# 11th St. Stephen's Lord of the Math

***Solution Booklet***

St. Stephen's High School

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# Contents

# Junior Division

## Junior Division Eliminations

**Easy**

JE-E1 **Problem** If $\dfrac{1}{x+5} = 4$, find the value of $\dfrac{1}{x+6}$.

**Answer** $\dfrac{4}{5}$

**Solution** $\dfrac{1}{(x+6)-1} = 4 \Rightarrow (x+6) - 1 = \dfrac{1}{4} \Rightarrow x + 6 = \dfrac{5}{4} \Rightarrow \dfrac{1}{x+6} = \dfrac{4}{5}$.

JE-E2 **Problem** Find the largest possible area of a quadrilateral with perimeter 8.

**Answer** 4 square units

**Solution** Let the four sides be $a$, $b$, $c$, $d$. From Bretschneider's formula, the area of the quadrilateral is

$$A = \sqrt{(4-a)(4-b)(4-c)(4-d) - abcd\cos^2\frac{\alpha+\beta}{2}},$$

where $\alpha$ and $\beta$ are opposite angles. Since $abcd\cos^2\dfrac{\alpha+\beta}{2} \geq 0$, then by AM-GM,

$$A^2 \leq (4-a)(4-b)(4-c)(4-d) \leq \left(\frac{(4-a)+(4-b)+(4-c)+(4-d)}{4}\right)^4 = \left(\frac{16-8}{4}\right)^4 = 16.$$

Therefore the maximum area is $\sqrt{16} = 4$, which can be attained when the quadrilateral is a square.

JE-E3 **Problem** How many seven-digit numbers have at most seven 7's?

**Answer** 9 000 000

**Solution** This is simply the number of seven-digit numbers, as a seven digit number cannot have more than seven 7's.

JE-E4 **Problem** What is the probability that the number of letters (written in English) of a positive integer less than or equal to 20 is prime?

**Answer** $\frac{9}{20}$

**Solution** From the table, the probability is $\frac{9}{20}$.

| Number | # of Letters | Number | # of Letters |
|---|---|---|---|
| one | $\boxed{3}$ | eleven | 6 |
| two | $\boxed{3}$ | twelve | 6 |
| three | $\boxed{5}$ | thirteen | 8 |
| four | 4 | fourteen | 8 |
| five | 4 | fifteen | $\boxed{7}$ |
| six | $\boxed{3}$ | sixteen | $\boxed{7}$ |
| seven | $\boxed{5}$ | seventeen | 9 |
| eight | $\boxed{5}$ | eighteen | 8 |
| nine | 4 | nineteen | 8 |
| ten | $\boxed{3}$ | twenty | 6 |

JE-E5 **Problem** If $A$ lies in the second quadrant and $3\tan A + 4 = 0$, what is the value of $2\cot A - 5\cos A + \sin A$?

**Answer** $\frac{23}{10}$

**Solution** We know that $\tan A = -\frac{4}{3}$. Since $A \in \text{QII}$, $\sin A > 0 > \cos A$. Thus,

$$\sin A = -\frac{\tan A}{\sqrt{1 + \tan^2 A}} = \frac{4}{5}$$

$$\cos A = -\frac{1}{\sqrt{1+\tan^2 A}} = -\frac{3}{5}$$
$$\cot A = \frac{1}{\tan A} = -\frac{3}{4}$$

Therefore, $2\cot A - 5\cos A + \sin A = 2\left(-\frac{3}{4}\right) - 5\left(-\frac{3}{5}\right) + \frac{4}{5} = \frac{23}{10}$.

JE-E6 **Problem** Find all real numbers $a$ such that $\Big|x + |x| + a\Big| + \Big|x - |x| - a\Big| = 2$ has exactly three real solutions in $x$.

**Answer** $-1$

**Solution** Note that if $x$ is a solution, then so is $-x$. Thus, in order to have an odd number of solutions, $x = -x \Rightarrow x = 0$ has to be a solution.

Substituting $x = 0$ results in $|a| + |-a| = 2 \Rightarrow 2|a| \Rightarrow |a| = \pm 1$.

If $a = 1$ and $x \geq 0$ then $|x + x + 1| + |x - x - 1| = |2x + 1| + 1 = 2 \Rightarrow x = 0$, so there is only one solution in $x$.

If $a = -1$ and $x \geq 0$, then $|x + x - 1| + |x - x + 1| = |2x - 1| + 1 = 2 \Rightarrow |2x - 1| = 1 \Rightarrow x = 0, 1$. Then the three solutions of the original equation are $x = 0, \pm 1$.

This means that $a = -1$ is the only possible value of $a$.

JE-E7 **Problem** An 80 m rope is suspended at its two ends from the tops of two 50 m-tall flagpoles. If the lowest point to which the midpoint of the rope can be pulled is 36 m from the ground, then what is the distance between the flagpoles?

**Answer** $12\sqrt{39}$ m

**Solution** In order for the rope to be at the lowest possible point, that point must be the middle of the rope. Thus, we are faced with solving a right-angled triangle with hypotenuse 40 m and one side of length $50 - 36 = 14$ m. By the Pythagorean theorem, the third side has length $\sqrt{40^2 - 14^2} = \sqrt{1404} = 6\sqrt{39}$, so the distance between the two flagpoles is $2 \times 6\sqrt{39} = 12\sqrt{39}$.

JE-E8 **Problem** Find all ordered pairs of integers $(a, b)$ satisfying $(a^3 + a^2 - 1) - (a - 1)b = 0$.

**Answer** $(0, 1)$ and $(2, 11)$

**Solution** Note that $a$ cannot be 1. Solving for $b$, we get $b = \dfrac{a^3 + a^2 - 1}{a - 1} = a^2 + 2a + 2 + \dfrac{1}{a - 1}$. Since $\dfrac{1}{a - 1}$ has to be an integer, then $a$ is either 0 or 2. If $a = 0$ then $b = 1$; if $a = 2$, then $b = 11$.

JE-E9 **Problem** Suppose a soccer game ends with a score of 7-5. How many possible half-time scores are there? (In soccer, the score is the number of goals each team scored.)

**Answer** 48

**Solution** There are 8 possible half-time scores for the first team, from 0 to 7, and 6 for the second team, from 0 to 5. Therefore there are $8 \times 6$ possible half-time scores.

JE-E10 **Problem** Let $x = \log_{17} \tan 1^\circ + \log_{17} \tan 2^\circ + \cdots + \log_{17} \tan 45^\circ$ and $y = \log_{17} \tan 46^\circ + \log_{17} \tan 47^\circ + \cdots + \log_{17} \tan 89^\circ$. What is $\dfrac{x}{y}$?

**Answer** $-1$

**Solution** Note that $\log_{17} \tan 45^\circ = \log_{17} 1 = 0$. Also,

$$\log_{17} \tan \theta + \log_{17} \tan(90^\circ - \theta) = \log_{17}(\tan \theta \tan(90^\circ - \theta)) = \log_{17} 1 = 0.$$

Then $\log_{17} \tan \theta = -\log_{17} \tan(90^\circ - \theta)$. This means that $x = -y$, leading to the desired answer of $-1$.

**Average**

JE-A1 **Problem** The angles $A$, $B$, and $C$ of $\triangle ABC$, where side $x$ is opposite angle $X$, are in arithmetic progression. If $2b^2 = 3c^2$, determine the angle $A$.

**Answer** $75^\circ$ or $\dfrac{5\pi}{12}$

**Solution** Since $A$, $B$, and $C$ are in arithmetic progression, $A+B+C = 3B = 180^\circ \Rightarrow B = 60^\circ$. Since $2b^2 = 3c^2$, by the sine law,

$$\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sin 60^\circ}{\sqrt{2}/2} = \frac{\sin B}{\sin C}.$$

Therefore, $C$ has measure $45^\circ$ or $135^\circ$. But $60^\circ + 135^\circ = 195^\circ > 180^\circ$, so $C$ has to be $45^\circ$. This means that $A = 180^\circ - 60^\circ - 45^\circ = 75^\circ$.

JE-A2 **Problem** What is the largest prime number that divides $(19! - 17!)$?

**Answer** 31

**Solution** $19! - 17! = 17!(18 \cdot 19 - 1) = 17!(341) = 17!(31)(11)$. Since the largest prime factor of $17!$ is 17, the largest prime factor of $19! - 17!$ is 31.

JE-A3 **Problem** What is the probability of getting a suit full house (three of a suit and two of another suit) when one draws a five-card hand from a standard deck of 52 playing cards?

**Answer** $\dfrac{429}{4165}$

**Solution** There are $\dbinom{13}{3}$ ways to choose three cards of one suit and $\dbinom{13}{2}$ ways to choose two cards of two suit, both taking into account only the rank of the card.

There are $4 \times 3$ ways to choose the suits of the three of a kind and the pair. Therefore, the probability is $\dfrac{12 \cdot \binom{13}{3} \cdot \binom{13}{2}}{\binom{52}{5}} = \dfrac{\cancel{12} \cdot \cancelto{11}{286} \cdot \cancelto{26}{78} \cdot \cancelto{12}{120}}{\cancelto{2}{52} \cdot \cancelto{17}{51} \cdot \cancelto{5}{50} \cdot 49 \cdot \cancelto{4}{48}} = \dfrac{11 \cdot 13 \cdot 3}{17 \cdot 5 \cdot 49} = \dfrac{429}{4165}$.

JE-A4 **Problem** Find $k$ in $k \sin 18^\circ \sin 42^\circ \sin 78^\circ = \cos 36^\circ$.

**Answer** 4

**Solution**

$$\begin{aligned}
\sin 18^\circ \sin 42^\circ \sin 78^\circ &= \sin 18^\circ \sin(60^\circ - 18^\circ) \sin(60^\circ + 18^\circ) \\
&= \sin 18^\circ \left((\sin 60^\circ \cos 18^\circ)^2 - (\cos 60^\circ \sin 18^\circ)^2\right) \\
&= \sin 18^\circ \left(\frac{3}{4}\cos^2 18^\circ - \frac{1}{4}\sin^2 18^\circ\right) \\
&= \frac{1}{4}\left(3\sin 18^\circ \cos^2 18^\circ - \sin^3 18^\circ\right) \\
&= \frac{1}{4}\left(3\sin 18^\circ(1 - \sin^2 18^\circ) - \sin^3 17^\circ\right) \\
&= \frac{1}{4}\left(3\sin 18^\circ - 4\sin^3 18^\circ\right) \\
&= \frac{1}{4}\sin 54^\circ = \frac{1}{4}\cos 36^\circ.
\end{aligned}$$

Thus, $k = 4$.

JE-A5 **Problem** For what values of $x$ does

$$2 + \cfrac{2 + \cfrac{2 + \frac{2}{x+1}}{2 + \frac{x+1}{2}}}{2 + \cfrac{2 + \frac{x+1}{2}}{2 + \frac{2}{x+1}}}$$

not have a real value?

**Answer** $-5, -2, -1, -7 \pm 2\sqrt{7}$

**Solution** The denominator of a fraction cannot be zero. Thus we have

$$x+1 \neq 0 \Rightarrow x \neq -1$$

$$2+\frac{2}{x+1} \neq 0 \Rightarrow 2x+4 \neq 0 \Rightarrow x \neq -2$$

$$2+\frac{2+\frac{x+1}{2}}{2+\frac{2}{x+1}} \neq 0 \Rightarrow x^2+14x+21 \neq 0 \Rightarrow x \neq -7 \pm 2\sqrt{7}$$

$$2+\frac{x+1}{2} \neq 0 \Rightarrow x+5 \neq 0 \Rightarrow x \neq -5$$

JE-A6 **Problem** Three congruent circles, centered at $(0,0)$, $(1,1)$, and $(2,1)$, have a common tangent. Find the radius of the circles.

**Answer** $\frac{\sqrt{5}}{10}$

**Solution** From an illustration it is clear that the common tangent is a common external tangent of the leftmost and rightmost circles, and a common internal tangent of the middle and rightmost circles. Then, the common tangent is parallel to the line passing through the centers of the leftmost and rightmost circles, and passes through the midpoint of the line segment connecting the middle and rightmost circles.

The slope of the line passing through the centers of the leftmost and rightmost circles is 0.5. Thus, let the line be $y = 0.5x + b$, for some $b$. Since the line passes through $(1.5, 1)$, we have $1 = 0.75 + b \Rightarrow b = 0.25$. Therefore, the equation of the common tangent is $2x - 4y + 1 = 0$.

The radius of the circles is the distance of any of the centers to this line is $\frac{|2(0)-4(0)+1|}{\sqrt{2^2+(-4)^2}} = \frac{1}{\sqrt{20}} = \frac{\sqrt{5}}{10}$.

JE-A7 **Problem** Jane rolls a fair, standard six-sided die repeatedly until she rolls a 1. She begins with a score of 1, and each time she rolls $x$, her score is divided by $x$. What is the expected value of her final score?

