

Lowering the exponent

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Valuation

Let p be a prime. The **p -adic valuation** of a positive integer n , written $\nu_p(n)$,¹ is the largest integer e such that $p^e \mid n$. In particular,

$$\nu_p(n) = e \iff p^e \mid n, p^{e+1} \nmid n.$$

Warmup

1. (Product rule) Prove that $\nu_p(ab) = \nu_p(a) + \nu_p(b)$. Use this to extend the definition of ν_p to $\mathbb{Q}_{\neq 0} \rightarrow \mathbb{Z}$. By convention, we set $\nu_p(0) = \infty$.
2. (Sum rule) Prove that $\nu_p(a+b) \geq \min\{\nu_p(a), \nu_p(b)\}$. Prove that if $\nu_p(a) \neq \nu_p(b)$, then equality holds. Find an example where $\nu_p(a) = \nu_p(b)$, yet equality holds.
3. ([Local-global principle](#)) Prove that $\nu_p(a) \leq \nu_p(b)$ for all primes p if and only if $a \mid b$. What does this tell you about $\nu_p(\gcd(a, b))$ and $\nu_p(\text{lcm}(a, b))$?
4. ([Ostrowski's theorem](#)) Let $\nu: \mathbb{Z}_{\neq 0} \rightarrow \mathbb{N} \cup \{0\}$ be a non-constant function satisfying $\nu(ab) = \nu(a) + \nu(b)$ and $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$. Prove there's some prime p such that $\nu = \nu_p$.

These are all the ν_p properties we need. Some important results I won't discuss, but you should learn later: [Legendre's formula](#), [Kummer's theorem](#), [Lifting the exponent](#).

Examples

1. Positive integers a , b , and c satisfy $a \mid b^3$, $b \mid c^3$, and $c \mid a^3$. Prove that $abc \mid (a+b+c)^{13}$.

Sketch: It's enough to show for all primes p that $\nu_p(abc) \leq \nu_p((a+b+c)^{13})$, or that $\nu_p(abc) \leq 13 \min\{\nu_p(a), \nu_p(b), \nu_p(c)\}$. The given conditions translate to

$$a \mid b^3, b \mid c^3, c \mid a^3 \iff \nu_p(a) \leq 3\nu_p(b), \nu_p(b) \leq 3\nu_p(c), \nu_p(c) \leq 3\nu_p(a).$$

This means that

$$\nu_p(abc) = \nu_p(a) + \nu_p(b) + \nu_p(c) \leq \nu_p(a) + 3\nu_p(c) + 3\nu_p(a) \leq \nu_p(a) + 9\nu_p(a) + 3\nu_p(a),$$

which is $13\nu_p(a)$. Similarly, $\nu_p(abc) \leq 13\nu_p(b)$ and $\leq 13\nu_p(c)$, so it's less than their minimum.

Remark 1: Problems with divisibility, multiplication, exponentiation, or GCDs and LCMs, are good candidates for using ν_p on. It works less well with addition, but it can happen.

Remark 2: I'm going to start omitting the *it's enough to show that, for all primes p* part in the sketches, but you should definitely say it in full solutions.

¹The ν here is the Greek letter ν (spelled *nu*, pronounced “new”), presumably chosen for visual similarity to v , for *valuation*. When reading aloud, I pronounce ν_p as “vee pee”.

2. (BAMO 2018/4 [🔗](#)) Show that if a, b, c are nonzero integers such that $a/b + b/c + c/a$ is an integer, then abc is a perfect cube.

