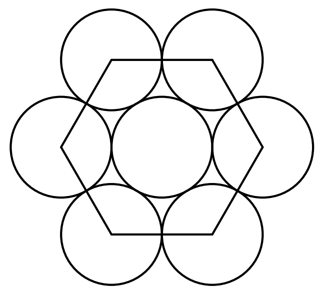
Oral Round

Tier 1

T1-1 *(20s)* The graphs of $x^2 + 4xy + 3x + 4y^2 + 6y - 4 = 0$ and $x^2 - 4xy - 7x + 4y^2 + 14y + 6 = 0$ enclose a region on the Cartesian plane. Determine the area of this region. $\left[\frac{25}{4}\right]$

T1-2 *(20s)* Consider a hexagonal close packing structure where the vertices of the hexagon are the centers of the circles each with radius $r = \frac{\sqrt{3}}{6}$. Find the sum of the regions outside the circles but outside the hexagon. $\left[\frac{\sqrt{3}}{2} - \frac{\pi}{4}\right]$



T1-3 *(15s)* What integer values of n will make the expression $\frac{n^4 - 1441}{n^2 - 1}$ a perfect square? $[n = -9, 9]$

Tier 2

T2-1 *(10s)* Jim wakes up alone in a hospital room after being unconscious for some time. He has no memory of when he was brought here. He sees his patient record and gathers the following information:

- Jim was first admitted to the hospital on a Monday.
- He had a birthday at most 2 weeks after he was admitted.
- His friends from MSA visited him for two whole days, once in his birthday and once more at least 3 weeks after he was admitted.
- He woke up on a Monday, at most 2 weeks after his birthday.

How many days was Jim confined in the hospital? [28 days]

T2-2 *(10s)* Two players take turns in rolling a fair die. The winner is the first to roll a number of the Fibonacci sequence. What is the probability that the first player wins two consecutive times? $\left[\frac{1}{2}\right]$

T2-3 *(20s)* Consider a unit circle centered at the origin. A random point (a, b) is selected among the points of this circle. Given that (a, b) is a point of the arc intercepted by a central angle with measure $0 \leq \theta < 2\pi$ such that $\cos\theta \geq -\frac{1}{2}$, what is the probability that $b \geq |a|$? $\left[\frac{5}{16}\right]$

Tier 3

T3-1 *(15s)* A figure constructed from a 5×5 square grid and four 3×3 square grid as follows. The corner squares of the 5×5 square grid are the center squares of the four 3×3 square grid. How many squares of sizes are there on the figure? [91]

T3-2 *(25s)* We define the Pabibo Najee sequence to consist of positive integers that is determined by the following recursive formula. For $k \geq 2$, $a_k = 2a_{k-1} + 9a_{k-2}$, where $a_0 = 1$, $a_1 = 2$. Find real numbers $m$and n such that $m > n$ and $a_k = \dfrac{m^{k+1} - n^{k+1}}{m - n}$ for all natural numbers k. $\left[m = 1 + \sqrt{10}, n = 1 - \sqrt{10}\right]$

T3-3 *(25s)* Let $p, q > 1$ such that $pq = p + q + k$ for some integer k. Find all possible values of k so that whenever $pq = 65$, $\dfrac{2}{pq} + \dfrac{3}{p^2q} + \dfrac{3}{pq^2} + \dfrac{4}{p^3q} + \dfrac{6}{p^2q^2} + \dfrac{4}{pq^3} + \dfrac{5}{pq^4} + \dfrac{10}{p^2q^3} + \cdots = \dfrac{1}{3}$. $[15]$

Tier 3 Clincher

T3-C1 *(30s)* A right triangle as area of 576 square units and the sum of the squares of the side lengths is 4640. Find the length of the two legs of the right triangle. [36, 32 units]

T3-C2 *(30s)* Find the set of real values satisfying $\dfrac{x+8}{x+7} - \dfrac{x+9}{x+8} = \dfrac{x+10}{x+9} - \dfrac{x+11}{x+10}$. $\left[\left\{-\dfrac{17}{2}\right\}\right]$

