Prize Question

Given a set of integers $\mathcal{S}$ such that $\mathcal{S} = \{101, 1001, 10\,001, 100\,010\,001, 1\,000\,100\,010\,001, \ldots\}$ where all integers after 10 001 are formed by appending 0001 to the previous integer. Determine the number of prime elements of $\mathcal{S}$. Show full solution and justify every statement.

Oral Round

Tier 1

T1-1 *(10s)* Find the range of $g(x) = \dfrac{\sin x \cos x}{\csc x + \csc x \cot x} + \dfrac{\sin^4 x}{\sin x + \cos x}$. $\left[[-1,1] \setminus \left\{\pm\dfrac{\sqrt{2}}{4}\right\}\right]$

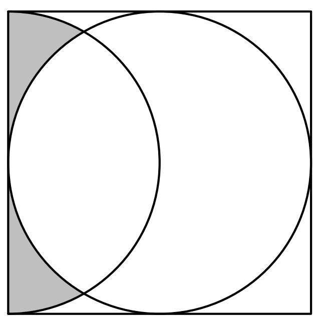
T1-2 *(15s)* In a party, all $15k$ persons gave and received exactly one gift to and from exactly $(5k+3)$ other persons, where $k \in \mathbb{Z}$. For any two persons, the number of gifts received by both is the same. How many persons were in the party? [15 persons]

T1-3 *(10s)* Rafael and Fermin are thinking of a random natural number. They whispered their numbers to Ranzer. Ranzer wrote the sum of their numbers on one board and their product on another. Ranzer then hid one board, then he showed the other. Looking at the board written with the number 2015, Rafael said that he was still unsure of Fermin's number. Fermin then exclaimed that he now knows Rafael's number. What number(s) could Fermin be thinking of? [5, 31, 65, 403, 2015]

Tier 1 Clincher

T1-C1 *(15s)* Find the sum of the digits of 1111^3. [28]

T1-C2 *(30s)* Find the area of the shaded region of the figure of a semicircle, and a circle inscribed in a square with sides of uniform length 12 cm. $\left[\left(18\sqrt{3} - 6\pi\right)\text{ cm}^2\right]$



Tier 2

T2-1 *(30s)* Three integers Y, M, and G have the same digit sum. What is the remainder when $14\,056Y - 5271M - 8785G$ is divided by 15 813? [0]

T2-2 *(10s)* In the Gregorian calendar, a leap year, which occurs once every 4 years, except if the year is divisible by 100 but not by 400, is a year where there is an extra day usually represented by February 29. Suppose the first day of the year $\overline{3MSA}$ is a Sunday, where M, S, A are single digits and $0 < M, S, A < 9$. If the year $\overline{3MSA} + 1$ is a leap year, find the closest year to $\overline{3MSA}$, except $\overline{3MSA} - 1$, such that the last day of that year is Saturday. $\left[\overline{3MSA} - 7\right]$

T2-3 *(20s)* To help Sasha sleep, she decides to imagine and count some sheep jumping over a fence. She imagines 2016 white sheep lining up before a fence, then one by one, a sheep jumps over the fence. Sasha suddenly summons a black sheep cutting in line after every 7 non-black sheep jump and a gray sheep cutting in line after every 9 non-gray sheep jump. When a black and a gray sheep cuts in line at the same time, they merge into a goat. After all the white sheep jump, Sasha fell asleep. How many goats and non-white sheep did Sasha count? [503]

Tier 3

T3-1 *(20s)* Let positive integers a, b and c be the sides of a right triangle, with c as the hypotenuse. Suppose that a is prime. Find the smallest positive integer n such that $n\left(\frac{(A+b)^2-1}{a+b-1}\right)$ is a perfect square. [2]