**Answer** $\dfrac{20}{91}$

**Solution** Let $k$ be the expected value of her final score. If she rolls a 1 the game ends; otherwise, she continues on, and since her score after the first roll will become $\dfrac{1}{x}$ (where $x \neq 1$ is the number rolled), then the expected value from the second roll onward is $\dfrac{k}{x}$. Since the probability of rolling a specific number from 1 to 6 is $\dfrac{1}{6}$, we have $k = \dfrac{1}{6} + \dfrac{k}{6}\left(\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \dfrac{1}{6}\right) \Rightarrow 6k = 1 + \dfrac{29}{20}k \Rightarrow k = \dfrac{20}{91}$.

## Difficult

JE-D1 **Problem** Find all real numbers $x, y, z$ satisfying the system of equations

$$\begin{cases} x + \lfloor y \rfloor + \{z\} = 1.1 \\ \lfloor x \rfloor + \{y\} + z = 2.2 \\ \{x\} + y + \lfloor z \rfloor = 3.3 \end{cases}$$

where $\lfloor k \rfloor$ is the greatest integer less than or equal to $k$ and $\{k\} = k - \lfloor k \rfloor$.

**Answer** $(x, y, z) = (0.1, 1.2, 2)$

**Solution** Using the fact that $\{k\} = k - \lfloor k \rfloor$, we can re-express the system as

$$\begin{cases} x + \lfloor y \rfloor + z - \lfloor z \rfloor = 1.1 & \ldots(1) \\ \lfloor x \rfloor + y - \lfloor y \rfloor + z = 2.2 & \ldots(2) \\ x - \lfloor x \rfloor + y + \lfloor z \rfloor = 3.3 & \ldots(3) \end{cases}$$

$(1) + (2) - (3)$ results in $2\lfloor x \rfloor + 2z - 2\lfloor z \rfloor = 0 \Rightarrow \lfloor x \rfloor + z - \lfloor z \rfloor = 0$. This implies that $z$ is an integer, or $z = \lfloor z \rfloor \Rightarrow \lfloor x \rfloor = 0 \Rightarrow 0 \le x < 1$.

Equation (1) then becomes $x + \lfloor y \rfloor = 1.1$. In other words, $0 \le 1.1 - \lfloor y \rfloor < 1$. The only integer $\lfloor y \rfloor$ that satisfies this is $\lfloor y \rfloor = 1$. This implies that $x = 1.1 - 1 = 0.1$, and $1 \le y < 2$.

Equation (2) now becomes $0 + y - 1 + \lfloor z \rfloor = y + z - 1 = 2.2 \Rightarrow y = 3.2 - z \Rightarrow 1 \le 3.2 - z < 2$. The only integer $z$ that satisfies this is $z = 2$. Then $y = 3.2 - z = 1.2$.

Therefore, the only solution is $(x, y, z) = (0.1, 1.2, 2)$.

JE-D2 **Problem** According to Benford's Law, for a set of numbers chosen in some specified procedure, the event that the leftmost digits of a number written in base 10 (removing leading zeros) is $n \in \mathbb{Z}^+$ has probability $\log_{10}\left(\frac{n+1}{n}\right)$. If the probability that the second digit from the left (removing leading zeros) of a set of numbers that satisfy Benford's Law is 2 can be written in the form $\log_{10} a - \log_{10} b$, where $a$ and $b$ are positive coprime integers, express $ab$ as a product of powers of prime numbers.

**Answer** $2^{18} \cdot 41 \cdot 43 \cdot 53 \cdot 73 \cdot 83$

**Solution** The probability that the second digit is 2 can be computed by adding the probabilities that the first two digits are $10x + 2$, from $x = 1$ to $x = 9$. Thus,

$$\begin{aligned} P(\text{second digit is 2}) &= \log_{10}\left(\frac{\cancel{13} \cdot \cancel{23} \cdot \cancel{33} \cdot 43 \cdot 53 \cdot \cancel{63} \cdot 73 \cdot 83 \cdot \cancel{93}}{\cancelto{4}{12} \cdot \cancelto{2}{22} \cdot 32 \cdot \cancelto{2}{42} \cdot \cancelto{4}{52} \cdot \cancelto{2}{62} \cdot \cancelto{8}{72} \cdot 82 \cdot \cancelto{4}{92}}\right) \\ &= \log_{10}\left(\frac{43 \cdot 53 \cdot 73 \cdot 83}{2^{18} \cdot 41}\right) \end{aligned}$$

Therefore, $a = 43 \cdot 53 \cdot 73 \cdot 83$ and $b = 2^{18} \cdot 41$. Their product, then, is $2^{18} \cdot 41 \cdot 43 \cdot 53 \cdot 73 \cdot 83$.

JE-D3 **Problem** Find the domain of definition of the function

$$f(x) = (|x-1| - \lfloor x \rfloor)^{-1/2} + \csc^{-1}\lfloor \sin x \rfloor + \sin^{-1}(1 + \sqrt{\lfloor |\sin x| \rfloor})$$

in the region $[-\pi, 2\pi]$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.

**Answer** $\left(-\pi, \frac{-\pi}{2}\right) \cup \left(\frac{-\pi}{2}, 0\right)$

**Solution** The function $f$ is defined if $f_1(x) = (|x-1| - \lfloor x \rfloor)^{-1/2}$, $f_2(x) = \csc^{-1}\lfloor \sin x \rfloor$ and $f_3(x) = \sin^{-1}(1 + \sqrt{\lfloor |x| \rfloor})$ are all defined.

Now, $f_1$ is defined when $|x-1| > \lfloor x \rfloor$. This is true for all $x < 1$. Therefore, $f_1$ is defined for $x \in [-\pi, 1)$.

We know that $\csc^{-1}\theta$ is defined when $|\theta| \geq 1$. Thus, $\csc^{-1}\lfloor \sin x \rfloor$ exists when either $\lfloor \sin x \rfloor \geq 1$ or $\lfloor \sin x \rfloor \leq -1$. $\lfloor \sin x \rfloor \geq 1 \Rightarrow x = \frac{\pi}{2}$; meanwhile, $\lfloor \sin x \rfloor \leq -1$ implies $x \in (\pi, 2\pi) \cup (-\pi, 0)$.

Finally, for any $x \in \mathbb{R}$, $|\sin x| \leq 1$. Now, when $|\sin x| < 1$, then $\lfloor |\sin x| \rfloor = 0 \Rightarrow \sqrt{\lfloor |\sin x| \rfloor} = 0 \Rightarrow f_3 = \sin^{-1} 1 = \frac{\pi}{2}$. On the other hand, when $|\sin x| = 1$, then $\sin^{-1}(1 + \sqrt{\lfloor |\sin x| \rfloor}) = \sin^{-1} 2$, which does not exist. Therefore, $f_3$ is defined for $x \in [-\pi, 2\pi] \setminus \left\{\pm\frac{\pi}{2}, \frac{3\pi}{2}\right\}$.

We get the intersection of the three sets to get the domain of definition:

$$\begin{aligned}
&[-\pi, 1) \cap \left(\left\{\frac{\pi}{2}\right\} \cup (-\pi, 0) \cup (\pi, 2\pi)\right) \cap \left([-\pi, 2\pi] \setminus \left\{\pm\frac{\pi}{2}, \frac{3\pi}{2}\right\}\right) \\
&= (-\pi, 0) \cap \left([-\pi, 2\pi] \setminus \left\{\pm\frac{\pi}{2}, \frac{3\pi}{2}\right\}\right) \\
&= (-\pi, 0) \setminus \left\{-\frac{\pi}{2}\right\}
\end{aligned}$$

JE-D4 **Problem** If

$$\log_2 \log_{1/2} \log_2 x = \log_5 \log_{1/5} \log_5 y = \log_7 \log_{1/7} \log_7 z,$$

arrange $x, y, z$ in increasing order.

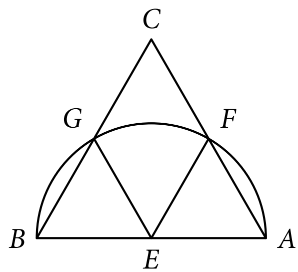
**Answer** $z < y < x$

**Solution** Without loss of generality set them all to 0. Then $\log_{1/2}\log_2 x = 1 \Rightarrow \log_2 x = \frac{1}{2} \Rightarrow x = 2^{1/2}$, and similarly, $y = 5^{1/5}$ and $z = 7^{1/7}$. But $x^{10} = 32 > 25 = y^{10}$ and $y^{35} = 78\,125 > 16\,807 = z^{35}$. Therefore, $z < y < x$.

JE-D5 **Problem** Triangle $ABC$ is an equilateral triangle. A point $D$ is randomly chosen in the triangle. What is the probability that triangle $ABD$ is obtuse?

**Answer** $\dfrac{9+\sqrt{3}\pi}{18}$

**Solution** Triangle $ABD$ is obtuse if and only if it is in the interior of the semicircle shown in the figure.




Thus, the desired probability is the area of the intersection of the semicircle and $\triangle ABC$, divided by the area of $\triangle ABC$. Let $AE = r$. Then the area of $\triangle ABC$ is $\frac{\sqrt{3}}{4}(4r^2) = \sqrt{3}r^2$.

The area of the intersection is equal to the area of equilateral triangles $CGE$, $AFE$, and the minor sector $GF$. The total area of the equilateral triangles is one-half of $[\triangle ABC]$, and the area of the sector is $\frac{\pi}{6}\cdot\frac{1}{4} = \frac{\pi}{24}$. The probability is then $\frac{1}{2} + \frac{\pi/24}{\sqrt{3}/4} = \frac{9}{18} + \frac{\sqrt{3}\pi}{18} = \frac{9+\sqrt{3}\pi}{18}$.

## Junior Division Team Finals

JTF-1 **Problem** Find all integer values of $x$ such that $x^2 + 19x + 88$ is a perfect square.

**Answer** $-7, -8, -11, -12$

**Solution** Let $m^2 = x^2 + 19x + 88$. Then the equation $x^2 + 19x + 88 - m^2 = 0$ must have integral solutions in $x$, that is, the determinant $19^2 - 4(88 - m^2)$ must be a perfect square, say $n^2$. The equation $19^2 - 4(88 - m^2) = n^2$ simplifies to $(n - 2m)(n + 2m) = 9$. Consider the following cases:

Case 1. $n - 2m = 3$ and $n + 2m = 3$. Then $n = 3$ and $m = 0$.

Case 2. $n - 2m = -3$ and $n + 2m = -3$. Then $n = -3$ and $m = 0$.

Case 3. $n - 2m = 9$ and $n + 2m = 1$. Then $n = 5$ and $m = -2$.

Case 4. $n - 2m = 1$ and $n + 2m = 9$. Then $n = 5$ and $m = 2$.

Case 5. $n - 2m = -9$ and $n + 2m = -1$. Then $n = -5$ and $m = 2$.

Case 6. $n - 2m = -1$ and $n + 2m = -9$. Then $n = -5$ and $m = -2$.

Either $m^2 = 0$ or $m^2 = 4$. If $m^2 = 0$, $x^2 + 19x + 88 = 0$ simplifies to $x = -8$ or $-11$. Meanwhile, if $m^2 = 4$, $x^2 + 19x + 84 = 0$ simplifies to $x = -12$ or $-7$. Therefore, the possible values of $x$ are $-7, -8, -11, -12$.

JTF-2 **Problem** Four nickels and six dimes are tossed, and the total number $N$ of heads is observed. If $N = 4$, what is the probability that exactly two nickels showed up heads?

**Answer** $\frac{3}{7}$

**Solution** The probability is $\dfrac{\binom{4}{2}\frac{1}{2^4}\binom{6}{2}\frac{1}{2^6}}{\binom{10}{4}\frac{1}{2^{10}}} = \dfrac{3}{7}$.

JTF-3 **Problem** What is the value of $\tan 101° + \tan 124° + \tan 101° \tan 124°$?

**Answer** 1

**Solution** We know that $\tan 225^\circ = 1$. Thus,

$$
\begin{aligned}
1 = \tan 225^\circ &= \frac{\tan 101^\circ + \tan 124^\circ}{1 - \tan 101^\circ \tan 124^\circ} \\
1 - \tan 100^\circ \tan 125^\circ &= \tan 100^\circ + \tan 125^\circ \\
\tan 100^\circ + \tan 125^\circ + \tan 100^\circ \tan 125^\circ &= 1.
\end{aligned}
$$

JTF-4 **Problem** Let $\omega$ be a non-real cube root of unity. Express $(1+\omega)^{19}$ as a linear function of $\omega$.