Sketch: Note that abc is a perfect cube if and only if $3 \mid \nu_p(abc)$ for all primes p . Let $x, y, z := \nu_p(a), \nu_p(b), \nu_p(c)$ respectively; we want to show $3 \mid x + y + z$. Because it's an integer, we have $\nu_p(a/b + b/c + c/a) \geq 0$. By the sum rule, we also have

$$\nu_p(a/b + b/c + c/a) \geq \min \{ \nu_p(a/b), \nu_p(b/c), \nu_p(c/a) \} = \min \{ x - y, y - z, z - x \},$$

an inequality we can't use yet. We can make it an equality if we think about distinctness. Doing casework on the number of distinct integers in $\{x - y, y - z, z - x\}$:

- Suppose all were distinct. This gives the equality case: $0 \leq \nu_p(a/b + b/c + c/a) = \min \{x - y, y - z, z - x\}$. As the three integers sum to 0, if they're all distinct, one of them must be negative. Then their minimum would be negative, contradicting the fact that $\nu_p(a/b + b/c + c/a) \geq 0$.
- Otherwise, two of them are equal. WLOG $x - y = y - z$. Then $x + y + z = 3y$, which is divisible by 3, as desired.

Remark: In general, an equality case of the sum rule is much more useful, and it's usually worth doing casework to make it happen.

3. (Poland 2nd 2019/3- [🔗](#)) Two sequences of rational numbers $q_1, q_2, \dots$, and $r_1, r_2, \dots$ satisfy

$$q_1 r_1 = 3, \quad q_{n+1} = q_n^5 + q_n, \quad \text{and} \quad r_{n+1} = r_n^7 + r_n.$$

Can there exist positive integers m and n such that $q_m = r_n$?

Sketch: No. We work entirely in ν_3 , and think about the sizes of terms in $\{\nu_3(q_n)\}$. We have

$$\nu_3(q_2) = \nu_3(q_1) + \nu_3(q_1^4 + 1) \geq \nu_3(q_1) + \min \{4\nu_3(q_1), 0\}.$$

We do casework on the sign of $\nu_3(q_1)$, so we can get an equality case.

- Suppose $\nu_3(q_1) = 0$. Then $q_1 \not\equiv 0 \pmod{3}$,² so by Fermat's little, $q_1^4 + 1 \equiv (q_1^2)^2 + 1 \equiv 2 \pmod{3}$, so $\nu_3(q_1^4 + 1) = 0$ as well. By the first equality above, we get $\nu_3(q_2) = \nu_3(q_1)$. We can then induct to get that $\{\nu_3(q_n)\}$ is constant.
- If $\nu_3(q_1) > 0$, we have $4\nu_3(q_1) \neq 0$, so we have the equality case. Then $\nu_3(q_2) = \nu_3(q_1) + \min \{4\nu_3(q_1), 0\} = \nu_3(q_1)$. Induction again shows $\{\nu_3(q_n)\}$ is constant.
- If $\nu_3(q_1) < 0$, we similarly get $\nu_3(q_2) = 5\nu_3(q_1) < \nu_3(q_1)$. In this case, induction shows $\{\nu_3(q_n)\}$ is decreasing.

Similar statements hold for $\{\nu_3(r_n)\}$.

In $\nu_3(q_1) + \nu_3(r_1) = 1$, the two addends are integers, so WLOG $\nu_3(q_1) > 0 \geq \nu_3(r_1)$. Then $\nu_3(q_m) = \nu_3(q_1) > \nu_3(r_1) \geq \nu_3(r_n)$, so no two terms can be equal.

Remark: This is a kinda contrived example, but the theme of “ p -adic valuation of sequence is monotonic” appears often enough that I've made a separate section of problems for it. For a decreasing sequence of *integers*, if there's a lower bound, it's eventually constant. This is helpful for ν_p , because it's always an integer, and the ν_p of an integer is always ≥ 0 .

²Here, $a \not\equiv 0 \pmod{p}$ means a and p are relatively prime.

