Easy

General

E1 Princess Judy is being courted by four royalties: the Duke, the Lord, the Prince, and the King (in increasing hierarchy), who come from various lands. In an effort to woo her as well as level the playing field, the four royalties kept their names and titles in secret, and told Princess Judy to choose based on what the royalties did for her love. Judy knows the following:

a) Homer was not the one who painted intricate fractals for Judy.

b) The Duke wooed the princess either directly before or directly after the King. Allan, who wrote a math poem, is neither royalty.

c) The one who gave 100! flowers to Judy is the one with the title of Lord and is not the last to woo Judy.

d) The second to woo Princess Judy rapped the digits of π to prove his love to Princess Judy. This person has a title two levels higher than Rency.

e) Alexis is not the King, nor is he the first nor last to woo Judy.

If Princess Judy will date the last royalty who wooed her, who will Princess Judy choose and what is their ttle? [Prince Allan]

E2 The Manila Bulletin has installed 77 CCTV cameras throughout their printing press. The CCTVs are numbered from 1 to 77 and are used as follows:

- During the first hour, all CCRVs are operating.
- During the second hour, the even-numbered CCTVs are turned off.
- During the third hour, the status of the CCTVs that are numbered with multiples of 3 is reversed, that is, the ones operating are turned off and the ones off are turned on.
- During the fourth hour, the status of the CCTVs that are numbered with multiples of 4 is reversed, and so on.
- After the 77th hour, the process repeats.

During the 77th hour, how many CCTVs are turned off? [69]

E3 In $\triangle MSA$, D is a point on SA such that MD is the bisector of $\angle M$. Given that $\angle S = 2\angle A$ and $MS = AD$, what is the measure of $\angle M$? [72°]

E4 Evaluate $(1 + \tan 1^\circ)(1 + \tan 2^\circ)(1 + \tan 3^\circ)\cdots(1 + \tan 45^\circ)$. [$2^{23}$]

E5 The Manila Bulletin has a special number encoder for confidential message sending. The machine takes a number, goes through each digit and replaces it with the remainder of the cube of that digit when divided by 10. Then it outputs this new number. For example, if we input 2369 in the machine, we get 8769. Let $n^3 = 384\,240\,583$. What number will the machine output if we feed n into it? [383]

E6 Kyle is trying to heal his friend Gordon, who is in a critical condition, by using his sister's power. He, carrying his sister on his back, is at the origin and is moving towards the hospital where Gordon is. He is moving either one unit northward or eastward to get to the hospital which is located at $(6, 6)$. However, their enemies Yumi and Shoka are stationary at $(4, 5)$ and $(5, 4)$ respectively. Assuming that Kyle's chances of moving northward or moving eastward are equal, what is the probability that the siblings will reach the hospital without encountering either Yumi or Shoka? $\left[\frac{2}{11}\right]$

Algebra and Probability

A1 Let D be the difference between the 15th coefficient of $(2x-3)^{25}$ and the 10th coefficient of $(2x+3y)^{25}$, expressed as a product of prime factors. Find the sum of the exponents of the factors of D. [30]

A2 Deci, Carol, and Albie played a card game using a standard deck of cards by removing the 2's, 3's, 4's, and the Queen of Spades. They start a round by shuffling and distributing the remaining cards evenly among them. An Ace is 4 points, a King is 3 points, a Queen is 2 points, and a Jack is 1 point. If $\frac{a}{b}$ is the probability that Deci won exactly 2 rounds out of 3 by having the most number of points such that a, b are relatively prime, what is the remainder when $a+b$ is divided by 100? (A tie is considered a win, e.g., Deci and Carol has 16 points while Albie has 6 points.) [11]

A3 Find all pairs of integers (a, b) such that $\frac{a+b}{a^2-ab+b^2} = \frac{3}{7}$. $[(4,5),(5,4)]$

Geometry and Trigonometry

G1 Let $ABCD$ be a rectangle. Point M is chosen inside $ABCD$ such that $BM = 8$ and $\angle ABM = 60^\circ$. If $AB = 15$ and $BC = 11$, find MD. $\left[\sqrt{290-88\sqrt{3}}\right]$