**Answer** $1 + \omega$

**Solution** Since $\omega^3 = 1$ and $\omega \neq 1$, we have

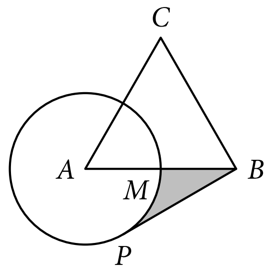
$$\omega^3 - 1 = 0 \Rightarrow (\omega - 1)(\omega^2 + \omega + 1) = 0 \Rightarrow 1 + \omega = -\omega^2.$$

Therefore, $(1+\omega)^{19} = -\omega^{38} = -\omega^{36} \cdot \omega^2 = -\omega^2 = 1 + \omega$.

JTF-5 **Problem** Each side of equilateral $\triangle ABC$ has length 2 units. A unit circle centered at $A$ cuts $AB$ at $M$. A tangent to the circle from $B$ and lying outside the triangle meets the circle at $P$. Find the area of the region bounded by $BP$, $BM$, and minor arc $MP$.

**Answer** $\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}$

**Solution** Refer to the following figure:




The area of the desired region is the difference between the area of $\triangle APB$, which is right-angled since $PB$ is a tangent of circle $A$, and the area of the minor sector $AMP$. Now, $PB = \sqrt{AB^2 - AP^2} = \sqrt{4-1} = \sqrt{3}$, so $[\triangle APB] = \frac{1}{2} \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{2}$. Also, the area of sector $AMP$ is $\frac{1}{2} \cdot 1^2 \cdot \frac{\pi}{3} = \frac{\pi}{6}$. Therefore, the area of the desired region is $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$.

JTF-6 **Problem** How many kilograms of water must be evaporated from 50 kilograms of a 3% salt solution so that the remaining solution will be 5% salt?

**Answer** 20 kilograms

**Solution** The salt in the solution is 1.5 kg. This is 5% of the new solution, which will be $\frac{1.5\,\text{kg}}{.05} = 30\,\text{kg}$. Thus 20 kg shall be evaporated.

JTF-7 **Problem** What is the value of $\frac{1}{2\sin 10^\circ} - 2\sin 70^\circ$?

**Answer** 1

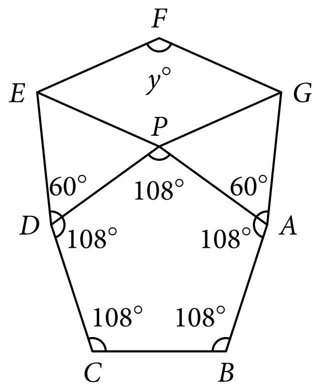
**Solution**

$$\begin{aligned}\frac{1}{2\sin 10^\circ} - 2\sin 70^\circ &= \frac{1}{2\sin 10^\circ} - 2\cos 20^\circ = \frac{1 - 4\cos 20^\circ \sin 10^\circ}{2\sin 10^\circ} \\ &= \frac{1 - 2\sin 30^\circ + 2\sin 10^\circ}{2\sin 10^\circ} = \frac{2\sin 10^\circ}{2\sin 10^\circ} = 1\end{aligned}$$

JTF-8 **Problem** A convex equilateral heptagon has angles that measure 168°, 108°, 108°, 168°, $x°$, $y°$, and $z°$, in clockwise order. What is $y$?

**Answer** 132

**Solution** Let $P$ be a point in the heptagon such that $ABCDP$ is a regular pentagon. Refer to the following figure:




Since $\angle DEP = \angle GAP = 60°$, it follows that $\triangle DEP$ and $\triangle GAP$ are equilateral. Then $EP = PG = EF = FG$, that is, $EFGP$ is a rhombus. This means that $y° = \angle EPG = 360° - 108° - 2(60°) = 132°$.

JTF-9 **Problem** There are 20 different amino acids in the human body, three of which have a positive charge (+1), 2 have a negative charge (−1) and the rest have no charge (0). A protein is a ordered sequence of amino acids whose charge is equal to the sum of the charges of its amino acids. How many proteins with negative charge are there that are four amino acids long?

**Answer** 33 352

**Solution** We tabulate the combinations as follows:

| Charges | Combinations | Rearrangements | Product |
|---|---|---|---|
| `----` | $2 \cdot 2 \cdot 2 \cdot 2 = 16$ | 1 | 16 |
| `---0` | $2 \cdot 2 \cdot 2 \cdot 15 = 120$ | 4 | 480 |
| `---+` | $2 \cdot 2 \cdot 2 \cdot 3 = 24$ | 4 | 96 |
| `--00` | $2 \cdot 2 \cdot 15 \cdot 15 = 900$ | 6 | 5400 |
| `--0+` | $2 \cdot 2 \cdot 15 \cdot 3 = 180$ | 12 | 2160 |
| `-000` | $2 \cdot 15 \cdot 15 \cdot 15 = 6750$ | 4 | 27 000 |

The sum, then, is $16 + 480 + 96 + 5400 + 2160 + 27\,000 = 33\,352$.

JTF-10 **Problem** Solve the equation $\sec x \cos 5x + 1 = 0$, $0 < x < 2\pi$.

**Answer** $\left\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{6}, \frac{7\pi}{4}, \frac{11\pi}{6}\right\}$

**Solution** The equation is equivalent to $\cos 5x = -\cos x = \cos(\pi - x)$, $\cos x \neq 0$. Thus $5x = 2n\pi \pm (\pi - x)$, where $n$ is an integer.

If $5x = 2n\pi + \pi - x$, then $x = \frac{(2n+1)\pi}{6}$; if $5x = 2n\pi - \pi + x$, then $x = \frac{(2n-1)\pi}{4}$. The values of $x$ in the interval $(0, 2\pi)$ are $\left\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{6}, \frac{7\pi}{4}, \frac{11\pi}{6}\right\}$.

JTF-11 **Problem** A *space diagonal* of a polyhedron is a line segment connecting two vertices of the polyhedron and is in the interior of the polyhedron. A dodecahedron is a polyhedron consisting of twelve pentagons such that three pentagons meet at a vertex. How many space diagonals does it have?

**Answer** 100

**Solution** The 12 pentagons give 60 sides and 60 vertices. Since three faces meet at a vertex, the dodecahedron has $60 \div 3 = 20$ vertices. Furthermore, two faces meet at an edge, so the dodecahedron has $60 \div 2 = 30$ edges. Among the 20 vertices, there are $\binom{20}{2} = 190$ line segments that can be formed. For each pentagonal face, there are 5 diagonals, so the dodecahedron has $5 \times 12 = 60$ face diagonals. Any line segment that is not an edge or a face diagonal has to be a space diagonal; thus, there are 190 − 60 − 30 = 100 space diagonals.

JTF-12 **Problem** What is the value of the nonnegative integer $n$ that satisfies $n! = 112\,296^2 - 79\,896^2$?

**Answer** 13

**Solution**

$$\begin{aligned} 112\,296^2 - 79\,896^2 &= (112\,296 - 79\,896)(112\,296 + 79\,896) \\ &= (32\,400)(192\,192) = 18^2 10^2 \cdot 192 \cdot 1001 \\ &= 2 \cdot 9 \cdot 3 \cdot 6 \cdot 2 \cdot 5 \cdot 10 \cdot 2 \cdot 8 \cdot 12 \cdot 7 \cdot 11 \cdot 13 \\ &= 2 \cdot 3 \cdot 2 \cdot 2 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \\ &= 13! \end{aligned}$$

JTF-13 **Problem** Solve for real numbers $x$ and $y$ in $4x^3 + 3x^2y + y^3 = 8$, $2x^3 - 2x^2y + xy^2 = 1$.

**Answer** $(1, 1), \left(\frac{\sqrt[3]{25}}{5}, \frac{3\sqrt[3]{25}}{5}\right), \left(\frac{\sqrt[3]{100}}{10}, \frac{2\sqrt[3]{100}}{5}\right)$

**Solution** Note that neither $x = 0$ nor $y = 0$ are solutions. Let $y = mx$. Then we have $x^3(4 + 3m + m^3) = 8$ and $x^3(2 - 2m + m^2) = 1$. Eliminating $x^3$ from this system results in $4 + 3m + m^2 = 8(2 - 2m + m^2) \Rightarrow m^3 - 8m^2 + 19m - 12 = 0 \Rightarrow (m - 1)(m - 3)(m - 4) = 0 \Rightarrow m = 1, 3, 4$.

If $m = 1$, then $x^3(2 - 2m + m^2) = x^3 = 1 \Rightarrow x = 1 \Rightarrow y = 1$.

If $m = 3$, then $x^3(2 - 2m + m^2) = 5x^3 = 1 \Rightarrow x = \frac{\sqrt[3]{25}}{5} \Rightarrow y = \frac{3\sqrt[3]{25}}{5}$.

If $m = 4$, then $x^3(2 - 2m + m^2) = 10x^3 = 1 \Rightarrow x = \frac{\sqrt[3]{100}}{10} \Rightarrow y = \frac{2\sqrt[3]{100}}{5}$.

JTF-14 **Problem** The sum of the squares of the positive factors of a positive integer is 1300, and the sum of the positive factors of the square of the integer is 2821. Find the square of the sum of the positive factors of the integer.

**Answer** 5184

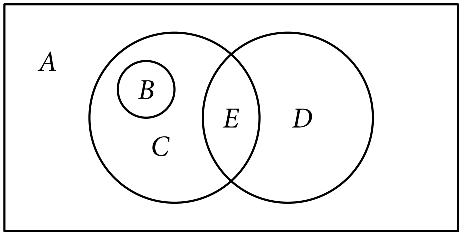
**Solution** Note that $1300 = 5 \times 10 \times 26 = (2^0 + 2^2)(3^0 + 3^2)(5^0 + 5^2)$ and $2821 = 7 \times 13 \times 31 = (2^0 + 2^1 + 2^2)(3^0 + 3^1 + 3^2)(5^0 + 5^1 + 5^2)$. Thus the integer is $2 \times 3 \times 5 = 30$. The sum of the positive factors of 30 is $(2^0 + 2^1)(3^0 + 3^1)(5^0 + 5^1) = 3 \times 4 \times 6 = 72$, and the square of that is $72^2 = 5184$.

JTF-15 **Problem** Two chimpanzees are playing a variation of tic-tac-toe. Instead of stopping when someone has formed a line, they continue and fill up the whole $3 \times 3$ grid. A chimpanzee wins if and only if it is able to form a line and the other is unable to. The game ends in a draw if either both or none of them form a line. Assuming these two chimpanzees have an equal chance of picking any of the empty squares available, and that a chimpanzee won, what is the probability that the first chimpanzee won?

**Answer** $\dfrac{31}{37}$

**Solution** Denote the first player's symbol as `O` and the second player's symbol as `X`. Consider the following sets of outcomes. Note that the second player cannot form two lines since the second player only has four turns.

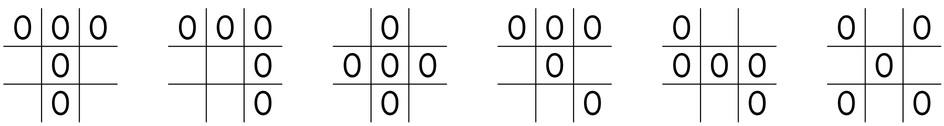
| | **set of outcomes where...** |
|---|---|
| $A$ | nobody forms a line |
| $B$ | the first player forms two lines |
| $C$ | the first player forms exactly one line and the second player does not form a line |
| $D$ | the second player forms exactly one line and the first player does not form a line |
| $E$ | both players form one line |




The desired answer is $\dfrac{|B| + |C|}{|B| + |C| + |D|}$.

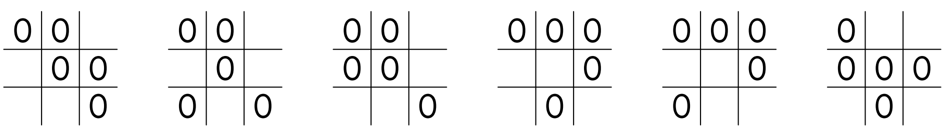
First, we find $|D|$. The only way for the second player to win is if it forms a diagonal line. There are two ways to form a diagonal, and there are six remaining spots to select the last X . All the other spots will be filled with O 's. This means that $|D| = 2 \times 6 = 12$.

If the first player completes two lines in six different ways (not yet counting reflections or rotations), as shown.



With rotations or reflections, there are a total of $|B| = 4 + 4 + 1 + 8 + 4 + 1 = 22$ ways to place the O 's.

Next, consider the case where the first player makes exactly one line and the second player does not form a line. The following are the ways (without accounting yet for reflections or rotations) for the O 's:



Taking rotations and reflections into account, there are a total of $|C| = 4+8+4+8+8+8 = 40$ configurations.

The probability, then, is $\frac{|B| + |C|}{|B| + |C| + |D|} = \frac{22 + 40}{22 + 40 + 12} = \frac{62}{74} = \frac{31}{37}$.