Problems

1. (India Regional 2013/Mumbai.5' [🔗](#)) Let a_1, b_1 , and c_1 be positive integers. Define

$$a_2 = \gcd(b_1, c_1), \quad b_2 = \gcd(c_1, a_1), \quad c_2 = \gcd(a_1, b_1),$$

and

$$a_3 = \text{lcm}(b_2, c_2), \quad b_3 = \text{lcm}(c_2, a_2), \quad c_3 = \text{lcm}(a_2, b_2).$$

Show that $a_2 \gcd(b_2, c_2) = a_3 \text{lcm}(b_3, c_3)$.

2. (Poland 1983/4 [🔗](#)) Positive integers a, b, c , and d satisfy $ab = cd$. Prove that

$$\frac{\gcd(a, c) \gcd(a, d)}{\gcd(a, b, c, d)} = a.$$

3. Let $n \geq 3$. Positive integers $a_1, \dots, a_n$ satisfy $\gcd(a_1, \dots, a_n) \text{lcm}(a_1, \dots, a_n) = a_1 \cdots a_n$. Prove that $a_1, \dots, a_n$ are pairwise relatively prime.

4. (Sozopol 2015/4.2 [🔗](#)) Let $a_0, a_1, \dots$ be a sequence of positive integers such that $a_n^2 \mid a_{n-1}a_{n+1}$ for all positive integers n . Suppose there exists some integer $k \geq 2$ such that a_1 and a_k are relatively prime. Show that $a_1 \mid a_0$. **Hint: 1**

5. (USEMO 2020/1 [🔗](#)) Which positive integers can be written in the form

$$\frac{\text{lcm}(x, y) + \text{lcm}(y, z)}{\text{lcm}(x, z)}$$

for positive integers x, y, z ? **Hint: 2**

6. (USEMO 2024/2 [🔗](#)) Let k be a fixed positive integer. For each integer $i \in \{1, 2, 3, 4\}$, let x_i and y_i be positive integers such that $\text{lcm}(x_i, y_i) = k$. Suppose that the four points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ are the vertices of a non-degenerate rectangle in the Cartesian plane. Prove that $x_1x_2x_3x_4$ is a perfect square. **Hint: 3**

Monotonicity problems

7. (ISL 2015/N1 [🔗](#)) Determine all positive integers M such that the sequence $a_0, a_1, a_2, \dots$ defined by

$$a_0 = M + \frac{1}{2} \quad \text{and} \quad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \dots$$

contains at least one integer term. **Hints: 4 5**

8. (Fake USAMO 2020/1 [🔗](#)) Let C be a positive integer, and $a_1, a_2, a_3, \dots$ an infinite sequence of positive integers. Suppose that for all positive integers n ,

$$a_{n+1} = \sqrt{a_n^3 - Ca_n}.$$

Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$. **Hints: 6 7**

9. (ISL 2024/N3 [🔗](#)) Determine all sequences $a_1, a_2, \dots$ of positive integers such that for any pair of positive integers $m \leq n$, the arithmetic and geometric means

$$\frac{a_m + a_{m+1} + \dots + a_n}{n - m + 1} \quad \text{and} \quad (a_m a_{m+1} \cdots a_n)^{\frac{1}{n - m + 1}}$$

are both integers. **Hints: 1 8**

10. (Thailand Online 2026/5 [🔗](#)) Determine all positive integers c for which the infinite sequence of positive integers $a_1, a_2, a_3, \dots$ such that $a_{n+1} = c \cdot \varphi(a_n)$ for all positive integers n is eventually periodic for all choices of a_1 . **Hints:** [9](#) [10](#) [11](#)
11. (IMO 2018/5 [🔗](#)) Let $a_1, a_2, \dots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Hints: [1](#) [12](#) [13](#)

