \frac{3}{8} \cdot \frac{1}{15} = \frac{7}{120}\end{aligned}$$

SF-3 **Problem** Let $f : \mathbb{Z}^+ \mapsto \mathbb{Z}^+$ be a piecewise function such that

$$f(n) = \begin{cases} 1 & n = 1 \\ f(\sqrt{n}) & n \text{ a perfect square} \\ f(n-1)+1 & \text{otherwise} \end{cases}.$$

What is the smallest n such that $f(n) = 100$?

Answer 2024

Solution We start with a few lemmas.

Lemma 1. *For all integers $n \geq 4$, $f(n) < n$.*

Proof. We proceed by strong induction. Note that $f(4) = f(2) = 2 < 4$. Assume for the sake of induction that $f(k) < k$ is true for all integers k from 4 to an integer $m \geq 4$. By definition, if $m+1$ is not a perfect square, then $f(m+1) = f(m)+1 < m+1$, and we are done.

On the other hand, if $m+1$ is a perfect square, then $f(m+1) = f(\sqrt{m+1})$. However, since $m \geq 4$, $m^2 - m - 1 > 0$. This means that $m > \sqrt{m+1}$. Therefore, $f(m+1) = f(\sqrt{m+1}) < \sqrt{m+1} < m < m+1$, and we are done. The inductive step is complete.

Thus, for all integers $n \geq 4$, $f(n) < n$. $\square$

Lemma 2. *For any positive integer n that can be expressed in the form $x^2 + y$, where $x^2 \leq n < (x+1)^2$, $x \in \mathbb{Z}^+$, and $0 \leq y \leq 2x$, $y \in \mathbb{Z}^{\geq}$, $f(n) = f(x) + y$.*

Proof. We proceed by induction on y. Note that if $y = 0$ $f(n) = f(x^2) = f(x) = f(x) + 0$. Assume for the sake of induction that if $n = x^2 + k$ for some integer k between 0 and $2x - 1$, then $f(n) = f(x) + k$. Then, since $n + 1$ is not a perfect square, $f(n+1) = f(n) + 1 = f(x) + (k+1)$, which completes the proof. $\square$

Lemma 3. *For any positive integer n that can be expressed in the form $x^2 + y$, where $x^2 \leq n < (x+1)^2$, $x \in \mathbb{Z}^+$, and $0 \leq y \leq 2x$, $y \in \mathbb{Z}^{\geq}$, $f(n) \leq 3x$.*

Proof. From Lemma 1 and Lemma 2, $f(n) = f(x) + y \leq x + 2x = 3x$. $\square$

From Lemma 3, we see that for all $n \leq 33^2 + 2 \cdot 33 = 1155$, $f(n) \leq 3 \cdot 33 = 99$. Thus, if $f(n) = 100$, and $n = x^2 + y$ (with x and y as in Lemma 2), then $x \geq 34$.

Consider all $n = 49^2 + y = 2401 + y$, where y is an integer from 0 to 98, inclusive. Then $f(n) = f(49) + y = f(7) + y = 5 + y = 100 \Rightarrow y = 95$. This means that $x = 49$ is possible. Thus, if $f(n) = 100$, then $34 \leq x \leq 49$.

However, if $x = 34$ or 35, then $f(n) = f(x) + y \leq 13 + 70 = 83 < 100$, since $0 \leq y \leq 70$, $f(34) = f(5) + 9 = 3 + 9 = 12$ and $f(35) = f(34) + 1 = 13$. Thus no n exists for this case.

If $x \geq 36$, let $x = 36 + p$ where $p \in \mathbb{Z}^{\geq}$. To minimize n, we need to minimize x, and in turn, p.

We have $f(n) = f(x) + y = f(36 + p) + y = f(6) + p + y = 4 + p + y = 100$, due to Lemma 2. To minimize p, we maximize y. (Note that increasing x by a unit amount leads to a greater increase in n than increasing y by a unit amount.) The maximum possible y is $y = 2x = 2(36 + p)$. Then,

$$4 + p + 2(36 + p) = 100 \Rightarrow 3p + 76 = 100 \Rightarrow p = 8$$

This implies that $x = 36 + p = 36 + 8 = 44$, $y = 2x = 88$, and $n = x^2 + y = 44^2 + 88 = 2024$.

SF-4 **Problem** Define $\{S_i\}_{i=0}^{\infty}$ to be a sequence of sets such that $S_0 = \{0\}$, and $S_i = \bigcup_{k=0}^{i-1} \{S_k\}$, the union of the set containing S_0, the set containing S_1, until the set containing S_{i-1}. How many opening braces { are needed to write down the power set of S_6 in expanded form, if the empty set $\varnothing$ is to be written as { }?

Answer 2081

Solution Note that $S_0 = \{0\}$, $S_1 = \{S_0\} = \{\{0\}\}$, $S_2 = \{S_0, S_1\} = \{\{0\}, \{\{0\}\}\}$, and so on.

