Easy

General

E1 Evaluate: $\frac{1}{(23)(25)} + \frac{1}{(25)(27)} + \frac{1}{(27)(29)} + \cdots + \frac{1}{(67)(69)}$. $\left[\frac{1}{69}\right]$

E2 The product of two numbers is 616 054 275 while their greatest common factor is 84 681. How many factors does the least common multiple of the two numbers have? [12]

E3 Consider a rectangle whose vertices are located at $T(2,-6)$, $E(13,-6)$, $F(13,4)$, and $I(2,4)$. How many points with integer coordinates are located in the interior of rectangle $TEFI$? [90 points]

E4 Evaluate $\frac{1}{25} + \frac{2}{625} + \frac{3}{15\,625} + \cdots$. $\left[\frac{25}{576}\right]$

E5 Eight teams joined a double round-robin tournament, i.e. each team plays against the other teams twice. Thus, each team is scheduled to play fourteen games, where each game ends in either a win or loss. At the end of the tournament, the four teams with the best records will join the players. What is the best possible record for a team not making the playoffs? Write your answer in the form w-ℓ where w is the number of wins and ℓ is the number of losses of the said team. [12-4]

E6 Jackie has an infinite army of monkeys with typewriters and she has ordered them to type out 2012 copies of Shakespeare's *Hamlet* in 25 hours. She knows that every kth monkey is only $\frac{3}{4}$ as efficient as the $(k-1)$th one. How many copies of *Hamlet* can the first three monkeys finish in 400 hours? [18 611 copies]

Algebra and Probability

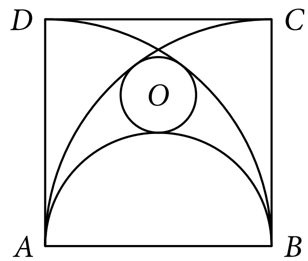
A1 Suppose you have 8 balls where 2 are blue, 2 are red, 3 are black, and 1 is white. You pick a ball randomly without replacement. What is the probability that you get your second red ball in your third pick? $\left[\frac{3}{13}\right]$

A2 Jasmine is a nail polish lover. She has a total of 90 bottles of nail polish! Of those, 45 bottles are of Top Speed and 75 bottles are of varying shades of red. If she randomly scrounges around her drawer of nail polish and gets one bottle at a time, what is the probability that the first bottle she gets is a bottle of red polish, the second to be a bottle of Top Speed, and the third to be both, without any replacement? $\left[\frac{1501}{11748}\right]$

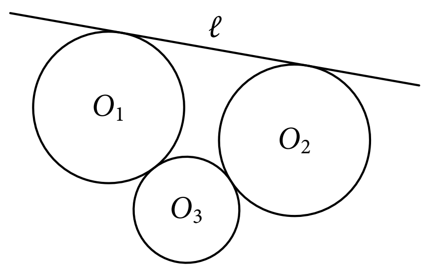
A3 Badette is thinking of a polynomial of degree 2. Alec guesses what polynomial it is. First he guesses $f(x) = 3x^2 + 6x - 1$. Badette replies by saying his guess is greater than the polynomial she's thinking. Again Alec guesses $f(x) = x^2 + 2x - 9$. Badette replies by syying his guess is less than the polynomial she's thinking. If she gives a hint that $f(-3) = -4$ and Alec is able to get the polynomial, solve for $f(7)$. [44]

Geometry and Trigonometry

G1 In the figure below, $ABCD$ is a square with side R. The circle centered at O is tangent to the arcs AC, with center B; BD, with center A; and semicircle BD, with diameter BD. If the radius of the circle is r, find $\frac{r}{R}$. $\left[\frac{1}{6}\right]$



G2 Two circles O_1 and O_2 have radii equal to 10 cm and both of them are tangent to line ℓ. O_3 is tangent to both O_1 and O_2 as shown in the figure; also, the centers of O_1 and O_2 are 25 cm apart. Solve for the distance between the center of O_3 and line ℓ if O_3 has a radius of 7 cm. $\left[10+\dfrac{\sqrt{531}}{2}\text{ cm}\right]$



G3 Evaluate: $\sec\left(\dfrac{1}{2}\cos^{-1}\dfrac{1}{8}\right)$. $\left[\dfrac{4}{3}\right]$

Number Theory and Combinatorics

N1 Find all ordered pairs (m,n) of positive integers that satisfy the equations $p = mn$, $q = 5n + m$, where p and q are prime numbers. $[\{(1,2)\},\{(2,1)\}]$