Harder problems

12. (ISL 2007/N2 [🔗](#)) Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k . Prove that $b = A^n$ for some integer A . **Hint:** [14](#)
13. (APMO 2017/4 [🔗](#)) Call a rational number r *powerful* if r can be expressed in the form p^k/q for some relatively prime positive integers p, q and some integer $k > 1$. Let a, b, c be positive rational numbers such that $abc = 1$. Suppose there exist positive integers x, y, z such that $a^x + b^y + c^z$ is an integer. Prove that a, b, c are all powerful. **Hints:** [15](#) [16](#) [17](#)
14. (Balkan 2026/4 [🔗](#)) Let $n \geq 2$ be an integer. Initially, the number 1 is written n times on a blackboard. An operation consists of choosing two numbers a and b currently on the blackboard, not both zero, and replacing them with the numbers

$$\frac{(a-b)^2}{a+b} \quad \text{and} \quad \frac{4ab}{a+b}.$$

Determine all integers n for which it is possible, after a finite number of operations, for the number n to appear on the blackboard. **Hints:** [18](#) [19](#) [20](#)

15. (China TST 2019/1.4 [🔗](#)) Call a sequence of positive integers $\{a_n\}$ good if for any distinct positive integers m and n , one has

$$\gcd(m, n) \mid a_m^2 + a_n^2 \quad \text{and} \quad \gcd(a_m, a_n) \mid m^2 + n^2.$$

Call a positive integer a to be k -good if there exists a good sequence such that $a_k = a$. Does there exist a k such that there are exactly 2019 k -good positive integers? **Hints:** [21](#) [22](#) [23](#)

16. (TSTST 2019/7 [🔗](#)) Let $f : \mathbb{Z} \rightarrow \{1, 2, \dots, 10^{100}\}$ be a function satisfying

$$\gcd(f(x), f(y)) = \gcd(f(x), x - y)$$

for all integers x and y . Show that there exist positive integers m and n such that $f(x) = \gcd(m + x, n)$ for all integers x . **Hints:** [24](#) [25](#) [26](#)

17. (IMO 2010/3 [🔗](#)) Find all functions $g : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$(g(m) + n)(g(n) + m)$$

is a perfect square for all $m, n \in \mathbb{N}$. **Hints:** [27](#) [28](#) [29](#) [30](#)

Hints

1. Telescope the conditions.
2. To prove odd impossible, take ν_2 .
3. Do casework on number of x_i such that $\nu_p(x_i) = \nu_p(k)$.
4. Show ν_2 of something is strictly decreasing.
5. Look at $\nu_2(a_i - 3/2)$.
6. Show $a_n \mid C$.
7. If $a_n > \nu_p(C)$, show $\{\nu_p(a_i)\}_{i \geq n}$ is decreasing.
8. Divide geometric mean conditions for $(1, d+1)$ and $(2, d+2)$, and note d is arbitrary.
9. Show $\nu_p(c) > 1$ is impossible due to eventual periodicity.
10. Show that c having two distinct prime divisors is impossible when $a_1 = c$.
11. Show c divisible by an odd prime > 3 is impossible.
12. Simplify the difference of terms. If you subtract 1, the numerator factors.
13. Consider the sequences $\{\gcd(a_1, a_n)\}$ and $\{\text{lcm}(a_1, a_n)\}$.
14. Let $e := \nu_p(b)$, and take $k = p^{e+1}$.
15. This is similar to BAMO 2018/4.
16. If $\nu_p(a) > 0$, WLOG $\nu_p(b) < 0$. Show $\nu_p(c) < 0$.
17. If $\nu_p(a) > 0$, show $y\nu_p(b) = z\nu_p(c)$.
18. What are the invariants?
19. The sum is invariant, and all numbers are non-negative.
20. At the start, all $\nu_p \leq 0$; at the end, all $\nu_p > 0$.
21. Prove a sequence is good iff $n \mid a_n^2$ and $a_n \mid n^2$.
22. Combine $(n, 2n)$, $(2n, 3n)$, $(n, 3n)$ to get $n \mid 2a_n^2$.
23. Choose (m, n) to try getting an equality case for the sum rule.
24. Take x_p such that $e_p := \nu_p(f(x_p))$ is maximal.
25. Choose n such that $\nu_p(n) = e_p$.
26. Use CRT to choose m such that $\nu_p(x - x_p) = \nu_p(x + m)$.
27. Show g is injective mod p .
28. Equivalently, show $p \mid g(b) - g(a)$ implies $p \mid a - b$.
29. Choose m to force $\nu_p(g(a) + m)$ and $\nu_p(g(b) + m)$ to both be odd.
30. By injectivity, $p \nmid g(n+1) - g(n)$.