G2 Find the area of the rhombus formed by the intercepts of the parabolas $y = bx^2 - 2$ and $y = -bx^2 + 2$, given that the distance between the two x-intercepts is equal to the length of the side of the rhombus. $\left[\frac{2\sqrt{3}}{3}\text{ units}^2\right]$

G3 In trapezoid $ABCD$, $BC \parallel AD$. If $AB = 13$, $BC = 15$, $CD = 14$, and $AD = 30$, find the area of trapezoid $ABCD$. [252]

Number Theory and Combinatorics

N1 How many sequences of consecutive integers sum to 1 000 000? [14]

N2 Define a *cipher* to be a mapping of letters to transform a string of letters into another string. For example, the cipher $F \to F$, $E \to A$, $A \to R$, $R \to E$ transforms $FEAR$ into $FARE$. Note that applying this cipher two more times to $FARE$ gives back the original string $FEAR$.

If Ω is the set of all ciphers containing the letters $M, A, T, H, R, N, G, I, B, Y$, how many ciphers are there from Ω such that they will need to be applied exactly 30 times to the string $MATHIRANG$ $MATHIBAY$ so that the result will be the original string only at the 30th application and not earlier? [2520]

N3 Find the least natural number k such that $27! \cdot k$ is a perfect cube. Express your answer in prime factored form. $[2 \cdot 3^2 \cdot 11 \cdot 13 \cdot 17^2 \cdot 19^2 \cdot 23^2]$

Average

General

E1 The *floor function* of x is defined as $\lfloor x \rfloor = n$ whenever n is an integer such that $n \le x < n+1$. Find the value of $\lfloor 2^k \rfloor$ if $\dfrac{1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots}{1-\frac{1}{2^k}+\frac{1}{3^k}-\frac{1}{4^k}+\cdots} = 2015.$ [2]

Algebra and Probability

A1 Define the *fancy moustache product* operation $\rfloor A \int$ on a set A of positive integers such that if $A = \{a_0, a_1, a_2, \dots, a_n\}$ where $a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n$, then $\rfloor A \int = \sum_{i<j} a_i a_j$.

For example, if $B = \{1,2,3,6\}$, then $\rfloor B \int = 1(2)+1(3)+1(6)+2(3)+2(6)+3(6) = 2+3+6+6+12+18 = 47$. Consider the set $G = \{3,7,11,15,19,\dots,399\}$. What is $\rfloor G \int$? [199 318 350]

A2 A team of contestants has decided to join the 27th Mathirang Mathibay. Their trainer calculates that the probability that they will advance to the Survival Round, a, is too low and that they need to do more training, thereby increasing the probability by b for every day they study. At any given day, it is equally likely that they will study as not. Knowing this, the trainer determines that if he gives them MSA Academic Advancement Institute reviewers six days before the contest, the probability that they will advance is 0.45. If he gives them the reviewers twelve days before, however, they would have 0.6 chance of advancing. If $a + b$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime, find $m + n$. [27]

A3 Let m be a real number such that $|x^2 - 5x + 6| = mx$ has 4 **distinct** real roots $\alpha, \beta, \delta, \gamma$. Find k such that k is the least natural number satisfying $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\delta^2} + \frac{1}{\gamma^2} \le k$. [12]

Geometry and Trigonometry

G1 Harry Potter is studying an ancient depiction of the Deathly Hallows in his Arithmancy class. The image is of a circle inscribed in $\triangle ABC$, with points of tangency marked at D, E, F. A line is drawn from A to E, to signify the Elder Wand. G is the line's intersection with the circle. In the ancient image, Harry measures what he marks as $\angle A$ to be equal to a magical value so that $\cos A = \frac{383}{385}$. He also knows that the ratio $AG : GE = 1 : 7$, and that $BD = \frac{3\sqrt{2}}{2}$ units, $FC = \frac{\sqrt{2}}{2}$ units. What is the length of the Elder Wand, AE, in the image?* [64 units]