## Junior Division Individual Finals

JIF-1 **Problem** Find the maximum value of $4x - x^4$, where $x$ is a real number.

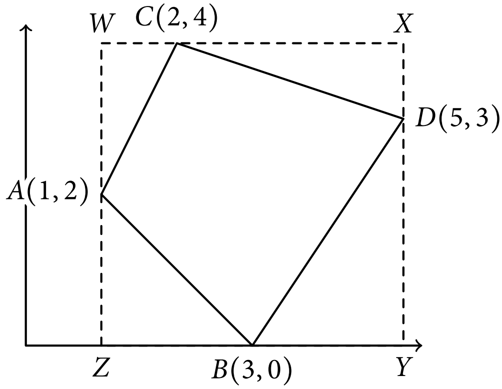
**Answer** 3

**Solution** Consider $x^4 - 4x + 3 = (x^4 - 2x^2 + 1) + (2x^2 - 4x + 2) = (x^2 - 1)^2 + 2(x - 1)^2 \geq 0$. This implies $4x - x^4 \leq 3$.

JIF-2 **Problem** Find the area of the convex quadrilateral whose vertices are $A(1, 2)$, $B(3, 0)$, $C(2, 4)$, and $D(5, 3)$.

**Answer** $\frac{17}{2}$

**Solution**




From the figure, $[ABCD] = [WXYZ] - [WAC] - [CXD] - [DBY] - [BZA] = 16 - \frac{2}{2} - \frac{3}{2} - \frac{6}{2} - \frac{4}{2} = 16 - \frac{15}{2} = \frac{17}{2}$.

JIF-3 **Problem** Find all ordered triples of positive integers $(a, b, c)$, $a \leq b \leq c$, that satisfy $a+b+c = 100$ and $a^2 + b^2 + c^2 = 1917$.

**Answer** $\{\}$

**Solution** The first equation implies $c = 100 - a - b$. Substituting into the second, we have

$$\begin{aligned} a^2 + b^2 + (100 - a - b)^2 &= 1917 \\ a^2 + b^2 + 10\,000 + a^2 + b^2 - 200a - 200b + 2ab &= 1917 \\ 2(a^2 + b^2 + ab - 100a - 100b + 5000) &= 1917 \end{aligned}$$

Since the variables are all integers, the left-hand side is always even and thus cannot be equal to 1917, an odd number.

JIF-4 **Problem** $A$ and $B$ are two events such that $P(A^{\mathrm{C}}) = 0.3$, $P(B) = 0.4$, and $P(A \cap B^{\mathrm{C}}) = 0.5$. What is the value of $P\left(B \mid A \cup B^{\mathrm{C}}\right)$?

**Answer** 0.25

**Solution**

$$\begin{aligned} P\left(B \mid A \cup B^{\mathrm{C}}\right) &= \frac{P(B \cap (A \cup B^{\mathrm{C}}))}{P(A \cup B^{\mathrm{C}})} = \frac{P(B \cap A)}{P(A) + P(B^{\mathrm{C}}) - P(A \cap B^{\mathrm{C}})} \\ &= \frac{P(A) - P(A \cap B^{\mathrm{C}})}{2 - P(A^{\mathrm{C}}) - P(B) - P(A \cap B^{\mathrm{C}})} \\ &= \frac{1 - P(A^{\mathrm{C}}) - P(A \cap B^{\mathrm{C}})}{2 - P(A^{\mathrm{C}}) - P(B) - P(A \cap B^{\mathrm{C}})} \\ &= \frac{1 - 0.3 - 0.5}{2 - 0.3 - 0.4 - 0.5} = \frac{0.2}{0.8} = 0.25 \end{aligned}$$

JIF-5 **Problem** Which point on the circle $(x - 11)^2 + (y - 13)^2 = 116$ is farthest from the point $(41, 25)$?

**Answer** $(1, 9)$

**Solution** The farthest point lies on the intersection of the line passing through the center of the circle $(11, 13)$ and $(41, 25)$. Note that the distance from the center to $(41, 25)$ is $6\sqrt{29}$

while the radius of the circle is $2\sqrt{29}$. Since the difference between the $x$ and $y$ coordinates are 30 and 12, respectively. Then the farther intersection is $\left(11 - \frac{30}{3}, 13 - \frac{12}{3}\right) = (1, 9)$.

JIF-6 **Problem** Simplify $\sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}$.

**Answer** $\cos 2x$

**Solution** Since $\sin^2 x$ and $\cos^2 x$ are both in the interval $[0, 1]$, the expression simplifies to

$$\begin{aligned}\sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x} &= \sqrt{\sin^4 x - 4\sin^2 x + 4} - \sqrt{\cos^4 x - 4\cos^2 x + 4} \\ &= (2 - \sin^2 x) - (2 - \cos^2 x) \\ &= \cos^2 x - \sin^2 x = \cos 2x\end{aligned}$$

JIF-7 **Problem** Let $n = 2^4 3^5 4^6 6^7$. How many positive integer factors does $n$ have?

**Answer** 312

**Solution** We have $n = 2^{23} 3^{12}$. Thus it has $(23 + 1)(12 + 1) = 24 \cdot 13 = 312$ factors.

JIF-8 **Problem** Find the sum of all the coefficients of the terms, excluding the constant, in the expansion of $\left(1 + x + \frac{2}{x}\right)^6$.

**Answer** 3515

**Solution** The constant term is

$$\begin{aligned}&\frac{6!}{6!0!0!}2^0 + \frac{6!}{4!1!1!}2^1 + \frac{6!}{2!2!2!}2^2 + \frac{6!}{0!3!3!}2^3 \\ &= 1 + 6 \cdot 5 \cdot 2 + 6 \cdot 5 \cdot 3 \cdot 4 + 5 \cdot 4 \cdot 8 \\ &= 1 + 60 + 360 + 160 = 581.\end{aligned}$$

The sum of all coefficients is $\left(1 + 1 + \frac{2}{1}\right)^6 = 4^6 = 4096$. Therefore, the sum of all the coefficients, excluding the constant, is 4096 − 581 = 3515.

JIF-9 **Problem** If the quadratic equation

$$4^{\sec^2\alpha}x^2 + 2x + \left(\beta^2 - \beta + \frac{1}{2}\right) = 0$$

has real roots in $x$, then what are the possible values of $\cos\alpha + \cos^{-1}\beta$?

**Answer** $\dfrac{\pi}{3} \pm 1$

**Solution** The discriminant of the equation is $4 - 4\cdot 4^{\sec^2\alpha}\left(\beta^2 - \beta + \frac{1}{2}\right)$. If the discriminant is non-negative, then we have $4^{\sec^2\alpha}\left(\beta^2 - \beta + \frac{1}{2}\right) \le 1$. However, $\sec^2\alpha \ge 1 \Rightarrow 4^{\sec^2\alpha} \ge 4$ and $\beta^2 - \beta + \frac{1}{2} = \left(\beta - \frac{1}{2}\right)^2 + \frac{1}{4} \ge \frac{1}{4}$. This means that $4^{\sec^2\alpha}\left(\beta^2 - \beta + \frac{1}{2}\right) \ge 1$, implying $\sec^2\alpha = 1$ and $\left(\beta - \frac{1}{2}\right)^2 = 0$.

Then $\cos\alpha = \pm 1$ and $\cos^{-1}\beta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$, so $\cos\alpha + \cos^{-1}\beta = \frac{\pi}{3} \pm 1$.

JIF-10 **Problem** If $z_1$, $z_2$, and $z_3$ are three complex numbers such that

$$|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1,$$

find $|z_1 + z_2 + z_3|$.

**Answer** 1

**Solution** We have $|z_1| = 1 \Rightarrow |z_1^2| = 1 \Rightarrow z_1\overline{z_1} = 1 \Rightarrow \frac{1}{z_1} = \overline{z_1}$. Similarly $\frac{1}{z_2} = \overline{z_2}$ and $\frac{1}{z_3} = \overline{z_3}$. Thus $\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = |\overline{z_1} + \overline{z_2} + \overline{z_3}| = |\overline{z_1 + z_2 + z_3}| = |z_1 + z_2 + z_3| = 1$.

JIF-11 **Problem** What is the length of the longest median of a triangle whose sides have length 4, 7, and 9?

**Answer** $\sqrt{61}$

**Solution** The longest median is the one to the shortest side. Using Apollonius's theorem, we have $7^2 + 9^2 = 2(2^2 + x^2)$, where $x$ is the length of the median. The equation simplifies to $x^2 = 61 \Rightarrow x = \sqrt{61}$.

JIF-12 **Problem** If $a$ and $b$ are positive integers that leave respective remainders of 6 and 1 when divided by 14, and $x$ is a positive integer solution to $x^2 - 2ax + b = 0$, find the remainder when $x$ is divided by 14.

**Answer** 13

**Solution** Let $a = 14m + 6$, $b = 14n + 1$, $x = 14p + k$ for some integers $m, n, p, k$ and $0 \le k \le 13$. Then $x^2 - 2ax + b = (14p + k)^2 - 2(14m + 6)(14p + k) + (14n + 1) = (196p^2 + 28pk + k^2) - 2(196mp + 84p + 14mk + 6k) + (14n + 1) = 14q + k^2 - 12k + 1$, for some integer $q$. Now $k^2 - 12k + 1$ and $k^2 - 12k - 13$ leave the same remainder when divided by 14, and $k^2 - 12k - 13 = (k + 1)(k - 13) = 0$. Then $k = 13$ so that $x^2 - 2ax + b$ is 0 (a multiple of 14). Thus, the remainder when $x = 14p + k = 14p + 13$ is divided by 14 is 13.

JIF-13 **Problem** For any positive integer $n$, let $d(n)$ be the sum of its digits. Find $n$ if $n + d(n) = 1\,000\,000\,000$.

**Answer** 999 999 932

**Solution** Since $d(n) > 0$, we have $n \le 999\,999\,999$, so $n$ has at most 9 digits. This means that $d(n) \le 9 \times 9 = 81$, or $n \ge 999\,999\,919 \Rightarrow n = 999\,999\,900 + 10x + y$, for some digits $x$ and $y$. In terms of $x$ and $y$, $d(n) = 63 + x + y$. We now have $n + d(n) = (999\,999\,900 + 10x + y) + (63 + x + y) = 1\,000\,000\,000$, or $11x + 2y = 37$. The only digits that satisfy this are $x = 3$, $y = 2$, so we have $n = 999\,999\,932$.

JIF-14 **Problem** If the answer to this question is a real number $x$, find the value of

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\frac{k!x^{-k}}{i!j!(k-i-j)!}.$$

**Answer** 4

**Solution** We know that $\sum_{j=0}^{k}\sum_{i=0}^{k-j}\frac{k!}{i!j!(k-i-j)!}=3^k$, as it is the number of ways to distribute $k$ items into three groups (any of which can be empty).

Thus, the given equation is equivalent to $x=\sum_{k=0}^{\infty}\left(\frac{3}{x}\right)^k=\frac{1}{1-\frac{3}{x}}$. The only real number $x$ that satisfies this is $x=4$.

JIF-15 **Problem** Let $D$ be a point inside acute $\triangle ABC$ such that $\angle ADB=\angle ACB+90^\circ$ and $AC\cdot BD=AD\cdot BC$. Find $\frac{AB\cdot CD}{AC\cdot BD}$.

**Answer** $\sqrt{2}$

**Solution** Let $E$ be the point on the plane such that $\overline{DE}$ is perpendicular to $\overline{DB}$ and intersects $\overline{AB}$, and $DE$ and $DB$ have the same length.

$\angle ADB=\angle ACB+90^\circ$ implies $\angle ADE=\angle ACB$. Also, since $DE=DB$, the given condition $AC\cdot BD=AD\cdot BC$ implies $\frac{AD}{DE}=\frac{BD}{BC}=\frac{DE}{CB}$. By SAS, $\triangle ADE\approx\triangle ACB$, which means $\frac{AE}{AD}=\frac{AB}{AC}$.

Now, $\angle CAB=\angle DAE\Rightarrow\angle CAD=\angle BAE$. Together with $\frac{AE}{AD}=\frac{AB}{AC}$, this implies $\triangle AEB\sim\triangle ADC$. Therefore, since $\triangle DEB$ is an isosceles right triangle,

$$\frac{AB}{AC}=\frac{BE}{CD}=\frac{\sqrt{2}BD}{CD}\Rightarrow\frac{AB\cdot CD}{AC\cdot BD}=\sqrt{2}.$$

# Senior Division

## Senior Division Eliminations

**Easy**

SE-E1 **Problem** How many 4-digit numbers with nonzero digits are divisible by 4 but not by 8?

**Answer** 729

**Solution** For any three-digit number $ABC$ with nonzero digits, exactly one of $ABC2$, $ABC4$, $ABC6$, and $ABC8$ is divisible by 4 but not by 8. Therefore, there are $9^3 = 729$ of them.