Sketches

1. Let $a, b, c := \nu_p(a_1), \nu_p(b_1), \nu_p(c_1)$ respectively. WLOG $a \leq b \leq c$. Then $\nu_p(a_2) = b$ and $\nu_p(b_2) = \nu_p(c_2) = a$. Similarly, $\nu_p(a_3) = a$ and $\nu_p(b_3) = \nu_p(c_3) = b$. So $\nu_p(a_2 \gcd(b_2, c_2)) = a + b = \nu_p(a_3 \operatorname{lcm}(b_3, c_3))$.
2. Let $w, x, y, z := \nu_p(a), \nu_p(b), \nu_p(c), \nu_p(d)$ respectively. Want to show $\min\{w, y\} + \min\{w, z\} = w + \min\{w, x, y, z\}$. Per $ab = cd$, RHS here is $w + \min\{w, y + z - w, y, z\}$. Casework on relative sizes of $\{w, y, z\}$ finishes: symmetry of y and z rule out half the cases.
3. Condition is $\min_i \{\nu_p(a_i)\} + \max_i \{\nu_p(a_i)\} = \sum_i \nu_p(a_i)$. WLOG, suppose $\nu_p(a_1)$ is the minimum and $\nu_p(a_n)$ is the maximum. Then this is $\sum_{1 < i < n} \nu_p(a_i) = 0$, so all the middle a_i (of which there are ≥ 1 , because $n \geq 3$) must all be 0. So the minimum must also be 0. This means, for each p , there's only one a_i such that $p \mid a_i$; this proves the result.
4. Take $p \mid a_1$ and add the inequalities $2\nu_p(a_n) \leq \nu_p(a_{n+1}) + \nu_p(a_{n-1})$ for $n = 1, 2, \dots, k-1$ to get $\nu_p(a_1) + \nu_p(a_{k-1}) \leq \nu_p(a_0) + \nu_p(a_k)$. But $a_1 \perp a_k$ so $p \nmid a_k$, so $\nu_p(a_1) \leq \nu_p(a_0)$.
5. Evens: $(1, \frac{n}{2}, 1)$. We show odd is impossible. Let $a, b, c := \nu_2(x), \nu_2(y), \nu_2(z)$, respectively. If expression was odd, we have $\min\{\max\{a, b\} + \max\{b, c\}\} = \max\{a, c\}$. By symmetry, WLOG $a \leq c$. Do casework on relative sizes of $\{a, b, c\}$. For the case where b isn't between a and c , we have to go back to the original expression to handle the cancellation properly.
6. From parallelogram, $x_1 + x_3 = x_2 + x_4$ and $y_1 + y_3 = y_2 + y_4$. From equal-length diagonals, $(x_1 - x_3)^2 + (y_1 - y_3)^2 = (x_2 - x_4)^2 + (y_2 - y_4)^2$; combine with the first two equations to get $x_1x_3 + y_1y_3 = x_2x_4 + y_2y_4$. Do casework on there being $\{0, 1, 2, 3, 4\}$ values of i such that $\nu_p(x_i) = \nu_p(k)$, show in each case $2 \mid \nu_p(x_1x_2x_3x_4)$.