G2 Let $\theta = \sum_{k=0}^{2015} \sec^{-1} \frac{(k+1)(k+2)}{1+\sqrt{k(k+1)(k+2)(k+3)}}$. Find the value of $\tan^2\theta$. [4 068 288]

G3 Nika the Ant lives on one point on the edge of the larger base of a right frustum of a cone: a space figure with two circular bases formed by slicing a right circular cone parallel to the base. This morning she feels like visiting her *beybs* ant who is waiting for her at a diametrically opposite point on the edge of the smaller base. If the radii of the bases are 1 cm and 2 cm and the lateral height of the frustum is 3 cm, find the length of the shortest path Nika can take to visit her *beybs* and return home before dark. [$6\sqrt{3}$ cm]

Number Theory and Combinatorics

N1 Find the sum of all integers n such that $\sqrt{\frac{25n-12}{n+20}}$ is rational. [330]

* The figure is impossible. (Following from the answer, the radius of the incircle should have been $2\sqrt{3}/3$ and not 28.) In the figure, the following are collinear: $\{A,D,B\}$, $\{A,G,E\}$, $\{A,F,C\}$, $\{B,E,C\}$.

N2 For $r, s > 0$ define the symbol the *s-incremental factorial of r* to be the product

$$R!_{(s)} \stackrel{\text{def}}{=} r(r-s)(r-2s)(r-3s)\cdots(r-ms),$$

where m is a natural number satisfying $r - ms > 0$ and $r-(m+1)s \le 0$. If $5!_{(1/403)}13!_{(1/155)}31!_{(1/65)}$ is a fraction $\frac{m}{n}$ in lowest terms, how many times does 2015 divide n? [3829]

N3 What is the sum of all possible integer values of k such that $18k-8$ and $50k-275$ are both perfect squares? [124]

Difficult

General

E1 Let a, b, and c be positive real numbers. Find the maximum possible value of $\dfrac{abc(a+b+c)}{(a+b)^2(b+c)^2}$. $\left[\frac{1}{4}\right]$

E2 Find $A^B + C$, if $A = \sqrt{11+\sqrt{13+2\sqrt{15+3\sqrt{17+4\sqrt{\cdots}}}}}$, $B = \sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\sqrt{5}}}}}$, and $C = \sqrt{10+\sqrt{12+3\sqrt{14+5\sqrt{16+7\sqrt{\cdots}}}}}$. [260]

Algebra and Probability

A1 A sequence $x_1, x_2, \ldots$ satisfies the following equations:

$$\begin{cases} x_1 = 1 \\ \dfrac{x_n^2}{x_{n+1}^2} = x_1 + \dfrac{x_n^2}{n!} \cdot \dfrac{n + \frac{1}{n+1}}{n(n+1)}, & n > 1 \end{cases}$$

Find $\dfrac{x_{2015}^2}{2x_{2015}^2 - 1} + 2015!$. [4031(2015!)]

Geometry and Trigonometry

G1 You can draw an ugly spiral with center C by drawing a series of semicircles starting from C with increasing radii, i.e., one end of the first semicircle is at C and the other is the starting point of the second semicircle with a larger radius which ends at the starting point of the third semicircle with an even larger radius and so on and so forth. Call it a "really ugly spiral" if the first semicircle is of radius 1 and every increment in the radii is 1. Given a circle of center O and radius $2015\sqrt{\frac{4\,060\,226}{4\,060\,225}}$, if T is the length of the trace of the really ugly spiral with center at O which ends once it intersects the circle, what is $\frac{T}{\pi}$? $\left[\frac{2015 \cdot 1007\pi}{4}\right]$

Number Theory and Combinatorics

N1 Magic 89.9 is giving away free movie tickets to an advanced screening of ***Mathirang Mathibay: the Survival Rounds*** if any one of their listeners can answer this "simple" question. "DJ Brian James has a list containing k numbers, which contains the possible positive divisors of 27!. DJ Brian loves the number 22, however, and has replaced all but the multiples of 22 on the list with "MAGIC". If MAGIC appears a times in the list, find the value of $c+d$, given that $\frac{a}{k} = \frac{c}{d}$, where c, d are relatively prime." [49]