T3-2 *(15s)* I left a number of red green, and pink candies at home. Seventy-five percent of the red and green candies was at least 80 percent of the pink candies. After a long day in school, I noticed that there were 27 less red candies, 21 less green candies, and 25 less pink candies. To help me count them, I compiled them into groups of 18 with each group having 4 more green than red candies and 4 more pink than green candies. What is the greatest possible number of candies I left at home? [217 candies]

T3-3 *(20s)* Patrick the Jejehackerz hacks into the YourMathGuru.com mainframe, and transforms a string code of 0's and 1's into JE's. If the digit is 1, it becomes JE while if it is 0, it becomes JEJE. For example, 100 101 becomes JE/JEJE/JEJE/JE/JEJE/JE = JEJEJEJEJEJEJEJEJE, 9 JE's. How many possible number strings correspond to JEJEJEJEJEJEJEJEJEJE which has 10 JE's? [89 strings]

Tier 4

T4-1 *(10s)* The roots of the equation $x^2 + bx + c = 0$ are $\sin\frac{\pi}{2016}$ and $\cos\frac{\pi}{2016}$. Find the value of $2016b^2 - 4032c - 2017$. [−1]

T4-2 *(15s)* Express $\prod_{k=2}^{2016} 2\cos\frac{\pi}{2^k}$ as $\csc x$ where $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. $\left[\csc\frac{\pi}{2^{2016}}\right]$

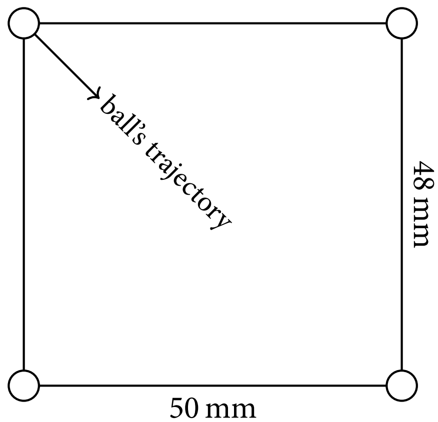
T4-3 *(10s)* Let $f(a,b) = \frac{1}{a+b}$, where $a + b \neq 0$. Suppose that y, m, g are non-zero distinct integers such that $y + +m + g = 2016$ and $f((y,m),g) = f(y, f(m,g))$. (Both sides of the equation exist and are well-defined.) Find m. [−2016]

Tier 5

T5-1 *(15s)* In Turnaround Town, cars are required to turn around whenever they meet an incoming vehicle. On one long highway, 50 cars are traveling individually from one end, while another 50 cars are traveling individually from the other end. If all of the cars traveled at the same constant speed, how many meetings took place before all of the cars have returned to both ends? [2500]

T5-2 *(20s)* Nika was holding a pigeon but it got away. She ran 10 km/h eastward to get the pigeon while the pigeon was flying 25 km/h in the same direction. But there was a giant wall 1 km ahead so the pigeon zigzagged back and forth between Nika and the wall until Nika was able to corner and capture the pigeon. How many hours did the pigeon travel eastward? $\left[\frac{7}{100}\text{ hours}\right]$

T5-3 *(30s)* Jenisne is practicing her precision in a $50 \times 48\ \text{mm}^2$ rectangular billiard table with 4 pockets on each corner. She places the cue ball at a corner and hits the ball at a 45° angle. How many times would the ball bounce until it shoots in a pocket? [47]



Tier 6

T6-1 *(30s)* Brian James would like to visit the Great Wall of Macho which extends indefinitely. If we are to imagine the situation in the Cartesian plane, the wall is the curve $y = \sqrt{x} + 3$ and Brian James is situated at $\left(\frac{20}{21}, 3\right)$. What is the shortest distance Brian James can travel to go to the Wall of Macho?