7. It's $M \geq 2$. We show $\nu_2(a_i - 3/2)$ decreases until it's 0. Write $M = 2^e k + 1$. If $e = 0$, then a_1 is an integer and we're done. Otherwise $a_1 = M^2 + M/2 = 2^{2e}k^2 + 2^{e+1}k + 2^{e-1}k + 3/2$, and $\nu_2(a_0 - 3/2) = \nu_2(M - 1) = e > e - 1 = \nu_2(a_1 - 3/2)$, as desired.
8. Note $2\nu_p(a_{n+1}) = \nu_p(a_n) + \nu_p(a_n^2 - C)$. We show $a_n \mid C$. Suppose otherwise: then $\nu_p(a_n) > \nu_p(C)$, and RHS is $\nu_p(a_n) + \nu_p(C)$, and we must have $\nu_p(C) < \nu_p(a_{n+1}) < \nu_p(a_n)$. This would mean $\{\nu_p(a_i)\}_{i \geq n}$ is decreasing, while $\nu_p(C)$ is constant, contradiction.
9. We show sequence is constant. Scale to remove common factors, then take $p \mid a_1$. Take $(m, n) = (1, d+1)$ and $(2, d+2)$ and divide the geometric mean conditions to get $d \mid \nu_p(a_{d+1}) - \nu_p(a_1)$. For arbitrarily large d , RHS must be zero, so $\nu_p(a_{d+1}) = \nu_p(a_1) > 0$, so the sequence is eventually 0 mod p . But from $(m, n) = (i, p+i)$, the sum of every p consecutive terms is 0 mod p , so the whole sequence is 0 mod p , contradiction.
10. Note $c \in \{1, 2, 3\}$ work for size reasons; we claim that's it. Note $\nu_p(\varphi(n)) \geq \nu_p(n) - 1$, so $\nu_p(a_{n+1}) - \nu_p(a_n) \geq \nu_p(c) - 1$. If $\nu_p(c) > 1$, then $\nu_p(a_n)$ is increasing, and can't be eventually periodic. Similarly, if c has more than one distinct odd prime divisor, then taking $a_1 = c$, we get $\nu_2(a_n)$ is increasing, and can't be eventually periodic. If c is an odd prime > 3 , then eventually all prime divisors of a_n are at most c ; from that point it's increasing, and can't be eventually periodic. Finally, if $c = 6$, taking $a_1 = 6$ gives an increasing sequence, which can't be eventually periodic.