N2 Let M be the set of integers from 1 to 6000 not divisible by 2, S be the set of integers from 1 to 6000 not divisible by 3, and A be the set of integers from 1 to 6000 not divisible by 5. Consider $m \in M$, $s \in S$, and $a \in A$. How many ordered triples (m,s,a) are there if m, s, and a are all equal? [3200]

N3 Patty adds numbers starting from 1. After a while, Geo comes and asks Patty what number it is she is at. Geo then starts to add numbers starting from Patty's last number, after which, Patty and Geo add until 2012. Jayson wants to figure out what number it is that Geo started adding from knowing it divides the difference between Patty's sum and Geo's sum, however, he recognizes there may be more than one such number. Determine the mean of all numbers Jayson suspects Geo started adding from. [1006]

Average

General

E1 When Conan went home, he noticed that his analog clock showed 12:31:23 (HH:MM:SS), the first three digits are the same as the last three digits. After some time he noticed that it became 12:33:21, a palindrome, i.e. the last three digits are the reverse of the first three digits. Conan then begins to count the number of the first form or the second form that appear from 00:00:00 to 23:59:59. What number should Conan get if he counts correctly? [176]

Algebra and Probability

A1 Determine the sum of all real and complex solutions to the equation $x^2 + 2|x| - 6x + 15 = 0$. (Note: the modulus of a complex number $x = a + bi$ is $|x| = \sqrt{a^2 + b^2}$.) [6]

A2 Mia and Anjo's friends want to play a card game using a deck of 40 cards. In order to deal faster to their 6 friends who are seated in a circle, dealers Mia and Anjo split the deck randomly between the two of them and not necessarily in half, such that it is possible that Mia deals clockwise while Anjo deals counterclockwise. Determine the probability that both dealers will deal their last card to the same person (not necessarily the same person they started with). $\left[\frac{1}{3}\right]$

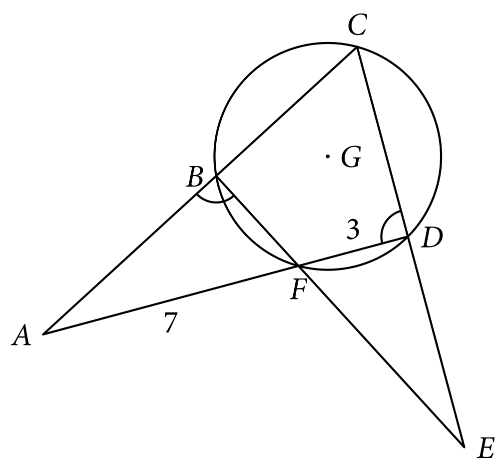
A3 Find the solution set of the equation $4\cos^3 2z - \cos^2 z + 3 = 16\cos^2 z \sin^2 z - \sin^2 z$, where $z \in (0, 2\pi)$.
$\left[z \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}\right]$

Geometry and Trigonometry

G1 Let MSA be a triangle with $MS = 13$, $MA = 14$, and $SA = 15$. Find the value of $\frac{\sin M + \sin S}{\cos M + \cos S} + \frac{\sin M + \sin A}{\cos M + \cos A} + \frac{\sin S + \sin A}{\cos S + \cos A}$. $\left[\frac{21}{4}\right]$

G2 Bobert is folding a piece of short bond paper with dimensions 8.5 in by 11 in. First, he folds along a diagonal, forming two overlapping triangles. Then, without opening the figure, he makes a fold on one of the triangles so that the topmost layer is an isosceles triangle sharing a vertex with the original triangle. Bobert then wonders what could be the area of the region which is three leaves thick. What is the area? $\left[\frac{1445}{176}\text{ in}^2\right]$

G3 In the figure below, let the marked angles be congruent, AF and FD as written, and G the center of the circle that contains points B, C, and D. If $\angle FCD = 30°$, what is the distance between A and G? $\left[\sqrt{79}\text{ units}\right]$



Number Theory and Combinatorics

N1 TEAM-ENERGY decides to list all 4-digit positive integers $\overline{abcd}$ satisfying the following:

(a) The first three digits, a, b, and c, form a geometric progression.

(b) The first, second, and fourth digits, a, b, and d, form an arithmetic progression.