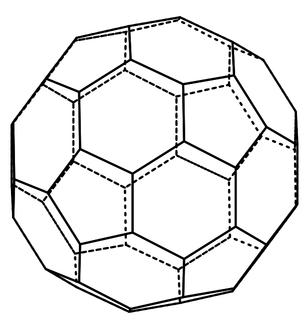
$\left[\frac{2\sqrt{5}}{5} \text{ units}\right]$

T6-2 *(30s)* A region $\mathcal{R}$ is defined by all points (x, y) on the Cartesian plane with x and y satisfying the inequalities $\frac{|y|}{|x|} \leq 2$ and $|x| \leq 4$. A point P with integer coordinates is randomly chosen inside the circle with radius 5 and center at the origin. What is the probability that P lies on $\mathcal{R}$? $\left[\frac{52}{69}\right]$

T6-3 Evaluate $\sum_{k=1}^{\infty} \frac{k(k+1)}{2} e^{(1-k)/429}$. $\left[\frac{e^{143}}{(e^{429}-1)^3}\right]$

Tier 7

T7-1 *(20s)* A truncated icosahedron is a solid with 20 regular hexagonal faces, 12 regular pentagonal faces, 60 vertices, and a shape similar to that of a football (soccer ball). Two Magic 89.9 DJs play a game on the said solid. Each player take turns writing his initials on an unoccupied face. The first player to successfully write his initials on three faces that share a common vertex is declared the winner. Assuming optimal play, what is the minimum number of turns such that there is a winner? Note that after Player 2 has made his first move, two turns will have already been made. [5 turns]



T7-2 *(25s)* An old grandfather clock functions in a particular way. Every time the minute hand and hour hand overlap, the numbers on the clock's face move one position counterclockwise (i.e., 12 moves to the position of 11, 1 moves to the position of 12, 2 moves to the position of 1, and so on). At 12:00 AM today, the numbers on the clock are on their proper positions (both minute hand and hour hand are pointing at 12). At 12:00 AM tomorrow, what number will the hour hand be pointing at? [10]

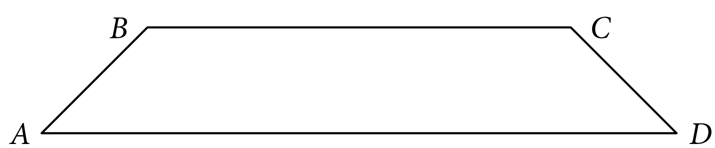
T7-3 *(30s)* Consider a triangle with vertices at points $Y(-x,-y)$, $M(x,0)$, and $G(0,y)$. If $\tan(\angle GYM) = \frac{3}{5}$ and $\tan(\angle YMG) = 3$. Find all possible coordinates of point Y. $[(\pm 1, 2), (\pm 1, -2)]$

Tier 8

T8-1 *(30s)* A player picks all the Jack, Queen, King and Ace suits out of a standard deck of cards and arranges them in a 4×4 grid. He does so in such a way that each row, column and diagonal of the grid contains one of each card value (Jack, Queen, King, Ace) and one of each suit (club, diamond, heart, spade). How many possible arrangements are there? [1152]

T8-2 *(30s)* Lara the leprechaun has a pot of 30 gold coins. She takes a random number of the coins (minimum 1, maximum 30) and throws them all up in the air. For every pair of coins that shoe 'heads' and 'tails', both coins run away from Lara. For every coin that shows 'tails' but has no 'heads' partner, the coin explodes. For every coin that shows 'heads' without 'tails' partner, the coin starts floating magically. What is the probability that after doing this scenario once, she has at least 10 floating coins? $\left[\frac{121}{496}\right]$

T8-3 *(30s)* Trapezoid $ABCD$ is such that the shorter base $\overline{BC}$ is $\frac{2}{3}$ the longer base $\overline{AD}$. The trapezoid is positioned inside the rectangle $AEFD$ whose area is 144 square units. Right triangle GHD (with the right angle at $\angle GHD$) has an area 60 square units and is positioned inside the rectangle in such a way that the leg $\overline{GH}$ passes through point B. Point G and point H lie on $\overline{EF}$ and $\overline{AD}$, respectively. The length of $\overline{BG}$ is $\frac{5}{8}$ the length of $\overline{AE}$. The length of $\overline{AB}$ and $\overline{DG}$ measure $3\sqrt{2}$ and 17 units, respectively. Let $[\cdot]$ denote area. Compute the value of $[ABCD] + [AEGB]$. $\left[\frac{129}{2} \text{ units}^2\right]$