11. For $n > N$, the difference $d_n := \frac{a_{n+1}}{a_1} - \frac{a_n}{a_1} + \frac{a_n}{a_{n+1}}$ is an integer. Then

$$\mathbb{Z} \ni d_n - 1 = \frac{a_{n+1}^2 - a_n a_{n+1} + a_n a_1 + a_1 a_{n+1}}{a_1 a_{n+1}} \implies a_1 a_{n+1} \mid (a_{n+1} - a_1)(a_{n+1} - a_n).$$

Note this means $\min\{\nu_p(a_1), \nu_p(a_n)\} \leq \nu_p(a_{n+1}) \leq \max\{\nu_p(a_1), \nu_p(a_n)\}$. Hence $\gcd(a_1, a_n) \mid a_{n+1} \mid \text{lcm}(a_1, a_n)$. So $\gcd(a_1, a_n)$ is increasing, $\text{lcm}(a_1, a_n)$ is decreasing, and both are bounded by a_1 , so they're eventually constant, and so is their product $a_1 a_n$.

12. Let $e := \nu_p(b)$ and take $k = p^{e+1}$. This means $\nu_p(b - a_k^n) \geq e + 1$. If $\nu_p(b) \neq \nu_p(a_k^n)$, then LHS is $\min\{\nu_p(b), \nu_p(a_k^n)\} \leq \nu_p(b) = e$, leading to $e \geq e + 1$, contradiction. So $\nu_p(b) = n\nu_p(a_k)$, and hence $n \mid \nu_p(b)$.
13. If $\nu_p(a), \nu_p(b), \nu_p(c)$ are all equal, then by $abc = 1$ they're all zero. Take $\nu_p(a) > 0$, WLOG $\nu_p(b) < 0$. If $\nu_p(c) \geq 0$, then $\nu_p(a^x + b^y + c^z) = \nu_p(b^y) < 0$, so it can't be an integer; thus $\nu_p(c) < 0$. Similarly, if $y\nu_p(b) \neq z\nu_p(c)$, we get $\nu_p(a^x + b^y + c^z) = \min\{y\nu_p(b), z\nu_p(c)\} < 0$. Because $y\nu_p(b) = z\nu_p(c)$, we have some d, y', z' such that $\nu_p(b) = -dz'$, $\nu_p(c) = -dy'$, and $\nu_p(a) = d(y' + z')$, which makes it powerful.
14. Only powers of 2: construct by pairing 1s, then pairing 2s, etc. Otherwise, some odd $p \mid n$. At the start, all numbers have $\nu_p \leq 0$. At the end, because sum is invariant and all numbers are non-negative, the final state is one n and $n - 1$ zeroes, where all numbers have $\nu_p > 0$. So, at some point, we erase some a and b , at least one with $\nu_p \leq 0$, to get two numbers, *both* with $\nu_p > 0$. But this is impossible: if both a and b have $\nu_p \leq 0$, then $\nu_p\left(\frac{4ab}{a+b}\right) \leq 0$, and if $\nu_p(a) \leq 0 < \nu_p(b)$, then $\nu_p\left(\frac{(a-b)^2}{a+b}\right) \leq 0$.
15. We show sequence is good iff $n \mid a_n^2$ and $a_n \mid n^2$; equivalently $\nu_p(n)/2 \leq \nu_p(a_n) \leq 2\nu_p(n)$. (If so, there's $2\nu_p(i) - \lfloor \nu_p(i)/2 \rfloor$ choices for $\nu_p(a_i)$, which is never divisible by 3, so there can't be 2019.) Combine $(m, n) = (n, 2n), (2n, 3n), (n, 3n)$ in statement to get $n \mid 2a_n^2$, equivalently $\nu_p(n) \leq \nu_p(2) + 2\nu_p(a_n)$. Choose m such that $\nu_p(n) \ll \nu_p(m)$ and, using previous, $\nu_p(a_n) < (\nu_p(m) - 1)/2 < \nu_p(a_m)$. Thus $\nu_p(n) = \nu_p(\gcd(m, n)) \leq \nu_p(a_m^2 + a_n^2) = \nu_p(a_n^2)$, and $\nu_p(a_n) = \nu_p(\gcd(a_m, a_n)) \leq \nu_p(m^2 + n^2) = \nu_p(n^2)$, as desired.
16. Take x_p such that $e_p := \nu_p(f(x_p))$ is maximal. Then $\nu_p(f(x)) = \nu_p(\gcd(f(x_p), f(x))) = \nu_p(\gcd(f(x_p), x - x_p)) = \min\{\nu_p(x - x_p), e_p\}$. Taking $n := \prod p^{e_p}$ and by CRT choose $m \equiv -x_p \pmod{p^{e_p}}$ for all p . Then $\min\{\nu_p(x - x_p), e_p\} = \min\{\nu_p(x + m), \nu_p(n)\}$, which works.
17. Only $g(n) = n + C$ works. We'll show g is injective mod p . Suppose $p \mid g(b) - g(a)$. We choose m to force $\nu_p(g(a) + m)$ and $\nu_p(g(b) + m)$ to be odd: if $p^2 \mid g(b) - g(a)$, take $m \equiv p - g(a) \pmod{p^2}$, otherwise take $m \equiv -g(b) \pmod{p^3}$. This shows $p \mid g(m) + a$ and $p \mid g(m) + b$, so $p \mid a - b$, done. From injectivity, we get $g(n + 1) - g(n) = \pm 1$. Taking $(m, n) = (n, n + 1)$ as a perfect square rules out the -1 case.

Acknowledgements

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