SE-E2 **Problem** What is the probability that a positive integer less than 100 is prime?

**Answer** $\frac{25}{99}$

**Solution** There are 99 positive integers less than 100, and there are 25 prime numbers from 1 to 99: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97. Thus the probability is $\frac{25}{99}$.

SE-E3 **Problem** A car ran five full laps on a circular track whose radius is 20 km, for 1 hour at a uniform speed. Find the average velocity of the car.

**Answer** 0 kph

**Solution** Since after five laps the car returned to its original position, the displacement is zero and so is the velocity.

SE-E4 **Problem** How many lattice points $(x, y)$ are there such that both $|x|$ and $|y|$ are less than 10, $x$ is divisible by $y$, and $y$ is divisible by $x$?

**Answer** 36

**Solution** The desired points are of the form $(x, x)$ or $(x, -x)$, where $x \in \{-9, -8, \ldots, 8, 9\} \setminus \{0\}$. Thus, we have $18 \times 2 = 36$ points.

SE-E5 **Problem** Solve for real $x$: $\tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) = \tan^{-1}(-7)$

**Answer** no solution

**Solution** Take the tangent of both sides of the equation:

$$\begin{aligned} \frac{\frac{x+1}{x-1} + \frac{x-1}{x}}{1 - \frac{x+1}{x-1} \cdot \frac{x-1}{x}} &= -7 \\ \frac{2x^2 - x + 1}{1 - x} &= -7 \\ 2x^2 - x + 1 &= -7(1 - x) \\ 2x^2 - 8x + 8 &= 0 \\ 2(x-2)^2 &= 0 \Rightarrow x = 2 \end{aligned}$$

However, the left-hand side when evaluated at $x = 2$ gives $\tan^{-1} 3 + \tan^{-1} \frac{1}{2}$, which is positive. On the other hand, $\tan^{-1}(-7) < 0$. Therefore $x = 2$ is a degenerate solution, and the given equation has no solutions.

SE-E6 **Problem** Find all possible values of $z \in \mathbb{R}$ that satisfy the inequality $\log_{\sqrt{3}} \frac{|z|^2 - |z| + 1}{2 + |z|} < 2$.

**Answer** $z \in (-5, 5)$

**Solution** The inequality is equivalent to $0 < \frac{|z|^2 - |z| + 1}{2 + |z|} < 3$. The left-hand side is always true, so we focus on the right-hand side. Also, $2 + |z| > 0$, so the inequality becomes

$$|z|^2 - |z| + 1 < 6 + 3|z|$$

$$
\begin{aligned}
|z|^2 - 4|z| - 5 &< 0 \\
-1 < |z| &\wedge |z| < 5 \\
|z| &< 5
\end{aligned}
$$

SE-E7 **Problem** Suppose $A_1A_2A_3 \cdots A_n$ is an $n$-sided regular polygon such that $\dfrac{1}{A_1A_2} = \dfrac{1}{A_1A_3} + \dfrac{1}{A_1A_4}$. Find $n$.

**Answer** 7

**Solution** Let $O$ be the circumcenter of the regular polygon. It is clear that $\angle A_1OA_2 = \dfrac{360^\circ}{n}$. Let $OA_1 = r$. Using the cosine law on $\triangle A_1OA_2$, we have $(A_1A_2)^2 = 2r^2\left(1 - \cos\dfrac{2\pi}{n}\right)$, or $A_1A_2 = 2r\sin\dfrac{\pi}{n}$. Similarly, using the cosine law on $\triangle A_1OA_3$ and $\triangle A_1OA_4$ results in $A_1A_3 = 2r\sin\dfrac{2\pi}{n}$ and $A_1A_4 = 2r\sin\dfrac{3\pi}{n}$, respectively.

Substituting into the given equation and removing common factors, we have

$$
\begin{aligned}
\frac{1}{\sin\frac{\pi}{n}} &= \frac{1}{\sin\frac{2\pi}{n}} + \frac{1}{\sin\frac{3\pi}{n}} \\
\frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}} &= \frac{1}{\sin\frac{2\pi}{n}} \\
\sin\frac{3\pi}{n} - \sin\frac{\pi}{n} &= \frac{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}}{\sin\frac{2\pi}{n}} \\
2\sin\frac{\pi}{n}\cos\frac{2\pi}{n}\sin\frac{2\pi}{n} &= \sin\frac{\pi}{n}\sin\frac{3\pi}{n} \\
\sin\frac{\pi}{n}\left(\sin\frac{4\pi}{n} - \sin\frac{3\pi}{n}\right) &= 0
\end{aligned}
$$

The only integer $n > 1$ that satisfies this is $n = 7$, as $\dfrac{4\pi}{n} = \pi - \dfrac{3\pi}{n}$.

SE-E8 **Problem** In $\triangle ABC$, $\tan A = \frac{3}{4}$ and $\tan B = \frac{21}{20}$. Find the ratio $\frac{AC}{BC}$.

**Answer** $\frac{35}{29}$

**Solution** From sine law, we have $\frac{AC}{BC} = \frac{\sin B}{\sin A}$. Then, $\sin A = \frac{\tan A}{\sec A} = \frac{\tan A}{\sqrt{1+\tan^2 A}} = \frac{3/4}{5/4} = \frac{3}{5}$. Similarly, $\sin B = \frac{\tan B}{\sqrt{1+\tan^2 B}} = \frac{21/20}{29/20} = \frac{21}{29}$. Then $\frac{AC}{BC} = \frac{3/5}{21/29} = \frac{35}{29}$.

SE-E9 **Problem** Find the coefficient of $x^{98}$ in the product $(x+1)(x+2)(x+3)\cdots(x+100)$.

**Answer** 12 582 075

**Solution** The coefficient of $x^{98}$ is the sum of the products of $1, 2, \ldots, 100$ taken two at a time. Therefore, the coefficient of $x^{98}$ is

$$\frac{1}{2}\left(\left(\sum_{i=1}^{100} i\right)^2 - \sum_{i=1}^{100} i^2\right) = \frac{1}{2}\left(\left(\frac{100\cdot 101}{2}\right)^2 - \frac{100\cdot 101\cdot 201}{6}\right)$$
$$= \frac{1}{2}(25\,502\,500 - 338\,350) = 12\,582\,075.$$

SE-E10 **Problem** Square $ABCD$ is inscribed in a unit circle. Let $P$ be the intersection of line $AB$ and the tangent of the circle at $C$. Find the length of segment $PD$.

**Answer** $\sqrt{10}$ units

**Solution** Since the circle has radius 1, then the sides of the square have length $\sqrt{2}$. Since $PC \perp AC$, $\angle PCB = 45^\circ$; also, $PA \perp BC$ implies $\angle CPB = 45^\circ$. Therefore, $\triangle PBC$ is isosceles, so $PB = BC = \sqrt{2}$. Furthermore, since $\triangle APD$ is a right triangle, we have $PD = \sqrt{AP^2 + AD^2} = \sqrt{(2\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{10}$.

**Average**

SE-A1 **Problem** Let $x = \sin^2 A$. Express $\sin A \sin 2A \sin 3A \sin 4A$ as a polynomial in $x$, in general form.

**Answer** $-64x^5 + 144x^4 - 104x^3 + 24x^2$

**Solution** The given expression is equal to

$$\begin{aligned}\sin A \sin 2A \sin 3A \sin 4A &= \sin A(2\sin A\cos A)(3\sin A - 4\sin^3 A)(2\sin 2A\cos 2A)\\ &= 2\sin^2 A\cos A\sin A(3-4\sin^2 A)4\sin A\cos A(1-2\sin^2 A)\\ &= 8\sin^4 A\cos^2 A(3-4\sin^2 A)(1-2\sin^2 A)\\ &= 8\sin^4 A(1-\sin^2 A)(3-4\sin^2 A)(1-2\sin^2 A)\\ &= 8x^2(1-x)(3-4x)(1-2x) = 8x^2(3-13x+18x^2-8x^3)\\ &= 24x^2 - 104x^3 + 144x^4 - 64x^5\end{aligned}$$

SE-A2 **Problem** What is/are the real value(s) of $x$ that satisfy the equation

$$(x^2\pi^2 + e^2)(x^{2^2}\pi^{2^2} + e^{2^2})(x^{2^3}\pi^{2^3} + e^{2^3})\cdots(x^{2^{2018}}\pi^{2^{2018}} + e^{2^{2018}}) = e^{2^{2019}-2}?$$

**Answer** 0

**Solution** We multiply both sides by $\left(x^2\pi^2 - e^2\right)$ and simplify to get $x^{2^{2019}}\pi^{2^{2019}} - e^{2^{2019}} = e^{2^{2019}-2}(x^2\pi^2 - e^2) = x^2\pi^2 e^{2^{2019}-2} - e^{2^{2019}-2+2}$. Then $x^{2^{2019}}\pi^{2^{2019}} = x^2\pi^2 e^{2^{2019}-2}$, implying

$$x^2\pi^2(x^{2^{2019}-2}\pi^{2^{2019}-2} - e^{2^{2019}-2}) = x^2\pi^2((x\pi)^{2^{2019}-2} - e^{2^{2019}-2}) = 0.$$

Then either $x = 0$ or $x = \pm\frac{e}{\pi}$ (as the exponent $2^{2019} - 2$ is even). It can be verified through substitution that $x = 0$ is a solution, and $x = \pm\frac{e}{\pi}$ are not.

SE-A3 **Problem** If no three diagonals of a convex decagon meet at a point, into how many line segments are the diagonals divided by their intersections?

**Answer** 455

**Solution** The total number of diagonals is $\binom{10}{2} - 10 = 35$. For every four distinct vertices of the decagon one can find exactly one intersection in the interior, as the decagon is convex. Thus, the total number of intersections is $\binom{10}{4} = 210$.

Since each intersection point produces two new segments, the 210 intersections add 420 new segments. Adding the 35 original segments corresponding to the diagonals, there is a total of $420 + 35 = 455$ segments.

SE-A4 **Problem** Let $a$, $b$, and $c$ be integers from 0 to 9, inclusive. How many triples $(a, b, c)$ are there such that the three-digit number $\overline{abc}$ is a prime and the function $f(x) = ax^2 + bx + c$ has at least one rational zero?

**Answer** 21

**Solution** If $a \neq 0$, then set $f(10) = 100a + 10b + c$. It is given that this prime, yet it can be factored into a product of two integers greater than 1, since $f(x)$ has at least one rational zero. Therefore, there is a contradiction, and we only need to consider $a = 0$.

If $a = 0$, as long as $b \neq 0$, we always have a rational zero $-\frac{c}{b}$. Since there are 21 primes between 10 and 99, there are also 21 triples $(a, b, c)$ that satisfy both conditions.

SE-A5 **Problem** Gretel has six paper clips, labeled 1 to 6, two cardboard boxes, and a fair die. She first puts all six paper clips into the first cardboard box. She then rolls the die, and she moves the paper clip whose number is shown face-up on the die, from the box it is currently in, to the other box. She rolls the die and moves paper clips repeatedly until both boxes have exactly 3 paper clips. On average, how many times will she toss the die before stopping?

**Answer** $\dfrac{23}{5}$

**Solution** Let $E_i$ denoted the expected number of tosses before having both boxes with equal paper clips given that the the first box originally had $i$ paper clips, $i \in \{6, 5, 4, 3\}$. The problem is finding $E_6$. We have the following system:

$$\begin{cases} E_6 = 1 + E_5 \\ E_5 = 1 + \frac{5}{6}E_4 + \frac{1}{6}E_6 \\ \\ E_4 = 1 + \frac{2}{3}E_3 + \frac{1}{3}E_5 \\ E_3 = 0 \end{cases} \Rightarrow \begin{cases} E_6 = \frac{23}{5} \\ \\ E_5 = \frac{18}{5} \\ \\ E_4 = \frac{11}{5} \\ E_3 = 0 \end{cases}$$

Therefore, the expected number of tosses is $\dfrac{23}{5}$.

SE-A6 **Problem** Solve for real values of $x$:

$$\sqrt[3]{(5+x)^2} + 4\sqrt[3]{(5-x)^2} = 5\sqrt[3]{25-x^2}$$

**Answer** 0 and $\dfrac{63}{13}$

**Solution** We take the cube of both sides to get

$$\begin{aligned} (5+x)^2 + 64(5-x)^2 + 12(25-x^2)^{2/3}\left(\sqrt[3]{(5+x)^2} + 4\sqrt[3]{(5-x)^2}\right) &= 125(25-x^2) \\ (5+x)^2 + 64(5-x)^2 + 12(25-x^2)^{2/3}\left(5\sqrt[3]{25-x^2}\right) &= 125(25-x^2) \\ (5+x)^2 + 64(5-x)^2 + 60(25-x^2) &= 125(25-x^2) \\ (5+x)^2 + 64(5-x)^2 &= 65(25-x^2) \end{aligned}$$

$$(25 + 10x + x^2) + 64(25 - 10x + x^2) = 65(25 - x^2)$$
$$130x^2 - 630x = 0$$

Then $x = 0$ or $\frac{63}{13}$.