Tier 9

T9-1 *(35s)* Bobert likes to make a sequence of numbers 6's and 9's. But, he does not want to have three consecutive 6's since he believes it is bad luck. Bobert knows the following:

- He can make 66 012 sequences of length 18.
- He can make 223 317 sequences of length 20.
- He can make 4 700 770 sequences of length 25.

How many sequences of length 19 can he make? There is one sequence with length 0. [121 415]

T9-2 *(45s)* While waiting for their scholarship application to be processed, YourMathGuru.com Chief Administrator Brian James Masalunga decided to give Abel and Tonya they could work on. You attempted to distract the two lovers by the only way you know how: to answer the following question faster than them. Find the numerical value of $S = \frac{\sin^2 35^\circ + \cos 35^\circ}{2 - 2\cos 35^\circ} + \frac{\sin^2 35^\circ - \cos 35^\circ}{2 + 2\cos 35^\circ} + \frac{2\cot 35^\circ}{\sin 220^\circ} +$

$2\cot 35^\circ \cot 220^\circ$. [2]

T9-3 *(30s)* A loaded die is a die with some of its faces more likely to land face up than others but none with zero probability. One loaded six-sided die is rolled twice. Suppose that the probability that the product of the two rolls is even is $\frac{40}{49}$, the probability that a roll is a 3 is $\frac{1}{7}$, and the probability that a roll is either a 1 or a 2 is $\frac{1}{3}$ of that when a roll is either a 4 or a 5 which is $\frac{3}{7}$. If among the six possible rolls, the most probable is x ans its probability is given be $\frac{a}{b}$, where a and b are relatively prime, what is the ordered triple (x, a, b)? [$(6, 2, 7)$]

Tier 10

T10-1 *(35s)* Evaluate $\sum_{n=0}^{2000} \frac{(-1)^n}{2} (n^2 + 3n + 2)$. [1 002 001]

T10-2 *(45s)* The *MSA sequence* is a special sequence such that the sum of any four consecutive terms starting from the nth is equal to n^2. If the 215th term equals 11 227 and the 216th term equals 11 540, what is the sum of the 6th and the 9th terms? [109]

T10-3 *(30s)* Suppose that $|x + y| + |x - y| = 2016$. What is the maximum value of $x^2 + 3x - 2y + y^2$? [2 037 168]

Tier 11

T11-1 *(30s)* We define a mutual friend C of A and B to mean that if C is A and B's friend, but A and B don't necessarily have to be friends. In a certain YourMathGuru.com forum group, the N participants in the group are special in that the following conditions hold:

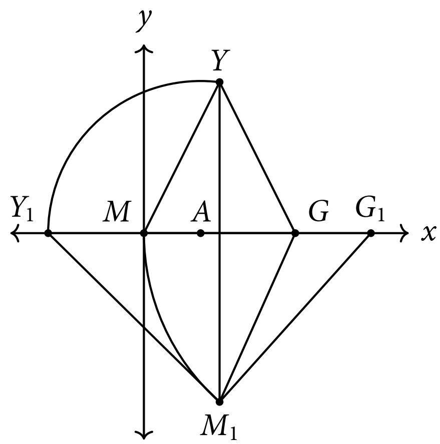
- For any two people in the group, there will be exactly 1 mutual friend between them who is also in the group.
- There exists a pair of people in the group that are not friends.