If Mojo, a member of the team, adds all the listed numbers, what sum would he get? [82 900]

N2 We define the "continued fraction expansion" of a number $x = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}\cdots$ as $[a_0, a_1, a_2, a_3, \ldots]$

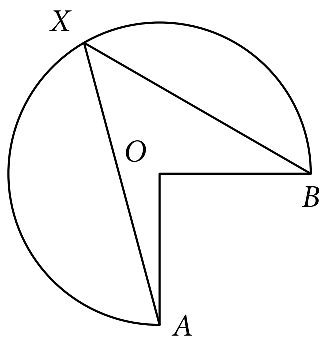
where a_n, for any nonnegative integer n, are the greatest integers. We denote $[1, \overline{2, 3}] = 1 + \frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\cdots$, i.e., if there are repeating entries, similar to a rational number in decimal form. What is the sum of the entries of the "continued fraction expression" of $3 - \sqrt{5}$? [8]

N3 Goma hates any number with 6 or 9, except when they appear consecutively as 69. He lists all natural numbers side-by-side, i.e. 1 2 3 4 5 7 8 10 . . ., but not the numbers he hates. Let a_n be the nth number in the list, and d_n the nth digit in the list. What is the value of $\left\lfloor \frac{a_{1974}}{d_{1974}} \cdot \frac{d_{1974}}{a_{1974}} \right\rfloor$? [0]

Difficult

General

E1 Dale has a frame in the shape of a 270° circular sector with center O and radius 2 cm. He attaches two ends of an elastic band to the points A and B and stretches the hand to a point X on the circumference as shown. What is the probability that the area enclosed by the elastic band is greater than or equal to $\sqrt{6}\,\text{cm}^2$? $\left[\frac{2}{9}\right]$



E2 Stan and Ollie play a game on a 2012×2012 grid, whose bottom left and upper right corners have coordinates $(0, 0)$ and $(2012, 2012)$ respectively. A stone is then placed on a point (x, y), $x, y \in \mathbb{Z}$, $0 \le x, y \le 2011$. Stan begins by moving the stone either up or right (but not both) to a point with integral coordinates as long as the stone stays in the grid. Ollie then plays with the same rules and they alternately make moves until one player moves the stone to the point $(2012, 2012)$, and he is declared winner. Let $S = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ be the set of all starting points for the stone where Ollie wins with optimal play. If $A = x_1 + x_2 + \cdots + x_n$, the sum of all the **abscissas** of S, and $B = y_1 + y_2 + \cdots + y_n$, the sum of all the **ordinates** of S, determine $A + B$. Let $z = 2011$, and leave your answer in terms of z. $[2\,024\,072z]$

Algebra and Probability

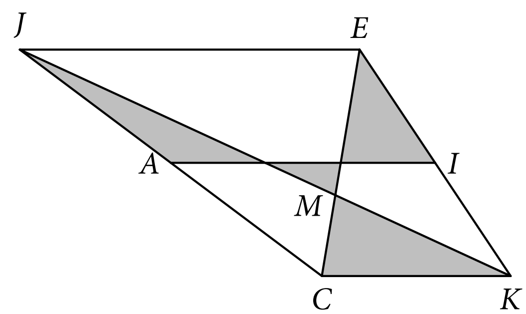
A1 Find all ordered pairs $(x, y) \in \mathbb{R}$ such that the following equations hold:

$$\sqrt{x}\left(\frac{x+y+1}{x+y}\right) = 2, \qquad \sqrt{y}\left(\frac{x+y-1}{x+y}\right) = 3$$

$$\left[\left(\frac{7+4\sqrt{3}}{4}, \frac{21+12\sqrt{3}}{4}\right)\right]$$

Geometry and Trigonometry

G1 In trapezoid $JCKE$, $JE > CK$, $\overline{JE}$ is parallel to $\overline{CK}$, lines $\overleftrightarrow{CE}$ and $\overleftrightarrow{JK}$ intersect at M, and $\overline{AI}$ is the median. If the area of CMK is a and the area of JME is b, find the area of the shaded region in terms of a and b. $\left[\dfrac{7a+b}{4}\right]$



Number Theory and Combinatorics

N1 We define the XOR ($\oplus$) of two numbers a and b as follows:

(a) Take the binary representations of a and b.

(b) Add each place value in the binary representation without "carrying".

For example, to get $6 \oplus 9$ we have:

$$\begin{array}{rr|rrrrr} & 6 & & 0 & 1 & 0 & 1 \\ \oplus & 9 & + & 1 & 0 & 0 & 1 \\ \hline & 12 & & 1 & 1 & 0 & 0 \end{array}$$

Let $f(x,y) = \begin{cases} xy, & \text{if } x \oplus y = 15 \\ 0, & \text{otherwise} \end{cases}$. Find the base-10 value of $\sum_{i=0}^{31}\sum_{j=0}^{31} f(i,j)$. [9056]