SE-A7 **Problem** Find the largest possible value of the five-digit number $\overline{PUMaC}$ in the cryptarithm shown below. Here, identical letters represent the same digits and distinct letters represent distinct digits.

$$\begin{array}{ccccc} & N & I & M & O \\ + & H & M & M & T \\ \hline P & U & M & a & C \end{array}$$

**Answer** 16 485

**Solution** It is clear that $P = 1$. Now $U$ cannot be 9 or 8; otherwise, $U$ will be equal to at least one of $N$ or $H$. Let $U = 7$, so $\{N, H\} = \{8, 9\}$. Without loss of generality, let $N = 9$, $H = 8$.

$$\begin{array}{ccccc} & 9 & I & M & O \\ + & 8 & M & M & T \\ \hline 1 & 7 & M & a & C \end{array}$$

Since the addition in the hundreds place does not have a carry-over to the thousands place, we conclude that $I = 0$. Then $M$ is either 2 or 3.

If $M = 3$, we have:

$$\begin{array}{ccccc} & 9 & 0 & 3 & O \\ + & 8 & 3 & 3 & T \\ \hline 1 & 7 & 3 & 6 & C \end{array}$$

The remaining digits are 2, 4, and 5, but no two of the three add up to the third.

If $M = 2$, we have:

$$\begin{array}{ccccc} & 9 & 0 & 2 & O \\ + & 8 & 2 & 2 & T \\ \hline 1 & 7 & 2 & 4 & C \end{array}$$

The remaining digits are 3, 6, and 5, but no two of the three add up to the third.

Next, we consider $U = 6$. Then either $\{N, H\} = \{9, 7\}$ or $\{N, H\} = \{8, 7\}$. Say $H = 7$.

If $N = 9$, then again $I = 0$. The largest possible value of $M$ is 4, so we have

$$\begin{array}{ccccc} & 9 & 0 & 4 & O \\ + & 7 & 4 & 4 & T \\ \hline 1 & 6 & 4 & 8 & C \end{array}$$

The remaining digits are 2, 3, and 5, so $C = 5$. Then $\overline{PUMaC} = 16\,485$.

If $N = 8$, then $I = 9$ as there is a carry-over. The only possible value for $M$ is 5. Then we have

$$\begin{array}{ccccc} & 8 & 9 & 5 & O \\ + & 7 & 5 & 5 & T \\ \hline 1 & 6 & 5 & 0 & C \end{array}$$

The remaining digits are 2, 3, and 4, but no two of the three add up to the third.

Since for $U = 6$ there is only one solution, we conclude that the largest possible value for $\overline{PUMaC}$ is $16\,485$.

## Difficult

SE-D1 **Problem** Evaluate $\cot \sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)$.

**Answer** $\frac{25}{23}$

**Solution** The expression is equal to

$$\begin{aligned}\cot \sum_{n=1}^{23} \cot^{-1}(n^2+n+1) &= \cot \sum_{n=1}^{23} \tan^{-1}\left(\frac{1}{n^2+n+1}\right)\\ &= \cot \sum_{n=1}^{23} \tan^{-1} \frac{(n+1)-n}{n(n+1)-1}\\ &= \cot \sum_{n=1}^{23} (\tan^{-1}(n+1) - \tan^{-1}(n))\\ &= \cot(\tan^{-1} 24 - \tan^{-1} 1)\\ &= \frac{1}{\tan(\tan^{-1} 24 - \tan^{-1} 1)}\\ &= \frac{1 + \tan\tan^{-1} 24 \tan\tan^{-1} 1}{\tan\tan^{-1} 24 - \tan\tan^{-1} 1}\\ &= \frac{1+24}{24-1} = \frac{25}{23}\end{aligned}$$

SE-D2 **Problem** In $\triangle ABC$, $\cos A \cos B \cos C = \dfrac{\sqrt{3}-1}{8}$ and $\sin A \sin B \sin C = \dfrac{3+\sqrt{3}}{8}$. Find

$$\frac{\tan A \tan B + \tan A \tan C + \tan B \tan C}{\tan A + \tan B + \tan C}.$$

**Answer** $\dfrac{9-2\sqrt{3}}{3}$

**Solution** A non-right triangle satisfies $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. Thus, $\tan A + \tan B + \tan C = \dfrac{3+\sqrt{3}}{8} \div \dfrac{\sqrt{3}-1}{8} = \dfrac{3+\sqrt{3}}{\sqrt{3}-1} = 3 + 2\sqrt{3}$.

Consider

$$\cos(A+B+C) = \cos((A+B)+C)$$

$$= \cos(A+B)\cos C - \sin(A+B)\sin C$$
$$= \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$$
$$= \cos A\cos B\cos C(1 - \tan A\tan B - \tan A\tan C - \tan B\tan C)$$

But $\cos(A+B+C) = \cos(180^\circ) = -1$ and $\cos A\cos B\cos C = \frac{\sqrt{3}-1}{8}$. Thus $\tan A\tan B + \tan A\tan C + \tan B\tan C = \frac{1}{(\sqrt{3}-1)/8} + 1 = 5 + 4\sqrt{3}$.

Therefore, the desired expression is $\frac{5+4\sqrt{3}}{3+2\sqrt{3}} = \frac{9-2\sqrt{3}}{3}$.

SE-D3 **Problem** How many prime factors does $\sum_{n=1}^{54}(-1)^{\lfloor(n+2)\div 3\rfloor}n^2$ have?

**Answer** 3

**Solution** We group every six terms:

$$\begin{aligned}\sum_{n=1}^{54}(-1)^{\lfloor(n+2)\div 3\rfloor}n^2 &= \sum_{n=1}^{9}\sum_{i=0}^{2}((6n-i)^2 - (6n-5+i)^2)\\ &= \sum_{n=1}^{9}\sum_{i=0}^{2}(12n-5)(5-2i)\\ &= \left(\sum_{n=1}^{9}(12n-5)\right)\left(\sum_{i=0}^{2}(5-2i)\right)\\ &= (6(9)(10)-5(9))(9) = 55(81) = 5\cdot 11\cdot 3^4\end{aligned}$$

Therefore the number has 3 prime factors.

SE-D4 **Problem** Let $y_0$ be chosen uniformly randomly from $\{0, 50\}$, $y_1$ from $\{40, 60, 80\}$, $y_2$ from $\{10, 40, 70, 80\}$, and $y_3$ from $\{10, 30, 40, 70, 90\}$. Let $P(x)$ be a polynomial with degree at most 3 such that $P(i) = y_i$ for $i \in \{0, 1, 2, 3\}$. Find the expected value of $P(4)$.

**Answer** 107

**Solution** Let $Q(x)$ be the expected value of $P(x)$. By linearity of expectation, the expected value of $P(4)$ is $Q(4)$. Now $Q(x)$ is also a polynomial of degree at most 3. We have

$$Q(O) = E(y_0) = \frac{0 + 50}{2} = 25$$
$$Q(1) = E(y_1) = \frac{40 + 60 + 80}{3} = 60$$
$$Q(2) = E(y_2) = \frac{10 + 40 + 70 + 80}{4} = 50$$
$$Q(3) = E(y_3) = \frac{10 + 30 + 40 + 70 + 90}{5} = 48$$

Since $Q(x)$ is a polynomial of degree at most 3, $R(x) = Q(x) - Q(x-1)$ is of degree at most 2, $S(x) = R(x) - R(x-1)$ is of degree at most 1, and $T(x) = S(x) - S(x-1)$ is constant. Then

$$\begin{aligned} T(4) = S(4) - S(3) &= (R(4) - R(3)) - (R(3) - R(2)) \\ &= R(4) - 2R(3) + R(2) \\ &= (Q(4) - Q(3)) - 2(Q(3) - Q(2)) + (Q(2) - Q(1)) \\ &= Q(4) - 3Q(3) + 3Q(2) - Q(1) = Q(4) - 144 + 150 - 60 \\ &= Q(4) - 54. \end{aligned}$$

Similarly, $T(3) = Q(3) - 3Q(2) + 3Q(1) - Q(0) = 48 - 150 + 180 - 25 = 53$. Since $T(4) = T(3)$, then $Q(4) - 54 = 53 \Rightarrow Q(4) = 107$.

SE-D5 **Problem** A graph is defined in polar coordinates by $r(\theta) = \cos\theta + \frac{1}{2}$. Find the smallest $x$-coordinate of a point on the graph.

**Answer** $-\frac{1}{16}$

**Solution** For polar coordinates, we have

$$\begin{aligned} x = r\cos\theta &= \left(\cos\theta + \frac{1}{2}\right)\cos\theta = \cos^2\theta + \frac{\cos\theta}{2} \\ &= \cos^2\theta + \frac{\cos\theta}{2} + \frac{1}{16} - \frac{1}{16} \\ &= \left(\cos\theta + \frac{1}{4}\right)^2 - \frac{1}{16} \end{aligned}$$

Hence, the smallest $x$-coordinate is $-\frac{1}{16}$, when $\theta = \cos^{-1}\frac{1}{4}$.

## Senior Division Semifinals

SSF-1 **Problem** The sum of squares of deviations of 10 observations from the mean 50 is 250. What is the coefficient of variation? Express as a percentage.

**Answer** 10%

**Solution** The standard deviation of the data is $\sigma = \sqrt{\frac{250}{10}} = 5$. Thus, the coefficient of variation is $\frac{5}{50} = 10\%$.

SSF-2 **Problem** Let $\lfloor k \rfloor$ denote the largest integer not exceeding $k \in \mathbb{R}$. If $x$ and $y$ are real numbers such that $y = 2\lfloor x \rfloor + 3 = 3\lfloor x - 2 \rfloor + 5$, find $\lfloor x + y \rfloor$.

**Answer** 15

**Solution** Let $\lfloor x \rfloor + \{x\} = x$. Then $y = 2\lfloor x \rfloor + 3 = 3\lfloor x \rfloor - 1$. This implies $\lfloor x \rfloor = 4$, and $y = 2 \times 4 + 3 = 11$. Then $\lfloor x + y \rfloor = \lfloor x \rfloor + y = 4 + 11 = 15$.

SSF-3 **Problem** In a market eggs are available by sets of 12's or 13's. What is the largest number of eggs that cannot be bought as a combination of 12's or 13's?

**Answer** 131

**Solution** This is a direct application of the Chicken McNugget theorem: $(13-1)(12-1)-1 = 131$.

SSF-4 **Problem** Simplify $\prod_{i=1}^{5} \tan \frac{(2i-1)\pi}{20}$.

**Answer** 1

**Solution**

$$\begin{aligned}\tan\frac{\pi}{20}\tan\frac{3\pi}{20}\tan\frac{5\pi}{20}\tan\frac{7\pi}{20}\tan\frac{9\pi}{20} &= \tan\frac{\pi}{20}\tan\left(\frac{\pi}{2}-\frac{\pi}{20}\right)\tan\frac{3\pi}{20}\tan\left(\frac{\pi}{2}-\frac{3\pi}{20}\right)\tan\frac{\pi}{4}\\ &= \tan\frac{\pi}{20}\cot\frac{\pi}{20}\tan\frac{3\pi}{20}\cot\frac{3\pi}{20} = 1\end{aligned}$$

SSF-5 **Problem** Find the monic polynomial, with rational coefficients and of least degree, such that

$$\sqrt{1+\sqrt{2+\sqrt{1+\sqrt{2+\cdots}}}}$$

is one of its zeros.

**Answer** $x^4 - 2x^2 - x - 1$

**Solution** Let $x$ be the above expression. Then

$$\begin{aligned} x &= \sqrt{1+\sqrt{2+x}}\\ x^2 &= 1+\sqrt{2+x}\\ x^2-1 &= \sqrt{2+x}\\ x^4-2x^2+1 &= 2+x\\ x^4-2x^2-x-1 &= 0\end{aligned}$$

Since $x^4 - 2x^2 - x - 1$ is not factorable, the degree cannot be reduced.