What is the smallest possible N that can achieve this? [5]

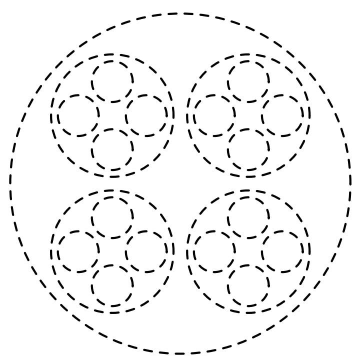
T11-2 *(60s)* Find the smallest integer greater than 1 which divides every element of the set of integers $\{7997, 799\,997, 79\,999\,997, 7\,999\,999\,997, \ldots\}$. [11]

T11-3 *(45s)* DJ Jay Spring of Magic 89.9 made a question for her co-workers to challenge their wits. She defines an α-triangle as a triangle with base $c = x$ and sides $a = x\sqrt[2000]{110^{123}}$ and $b = x\sqrt[1000]{110}$. The goal is to make a specific α-triangle using an algorithm. A move consists of choosing a side, a, b or c, of the α-triangle and making that chosen side a base c for a new α-triangle with a new a and b sides. If Jay Spring sets the starting α-triangle with base length $c = 1$, how many distinct ways are there to make an α-triangle with base length $c = 110^{183\,359}$ in exactly 123 456 moves? Same set of moves but with different positions are considered the same. [995]

T11-4 *(45s)* $\triangle YMG$ is a triangle in the Cartesian plane formed such that its altitude at point Y is equal to $\overline{MG}$, lying on the x-axis. From its initial position, point $M(0, 0)$ was moved to M_1 in a circular manner, centered at a point G_1 in the x-axis, so that $2\overline{YM_1} = 7\overline{MG}$. Meanwhile, Y moved to Y_1 in a similar manner such that the path is centered at a point A, also in the x-axis, such that $9\overline{Y_1M} = 16\overline{MG} = 88\overline{GG_1}$. If the area of $\triangle YMG$ is 2016 mm^2 smaller than that of $\triangle Y_1M_1G_1$, determine the numerical value of $\left|\overline{GG_1}\right| - \left|\overline{MA}\right|$.* $\left[\dfrac{734\sqrt{7}}{25}\right]$



T11-5 *(60s)* 2016 YourMathGuru.com scholarship applicants randomly sit in a certain seating arrangement such that all of the identical arrangements for 16 people, as shown, is configured in a circular manner. If the number of such arrangements can be expressed $a! \cdot 2016! \cdot \left(\dfrac{b}{2016}\right)^c$, where a, b, and c are positive real numbers, determine the value of $a + b + c$. All of the 2016 applicants get a chair of their own to sit. Similarly, all chairs are occupied by only one person. [254]



Final Round

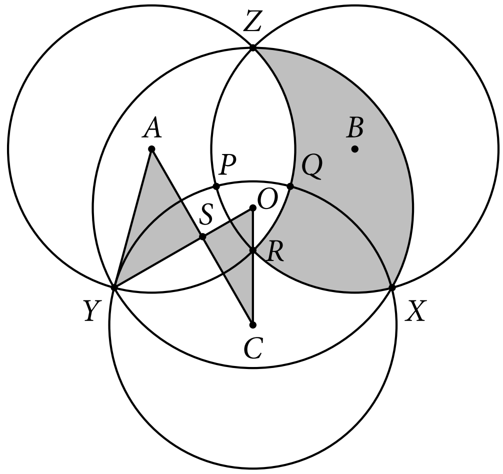
Wave 1

W1-1 Evaluate $\dfrac{\cos^2 0 + \cos^2 \frac{\pi}{36} + \cos^2 \frac{2\pi}{36} + \cdots + \cos^2 \frac{71\pi}{36}}{\tan^2 \frac{\pi}{36} \cdot \tan^2 \frac{2\pi}{36} \cdots \tan^2 \frac{17\pi}{36}}$. [36]

W1-2 Consider the unit circle with center at the origin. The circle is rotated π radians clockwise about the point $(1, 0)$. It is rotated again by π radians counterclockwise about $(2, 1)$, then $\dfrac{\pi}{2}$ radians clockwise about the point $(3, 2)$, then $\dfrac{\pi}{2}$ radians counterclockwise about $(4, 3)$, then lastly π radians clockwise about $(4, 1)$. Find the area of the region bounded by the x-axis and the path traced by the center of the circle. $\left[\dfrac{10 + \pi}{2} \text{ units}^2\right]$

* Figure not drawn to scale.