First, we show by strong induction that S_i has exactly 2^i opening braces. It is easy to see that S_0 has 1 opening brace and that S_1 has 2. Now say that for all i between 0 to n, S_i uses 2^i opening braces. Then for $S_{n+1} = S_0, S_1, S_2, \ldots, S_n$, we need one outermost opening brace, then add the number of opening braces for S_0 up to S_n. Thus the number of opening braces needed for S_{n+1} is $1+(1+2+\ldots+2^n) = 1+2^{n+1}-1 = 2^{n+1}$. Thus, we conclude that for all non-negative integers i, S_i has exactly 2^i opening braces.

The power set of S_6, denoted as 2^{S_6}, is the set of all subsets of S_6. Thus

$$2^{S_6} = \{\{\}, \{S_0\}, \ldots, \{S_5\}, \{S_0, S_1\}, \ldots, \{S_4, S_5\}, \ldots, \{S_0, \ldots, S_5\}\}.$$

2^{S_6} has 2^6 elements, since each element of S_6 can either be in a subset or not. First, there are $1 + 2^6 = 65$ opening braces used before expanding the S_i's, since one outermost opening brace is used, and for every subset of S_6, another opening brace is used.

Since each S_i are either in a subset or not, there is an equal number of subsets with S_i and of subsets without S_i, for all i. Thus, each S_i appears $2^5 = 32$ times. Since S_i has exactly 2^{i+1} opening braces, the total number of opening braces in 2^{S_6} is $65 + 32(1 + 2 + 4 + 8 + 16 + 32) = 65 + 32(63) = 65 + 2048 - 32 = 2081$.

SF-5 **Problem** Prove that for all positive integers n and m, F_{mn} is divisible by F_m. F_k is the kth Fibonacci number, defined by $F_1 = F_2 = 1$, $F_k = F_{k-1} + F_{k-2}$ for integers $k \geq 3$.

Solution

Proof. Extending the sequence, we see that $F_0 = 0$. We first show, by inducting on b, that $F_{a+b} = F_a F_{b+1} + F_{a-1} F_b$ for positive integers a and b.

It is easy to see that $F_{a+1} = F_a + F_{a-1} = F_a F_2 + F_{a-1} F_1$.

Say for some positive integer k, $F_{a+x} = F_a F_{x+1} + F_{a-1} F_x$ is true for all $x \leq k$. Then

$$\begin{aligned}
F_{a+x+1} &= F_{a+x} + F_{a+x-1} \\
&= F_a F_{x+1} + F_{a-1} F_x + F_a F_x + F_{a-1} F_{x-1} \\
&= F_a (F_x + F_{x+1}) + F_{a-1} (F_{x-1} + F_x) \\
&= F_a F_{x+2} + F_{a-1} F_{x+1}
\end{aligned}$$

Thus $F_{a+b} = F_a F_{b+1} + F_{a-1} F_b$ for all positive integers a and b.

Next, we show by induction on n that F_m divides F_{mn}.

It is clear that F_m divides F_m.

For the inductive step, say for some positive integer i, F_i divides F_{im}. We first substitute $a = m$ and $b = im$ in the previous identity. Then we have

$$F_{(i+1)m} = F_{im+m} = F_m F_{im+1} + F_{m-1} F_{im}.$$

Thus, $F_{(i+1)m}$ is also divisible by F_m. Thus for all positive integers m and n, F_{mn} is divisible by F_m. □

Mathematical Results

Angle Bisector Theorem

If D is a point on BC of $\triangle ABC$ such that AD is an angle bisector of the triangle, then $AC \cdot BD = AB \cdot CD$. *(p 4)*

Binomial Theorem

For integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

(p 4, 28)

Chinese Remainder Theorem

If r and s are coprime positive integers, then for all integers x and y, there exists an integer N such that $N \equiv x \pmod{r}$ and $N \equiv y \pmod{s}$. Furthermore there exists only one N that satisfies this and is between 1 and rs. *(p 6)*

De Moivre's Theorem

For any complex number $z = r \operatorname{cis}\theta$ and integer n, $z^n = r^n \operatorname{cis}(n\theta)$. *(p 5)*

Pythagorean Identity

For all θ, $\sin^2\theta + \cos^2\theta = 1$. *(p 22)*

Pythagorean Theorem

$\triangle ABC$ is a right triangle with right angle at B iff $AB^2 + BC^2 = AC^2$. *(p 9, 18, 19, 24)*

Stewart's Theorem

If D is a point on side BC in $\triangle ABC$, then

$$AC^2 \cdot BD + AB^2 \cdot DC = BC \cdot (BD \cdot DC + AD^2).$$

(p 4)

Vandermonde Identity

For non-negative integers m, n, and r,

$$\binom{n+m}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

(p 16)

Vieta's Formulas

Let $x_1, \cdots, x_n$ be the n roots of the polynomial $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0$. Also, let s_i be the sum of all possible products of i x_k terms where the k's are distinct. This means that $s_1 = x_1 + \cdots + x_n$, $s_2 = x_1x_2 + x_1x_3 + \cdots + x_1x_n + \cdots + x_{n-1}x_n$, and so on until $s_n = x_1x_2 \cdots x_n$. Then for all $i = 1, 2, \ldots, n$,

$$s_i = \frac{(-1)^i a_{n-i}}{a_n}.$$

(p 7)