SSF-6 **Problem** If $1, \omega, \omega^2$ are the three cube roots of unity, find

$$\prod_{j=1}^{2018} (1 - \omega^j + \omega^{2j}).$$

**Answer** $2^{1346}$

**Solution** When $j$ is divisible by 3, $\omega^j = \omega^{2j} = 1$, which implies $1 - \omega^j + \omega^{2j} = 1 - 1 + 1 = 1$. When $j$ is not divisible by 3, $\omega^j \neq 1$. Since $\omega^{3j} = 1$, we have

$$\omega^{3j} - 1 = 0 \Rightarrow (\omega^j - 1)(\omega^{2j} + \omega^j + 1) = 0 \Rightarrow \omega^{2j} + \omega^j + 1 = 0 \Rightarrow 1 - \omega^j + \omega^{2j} = -2\omega^j.$$

Therefore,

$$\begin{aligned}
\prod_{j=1}^{2018} (1 - \omega^j + \omega^{2j}) &= \left( \prod_{i=0}^{671} \prod_{j=1}^{3} (1 - \omega^{j+3i} + \omega^{2j+3i}) \right) (1 - \omega^{2017} + \omega^{4034})(1 - \omega^{2018} + \omega^{4036}) \\
&= \left( \prod_{i=1}^{672} \prod_{j=1}^{3} (1 - \omega^j + \omega^{2j}) \right) (-2\omega^{2017})(-2\omega^{2018}) \\
&= \left( \prod_{i=1}^{672} (-2\omega)(-2\omega^2)(1) \right) 4\omega^{4035} \\
&= 4^{672} \cdot 4 = 4^{673} = 2^{1346}
\end{aligned}$$

SSF-7 **Problem** Find the locus (equation describing the set) of points $(x, y, z)$ that are equidistant to the lines $x - y = 0, z = 1$, and $x + y = 0, z = -1$.

**Answer** $xy + 2z = 0$

**Solution** The two lines can be parametrized as $\ell_1 : (t, t, 1)$ and $\ell_2 : (t, -t, -1)$, for all $t \in \mathbb{R}$.

Consider an arbitrary point $(a, b, c)$. The distance of this point to $(t, t, 1)$ is

$$\begin{aligned}\sqrt{(a-t)^2+(b-t)^2+(c-1)^2} &= \sqrt{2t^2-2(a+b)t+(a^2+b^2+(c-1)^2)} \\ &= \sqrt{2\left(t-\frac{a+b}{2}\right)^2-\frac{(a+b)^2}{2}+(a^2+b^2+(c-1)^2)}\end{aligned}$$

This is minimized when $t = \dfrac{a+b}{2}$. Then, the distance of $(a, b, c)$ to the line $x - y = 0, z = 1$ is the distance of $(a, b, c)$ to $\left(\dfrac{a+b}{2}, \dfrac{a+b}{2}, 1\right)$. Similarly, the distance of $(a, b, c)$ to the line $x + y = 0, z = -1$ is the distance of $(a, b, c)$ to $\left(\dfrac{a-b}{2}, \dfrac{a-b}{2}, -1\right)$. Therefore, the locus of points can be solved by the equation

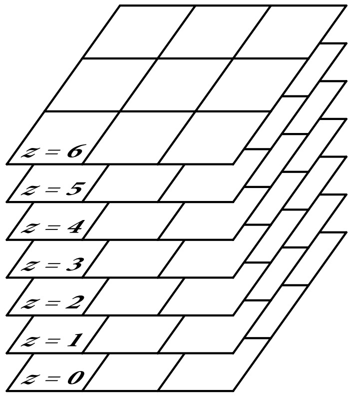
$$\sqrt{\left(x-\frac{x+y}{2}\right)^2+\left(y-\frac{x+y}{2}\right)^2+(z-1)^2} = \sqrt{\left(x-\frac{x-y}{2}\right)^2+\left(y-\frac{x-y}{2}\right)^2+(z+1)^2}.$$

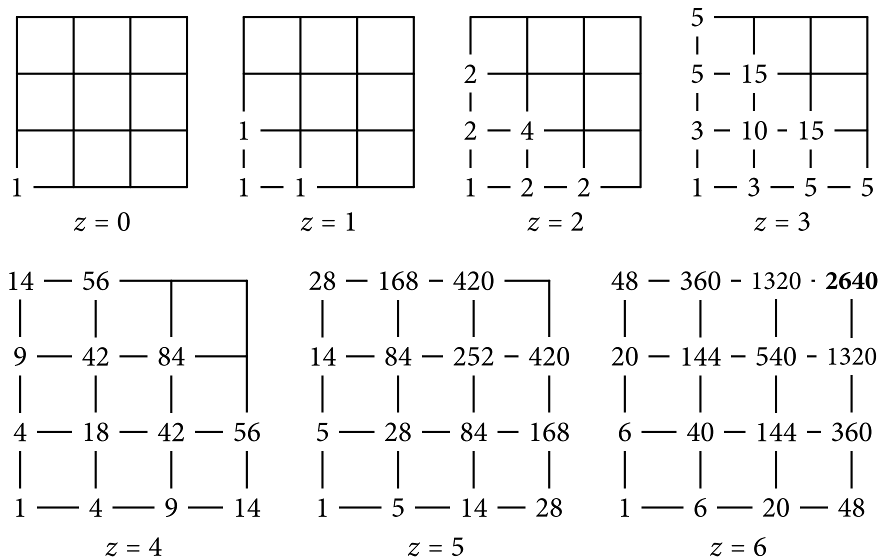
This simplifies to $xy + 2z = 0$.

SSF-8 **Problem** How many paths of shortest length are there from $(0, 0, 0)$ to $(3, 3, 6)$ if one can only go up, forward, or right by integral units, and the path can only pass through points $(x, y, z)$ that satisfy $x + y \leq z$?

**Answer** 2640

**Solution** We find the number of ways using a diagram. For any point $P$ we place a number $\#(P)$ denoting the number of ways one can go there from $(0, 0, 0)$ that satisfy the above condition. Thus $\#(3, 3, 6)$ is the answer. Now, if at least one of $x$, $y$, or $z$ is negative, then $\#(x, y, z) = 0$; $\#(0, 0, 0) = 1$, and for all other points $\#(x, y, z) = \#(x-1, y, z) + \#(x, y-1, z) + \#(x, y, z-1)$. Now we compute the values up to $(3, 3, 6)$.




Therefore, there are 2640 paths.

SSF-9 **Problem** Let $a_k$ the sum of the coefficients of $x^{4n}$, where $n$ is an integer from 0 to $\frac{k}{4}$, inclusive, in the expansion of $(x+1)^k$. Find $a_{2019} - 2a_{2018}$.

**Answer** $-2^{1008}$

**Solution** Consider $(x+1)^k$, and let $c_i$ be the coefficient of $x^i$ in its expansion, $i \in \{0, 1, \ldots, k\}$. Then if we substitute $x = \pm 1$ in $(x+1)^k = c_0 + c_1 x + \cdots + c_k x^k$, we have

$$\begin{cases} 2^k = c_0 + c_1 + \cdots + c_k \\ 0 = c_0 - c_1 + \cdots + (-1)^k c_k \end{cases} \Rightarrow c_0 + c_2 + c_4 + \cdots = 2^{k-1}$$

On the other hand, substituting $x = i$ gives

$$2^{k/2}\left(\cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4}\right) = (1+i)^k = c_0 + ic_1 - c_2 - ic_3 + c_4 + \cdots + i^k c_k$$

We then take the real parts of both sides to get $2^{k/2}\cos\frac{k\pi}{4} = c_0 - c_2 + c_4 + \cdots$. This implies $a_k = c_0 + c_4 + c_8 + \cdots = \frac{1}{2}\left(2^{k-1} + 2^{k/2}\cos\frac{k\pi}{4}\right)$.

Thus, $a_{2019} - 2a_{2018} = (2^{2017} - 2^{1008}) - 2 \cdot 2^{2016} = -2^{1008}$.

SSF-10 **Problem** Determine all functions $f : \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$ satisfying the relation $f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}$ for all values of $x \in \mathbb{R} \setminus \{0,1\}$.

**Answer** $f(x) = \dfrac{x+1}{x-1}$

**Solution** Let $y = \frac{1}{1-x} \Rightarrow x = 1 - \frac{1}{y}$. Then $f(x) + f(y) = \frac{2(1-2x)}{x(1-x)} = \frac{2}{x} - \frac{2}{1-x} = \frac{2}{x} - 2y$.

Also, $f(y) + f\left(\frac{1}{1-y}\right) = \frac{2}{xy} - \frac{2}{1-y}$. Let $z = \frac{1}{1-y}$. Then $f(y) + f(z) = \frac{2}{y} - 2z$.

Note, however, that $y = \frac{1}{1-x}$ and $z = \frac{1}{1-y}$ together imply $x = \frac{1}{1-x}$. Then as above, $f(z) + f(x) = \frac{2}{z} - 2x$.

Therefore, we have

$$
\begin{aligned}
f(x)+f(y)+f(z) &= \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-x-y-z \\
f(x) &= \frac{1}{x}-\frac{1}{y}+\frac{1}{z}-x-y+z \\
&= \cancel{\frac{1}{x}}-(\cancel{1}-\cancel{x})+\frac{x}{x-1}-\cancel{x}-\frac{1}{1-x}+\left(\cancel{1}-\cancel{\frac{1}{x}}\right) \\
f(x) &= \frac{x+1}{x-1}
\end{aligned}
$$

SSF-11 **Problem** Find the remainder when $2903^{2019}-803^{2019}-464^{2019}+261^{2019}+2019$ is divided by 1897.

**Answer** 122

**Solution** Note that $1897 = 7 \cdot 271$. Now,

$$
\begin{aligned}
2903-803=2100 &\mid 2903^{2019}-803^{2019} \\
464-261=203 &\mid 464^{2019}-261^{2019} \\
7 &\mid 2903^{2019}-803^{2019}-464^{2019}+261^{2019}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
2903-464=2439=271(9) &\mid 2903^{2019}-464^{2019} \\
803-261=542 &\mid 803^{2019}-261^{2019} \\
271 &\mid 2903^{2019}-803^{2019}-464^{2019}+261^{2019}
\end{aligned}
$$

Thus,

$$\begin{aligned}2903^{2019} - 803^{2019} - 464^{2019} + 261^{2019} + 2019 &\equiv 2019 \pmod{1897}\\ &\equiv 122 \pmod{1897}\end{aligned}$$

SSF-12 **Problem** In a class of 10 students, the probability that exactly $i$ ($i$ from 0 to 10) students passed an exam is directly proportional to $i^2$. If a student is selected at random, find the probability that s/he passed the exam.

**Answer** $\dfrac{11}{14}$

**Solution** Denote by $P(N_i) = \lambda i^2$ the probability that exactly $i$ students passed, where $\lambda$ is the constant of proportionality. Then

$$1 = \sum_{i=0}^{10} P(N_i) = \sum_{i=0}^{10} \lambda i^2 = \lambda \frac{\cancelto{5}{10} \cdot 11 \cdot \cancelto{7}{21}}{\cancel{6}} = 385\lambda \Rightarrow \lambda = \frac{1}{385}.$$

From law of total probabilities, the probability that student $A$ passed is

$$\begin{aligned}P(A) &= \sum_{i=0}^{10} P\left(A \mid N_i\right) P(N_i) = \sum_{i=0}^{10} \frac{i}{10} \cdot \frac{i^2}{385} = \frac{1}{3850} \sum_{i=0}^{10} i^3\\ &= \frac{1}{3850} \left(\frac{10 \cdot 11}{2}\right)^2 = \frac{11}{14}.\end{aligned}$$

SSF-13 **Problem** Find the largest prime factor of the sum of the products of the nonzero digits of the positive integers less than 1000.

**Answer** 103

**Solution** Denote by $p(n)$ the product of the nonzero digits of $n$. We first append two zeroes to the front of one-digit numbers and one zero to the front of two-digit numbers.

Now, the sum of the products of the digits of the transformed positive integers less than 1000, zero or otherwise, is equal to

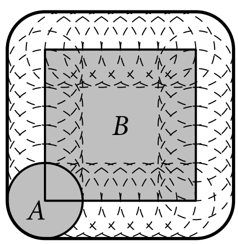
$$\begin{aligned}(0\cdot 0\cdot 0+0\cdot 0\cdot 1+\cdots+9\cdot 9\cdot 9)-0\cdot 0\cdot 0 &= (0+\cdots+9)(0+\cdots+9)(0+\cdots+9)-0\cdot 0\cdot 0\\ &= (0+\cdots+9)^3-0^3\end{aligned}$$

The sum of the products of the nonzero digits can be attained by changing all the zeroes above into ones. Therefore, the sum is $46^3-1=(46-1)(46^2+46+1)=45\cdot 2163=3^3\cdot 5\cdot 7\cdot 103$, so the largest prime factor is 103.

SSF-14 **Problem** Define the sum of any two points $(x_1, y_1)$, $(x_2, y_2)$ on the Cartesian plane to be the point $(x_1+x_2, y_1+y_2)$. Furthermore, for two sets $A$ and $B$, define their *Minkowski sum* to be the set $A+B=\{a+b \mid a \in A, b \in B\}$. Let $A=\{(x, y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ and $B=\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, |x| \le 2, |y| \le 2\right\}$. Find the area of the region enclosed by $A+B$.