W1-3 From the figure is the Sekken Sharingan, Circle O, made by drawing a circle through points X, Y, and Z, which are intersection points of three identical and mutually orthogonal circles A, B, and C. Two circles are *mutually orthogonal* if the tangent lines at each intersection point are perpendicular, i.e., $\angle AZB$, $\angle AYC$, and $\angle BXC$ are right angles. Let "$[\cdot]$" denote area. Given that $[\text{circle } A] = 113\,\text{m}^2$, $[\triangle ASY] = 9\,\text{m}^2$, $[\triangle OSC] = 5\,\text{cm}^2$ and $[ZQRX] = 92\,\text{m}^{2\dagger}$, find the area of the Sharingan, Circle O, rounded off to the nearest tens. $[140\,\text{m}^2]$



Wave 2

W2-1 Find the sum of all five-digit numbers with distinct digits from the set $\{5, 6, 7, 8, 9\}$. [9 333 240]

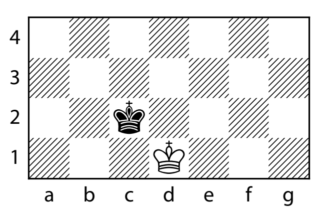
W2-2 Professor Young wants to organize his grading system by creating a grid of M rows and N columns, i.e., he has M students and N grading criteria. He fills up each entry of the grid with either a grade of 3.00 or 5.00. Being generous this semester, he also gave five grades of 2.00. After completing the grid, he notices that each column has at least seven 3.00's and each row has at least sixteen 5.00's. He also notes that he has obtained the least value for $M \cdot N$. How many students does he have? [16]

W2-3 Dobble Cards are cards with images of unique objects such that given a deck of N Dobble cards, any two cards from the deck has exactly one common object printed on both of them. Suppose that any Dobble card taken from a deck of N Dobble cards has 15 unique objects printed on it. Suppose as well that given any of the objects, there will be exactly three cards in the deck that will show it. How many unique objects are there all in all among the N Dobble cards? [155]

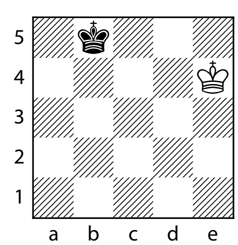
Wave 3

W3-1 A chess position is "illegal" if two kings check each other, i.e. the kings occupy squares which have a common edge or vertex. How many positions with only 2 kings on a 6×9 board are legal? [2516]

An illegal two-king position on a 7×4 board



A legal two-king position on a 5×5 board



† That is, the shaded part of the figure enclosed by the three arcs $\widehat{ZR}$, $\widehat{RX}$ and $\widehat{XZ}$.

W3-2 A restaurant serves 6 different flavors of ice cream—strawberry. mango, chocolate, vanilla, lemon, and pistachio. There are 150 regular customers who came here at random times each day, but each of them orders the same thing every day (person x always orders the flavor y). If there are already 130 customers served, there will always be 6 different flavors ordered. What is the minimum number of customers that must order to guarantee that there will be 4 distinct flavors of ice cream served?

[88]

W3-3 At the YourMathGuru.com conference, 20 geniuses were invited to a dinner party. Everyone was seated around a round table. However, Brainy Brian James, Danger Dumie, and King Kwan were also invited, and it was quite known that their rivalry between each other was strong and bitter. Quite much so that the hose of the party had decided to make sure that any two of the three were not seated together and that between any of them, there were at least three seats separating each from the other two. If there are $p\,(q!)$ possible arrangements that follow the above constraints such that p is prime and $|p - q| \neq 1$, find $p + q$. [23]

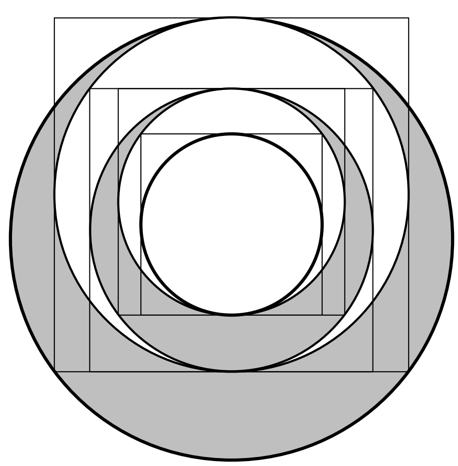
Wave 4

W4-1 Given five distinct points on a circle, what is the probability that a five-pointed star would be formed (a concave polygon made by extending the sides of a convex polygon until they meet) from a sequence of tracing from one point to a second point, that second point to a third, and so on up to the fifth, then from the fifth to the first? $\left[\frac{1}{12}\right]$

W4-2 Oli the software developer is starting to design a program. He designed it such that he inputs four points in the plane and the program forms a quadrilateral. He calls a unit square with integer coordinate for all vertices as a *box*. If a box is inside the quadrilateral, it will be displayed as a pixel. If a box has greater than or equal to half of its in area inside the quadrilateral, it will be displayed as a pixel. If the 4 points are $A : (4, 100)$, $B : (14, 150)$, $C : (64, 75)$, $D : (54, 25)$, how many points should Oli expect to be displayed? [3260]

W4-3 The pattern, shown below, goes with a larger circle than the base circle, passing through the square's top vertices, then followed by a larger circle passing through a larger square's lower vertices and so on. The pattern stops after having 2020 circles. What must be the length of a side of the smallest square so that the ratio of the 2nd largest unshaded crescent's perimeter to its area is $\frac{2^{2020}}{2020}$?

$\left[\frac{2^{2018}}{5^{2015}} \cdot 101\text{ m}\right]$



Wave 5

W5-1 When the positive integer N is multiplied to the square of its square, one obtains a seven-digit number ending in 7. Find the sum of all such positive integers. [17]

W5-2 Lucky Luke's lock lacks lockholes. It is a 4-digit code lock that runs digits from 0-9 in order. The current code shown is 1/2/3/4. The lock has four buttons that add or subtract from each digit. Here are the buttons with their displays:

A	+1	+2	−2	−3
B	0	+3	−1	+1
C	+3	−2	+1	−2
D	+3	0	0	0

For example, hitting A would change 1/2/3/4 into 2/4/1/1. Then hitting C after would change 2/4/1/1/ into 5/2/2/9. Lucky Luke's lock's code is actually 4/3/2/1. What is the fewest number of presses needed to change 1/2/3/4 into 4/3/2/1 if we must hit every button at least once? [16]

W5-3 Since Tonya has developed feelings of attraction for Abel and his functions after seeing all of his efforts, help her implicitly tell Abel this realization by evaluating the summation below. He defined the *Hard Hart Function*, defined by $\hat{h}(x)$, as follows:

$$\hat{h}(x) = \begin{cases} -\dfrac{m}{2}, & x \text{ is a multiple of any even square} \\ 0, & x \text{ is a multiple of an odd square except 1 and not divisble by 4} \\ n, & \text{else} \end{cases},$$

where m is the largest even divisor of x, and n denotes the number of positive integers less than or equal to x that are relatively prime with x. Evaluate $\sum_{i=1}^{41} \hat{h}(2015i)$. [423 250]