**Answer** $(12+\pi)$ square units

**Solution** We translate $A$ around the Cartesian plane such that the center of the circle is in $B$. Then the region enclosed by $A+B$ is as shown in the following figure:




Thus, the area of $A+B$ is equal to the sum of the area of square $B$, the four rectangles around $B$, and four quarter-circles on the corners: $2(2)+4(1)(2)+\pi(1^2)=12+\pi$.

SSF-15 **Problem** What is the value of

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2 n}{3^m(n3^m+m3^n)}?$$

**Answer** $\frac{9}{32}$

**Solution** Let $S$ be the given sum, and $a_n=\frac{3^n}{n}$. Then

$$S=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_m(a_m+a_n)}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_n(a_m+a_n)}$$

Adding both summations results in

$$\begin{aligned}2S&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{a_m(a_m+a_n)}+\frac{1}{a_n(a_m+a_n)}\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_m a_n}\\&=\left(\sum_{n=1}^{\infty}\frac{1}{a_n}\right)^2=\left(\sum_{n=1}^{\infty}\frac{n}{3^n}\right)^2=\left(\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{1}{3^n}\right)^2\\&=\left(\sum_{m=1}^{\infty}\frac{\frac{1}{3^m}}{1-\frac{1}{3}}\right)^2=\left(\frac{3}{2}\sum_{m=1}^{\infty}\frac{1}{3^m}\right)^2=\left(\frac{3}{2}\cdot\frac{\frac{1}{3}}{1-\frac{1}{3}}\right)^2\\&=\left(\frac{3}{4}\right)^2=\frac{9}{16}\Rightarrow S=\frac{9}{32}\end{aligned}$$

## Senior Division Finals

SF-1 **Problem** It is known that the only rational solution to the equation $10^x+11^x+12^x=13^x+14^x$ is $x=2$. Find all irrational solutions.

**Answer** There are no irrational solutions.

**Solution** Dividing both sides of the equation by $13^x \neq 0$ gives us

$$\left(\frac{10}{13}\right)^x + \left(\frac{11}{13}\right)^x + \left(\frac{12}{13}\right)^x = 1 + \left(\frac{14}{13}\right)^x.$$

The left-hand side is a decreasing function of $x$, and the right-hand side is an increasing function of $x$; therefore, the equation must have at most one solution, which is $x = 2$. Thus, there are no irrational solutions.

SF-2 **Problem** The three altitudes of a triangle have lengths $\frac{2}{9}$, $\frac{1}{5}$, and $\frac{2}{17}$. Find the inradius of the triangle.

**Answer** $\frac{1}{18}$

**Solution** Let $a$, $b$, and $c$ be the lengths of the sides corresponding to the altitudes with respective lengths $\frac{2}{9}$, $\frac{1}{5}$, and $\frac{2}{17}$. Then, if $A$ is the area of the triangle, we have $A = \frac{a}{9} = \frac{b}{10} = \frac{c}{17}$, so $a = 9A$, $b = 10A$, and $c = 17A$. Then the semiperimeter is $s = \frac{1}{2}(9A + 10A = 17A) = 18A$. Since the area of a triangle is equal to the product of its inradius $r$ and its semiperimeter, then $s = 18A = 18sr \Rightarrow r = \frac{1}{18}$.

SF-3 **Problem** Prove that any positive integer can be expressed as a sum of one or more positive Fibonacci numbers, no two of which are consecutive. (1 and 2 are considered consecutive Fibonacci numbers.)

**Solution**

*Proof.* We show this is true by induction. For $n = 1$, 2, and 3 this is true as all three are themselves Fibonacci numbers. Also we see that it is true for $4 = 1 + 3$. It is obvious that this is true for all Fibonacci numbers, so we consider numbers that are not. Now we show that if this is true for all integers less than a Fibonacci number $F_n$, then it is true for all integers less than the next one, $F_{n+1}$. Say we have a number $x$ between $F_n$ and $F_{n+1}$. Then we can express it as the sum of $F_n$ and $y = x - F_n$. Note that since $F_n < x < F_{n+1}$, $0 < y < F_{n+1} - F_n = F_{n-1}$. Since $y < F_n$, from the induction hypothesis, we can express it as a sum of one or more non-consecutive positive Fibonacci numbers. On the other hand, since $y < F_{n-1}$, the representation of $y$ will not contain $F_{n-1}$. Thus adding $F_n$ to the representation of $y$ (to yield $x$) will work. Therefore the proof is complete. □

SF-4 **Problem** Solve the equation $x^3 - 3x = \sqrt{x+2}$, where $x$ is a real number.

**Answer** $2, \frac{-1-\sqrt{5}}{2}, 2\cos\frac{4\pi}{7}$

**Solution** It is clear that $x \geq -2$. When $x > 2$, we have $x^3 - 4x = x(x^2 - 4) > 0 \Rightarrow x^2 - 3x > x$, and $x^2 - x - 2 = (x-2)(x+1) > 0 \Rightarrow x > \sqrt{x+2}$. Then $x^3 - 3x > x > \sqrt{x+2}$. Hence, $x \in [-2, 2]$.

Then let $x = 2\cos\alpha$, there $\alpha \in [0, \pi]$. We have

$$\begin{aligned} 8\cos^3\alpha - 6\cos\alpha &= \sqrt{2(\cos\alpha + 1)} \\ 2\cos 3\alpha &= \sqrt{4\cos^2\frac{\alpha}{2}} \\ \cos 3\alpha &= \cos\frac{\alpha}{2} \end{aligned}$$

Then either $3a - \frac{a}{2}$ or $3a + \frac{a}{2}$ is a multiple of $2\pi$. When $0 \leq a \leq \pi$, we have $x = 2\cos 0 = 2$, $x = 2\cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{2}$, and $x = 2\cos\frac{4\pi}{7}$.

SF-5 **Problem** A *tetromino* is a shape composed of four congruent squares such that each square shares a side with at least one of the other three. How many ways can one tile a $2 \times 18$ block with 9 tetrominoes, such that the tetrominoes fully fill the block and do not overlap?

**Answer** 3025

**Solution** First, we establish a lemma.

**Lemma 1.** *The squares of the Fibonacci numbers,* $F_1^2 = F_2^2 = 1$, $F_n^2 = (F_{n-1} + F_{n-2})^2$ *(*$n > 2$*), satisfy the recurrence relation*

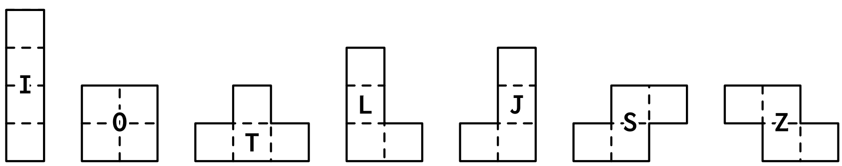
$$F_n^2 = 2F_{n-1}^2 + 2F_{n-2}^2 - F_{n-3}^2.$$

*Proof.* From the definition of the Fibonacci numbers, we have $F_n = F_{n-1} + F_{n-2}$ and $F_{n-1} = F_{n-2} + F_{n-3} \Rightarrow F_{n-3} = F_{n-1} - F_{n-2}$. Then,

$$\begin{aligned} F_n^2 &= F_{n-1}^2 + 2F_{n-1}F_{n-2} + F_{n-2}^2 \\ &= F_{n-1}^2 + F_{n-2}^2 + (F_{n-1}^2 - F_{n-1}^2) + (F_{n-2}^2 - F_{n-2}^2) + 2F_{n-1}F_{n-2} \\ &= 2F_{n-1}^2 + 2F_{n-2}^2 - (F_{n-1} - F_{n-2})^2 \\ &= 2F_{n-1}^2 + 2F_{n-2}^2 - F_{n-3}^2 \end{aligned}$$

□

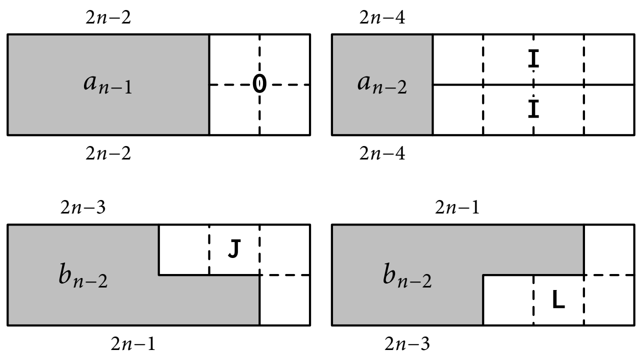
Denote the following letters to the seven tetrominoes.




First, note that the tiling may not have **T**, **S**, or **Z** tetrominoes, because if one of these tiles are placed anywhere in the 2 × 18 block, the number of squares on both the left sides of the tetromino will be odd and thus cannot be filled up by other tetrominoes. The tiling can only consist of **I**, **O**, **L** or **J** tetrominoes.

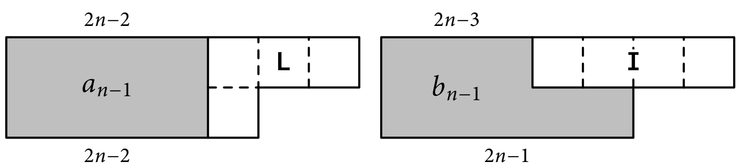
Let $a_n$ denote the number of ways to tile a $2 \times 2n$ block with tetrominoes, and $b_n$ denote the number of ways to tile a block that results from removing the two lower-leftmost squares from a $2 \times (2n + 1)$ block. Thus, we are finding $a_9$.

Given a block corresponding to $a_n$, there are four ways to fill the rightmost column. The shaded figures show the remaining area after placing the tetrominoes specified, and the number inside denotes the number of ways possible to tile the shaded area.




Therefore, $a_n = a_{n-1} + a_{n-2} + 2b_{n-2}$. Shifting the base by one unit results in $a_{n-1} = a_{n-2} + a_{n-3} + 2b_{n-1}$. Subtracting the two and combining like terms, we get $a_n = 2a_{n-1} - a_{n-3} + 2(b_{n-1} - b_{n-2})$.

Similarly, given a block corresponding to $b_n$, there are two ways to fill the two upper-rightmost cells.




Thus, $b_n = a_{n-1} + b_{n-1}$, or $b_n - b_{n-1} = a_{n-1}$. Shifting the base by one unit, we have $b_{n-1} - b_{n-2} = a_{n-2}$. This can now be substituted to the equation for $a_n$, giving $a_n = 2a_{n-1} + 2a_{n-2} - a_{n-3}$.

Note that this recurrence means that $a_n$ are squares of Fibonacci numbers, since $a_0 = a_1 = 1$ and $a_2 = 4$. Therefore, from Lemma 1, $a_n = F_{n+1}^2$. When $n = 9$, $a_9 = F_{10}^2 = 55^2 = 3025$.

# Mathematical Results

## AM-GM Inequality

For any positive real numbers $a_1, a_2, \ldots, a_n$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdots a_n}.$$

Equality holds iff $a_1 = a_2 = \cdots = a_n$. *(p 3)*

## Apollonius's Theorem

On $\triangle ABC$, if $D$ is the midpoint of $BC$, then $AB^2 + AC^2 = 2(AD^2 + BD^2)$. *(p 27)*

## Bretschneider's Formula

The area of any quadrilateral with side lengths $a$, $b$, $c$, and $d$ ($a$ and $c$, and $b$ and $d$ opposite) is

$$\begin{aligned} A &= \frac{1}{4}\sqrt{4p^2q^2 - (b^2 + d^2 - a^2 - c^2)^2} \\ &= \sqrt{(s-a)(s-b)(s-c)(s-d) - abcdx}, \end{aligned}$$

where $p$ and $q$ are the lengths of the diagonals, $x = \cos^2 \frac{\alpha + \beta}{2}$, $\alpha$ and $\beta$ are a pair of opposite angles, and $s = \frac{a+b+c+d}{2}$ is the semiperimeter. *(p 3)*

## Chicken McNugget Theorem

For two coprime numbers $p$ and $q$, the greatest integer that cannot be written in the form $ap + bq$ where $a$ and $b$ are nonnegative integers is $(p-1)(q-1) - 1$. *(p 42)*

### Cosine Law

For $\triangle ABC$, $c^2 = a^2 + b^2 - 2ab\cos C$. *(p 31)*

### Law of Total Probabilities

If $B_1, B_2, \ldots, B_n$ are non-empty sets such that any two are disjoint and $B_1 \cup B_2 \cup \cdots \cup B_n = B$, then

$$P(A) = \sum_{i=1}^{n} P(A \mid B_n) P(B_n).$$

*(p 48)*

### Pythagorean Theorem

$\triangle ABC$ is a right triangle with right angle at $B$ iff $AB^2 + BC^2 = AC^2$. *(p 5)*

### Sine Law

For $\triangle ABC$, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$. *(p 7, 32)*

**Disclaimer:** Not all of the problems here are original. Some are lifted from, or based on, other material. All information provided here is for educational purposes only.