T3-C3 (2mins) The sequence of numbers 1223334444555556666667777777888888 . . . is formed by writing the positive integers in order in such a way that each integer n is written n times. Give an ordered pair (x, y) where x and y are the 2016th and the 2017th digits in the sequence respectively. $[(4, 5)]$

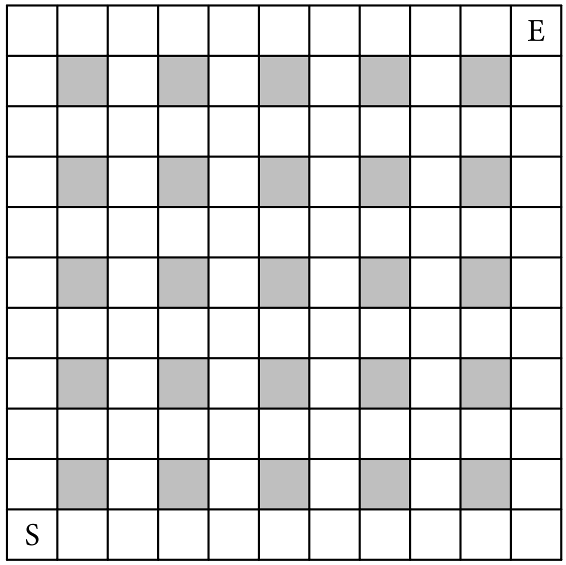
Tier 4

T4-1 *(15s)* An infinite number of poles is arranged such that the height of each pole and the distance between each pole decreases by half from one pole to another. A bird on top of the top first pole lands on the ground, then flies to the second pole. The bird lands on the ground again and then goes to the third pole. The bird repeats this process infinitely, regardless of the distance, as it can adapt to any size it needs to continue the process. If the height of the inital pole is 30 and the distance beween the first and second pole is 24, what is the minimum distance the bird will travel? [102]

T4-2 *(15s)* Find the product of all the real solutions of $\sqrt[4]{13 + x} + \sqrt[4]{4 - x} = 3$. $[-36]$

T4-3 *(20s)* In order to escape the trap-filled `WhenInManila.com` room, Steve uses a map, as given below, to avoid the booby traps. The booby traps would be activated when Steve double-steps on a tile or steps on a black tile. Given the rules below, determine the number of possible shortest paths for Steve to reach the exit.

1. Steve cannot skip a tile.
2. Steve cannot step on a tile that does not share a side with the tile he is stepping on.
3. The starting point, marked S, and the ending point, marked E, are known to be safe.



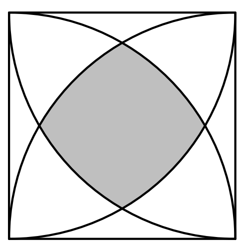
[252]

Tier 5

T5-1 *(20s)* The product of two of the four zeros of the quartic equation $x^4 - 4x^3 + kx^2 + 394x - 504$ is -18. Find k. $[k = -67]$

T5-2 *(25s)* A point lies on a diagonal of a square such that the point is one unit away from the other two vertices. The point is also the upper left vertex of a smaller square whose diagonals coinide with the diagonals of the larger square. The ratio between the area of the two squares is $\frac{3}{5}$. Determine the perimeter of the square. $\left[2\sqrt{5} \text{ square units}\right]$

T5-3 *(20s)* Consider a square of side length $\sqrt{3}$. Four quarter circles are drawn with radius $\sqrt{3}$ and centers at the corners of the square. Find the area of the shaded region. $\left[\pi + 3 - \sqrt{3}\right]$



Tier 6

T6-1 *(45s)* Consider a hexagonal close packing structure where a unit cell (hexagonal prism) is given by the figure. Here, each solid sphere is pairwise congruent and tangent with radius r. Find the ratio of the total area of solids enclosed in the unit cell with the volume of the unit cell.* $\left[\frac{\pi}{3\sqrt{2}}\right]$

T6-2 *(25s)* The queen sent 5100 coins and 2050 gems to form an army to kill the fabled MSA beast. The army is composed of four different troops: archers, musketeers, wizards, and witches. Hiring an archer costs 20 coins and 5 gems; hiring a musketeer costs 45 coins and 15 gems; hiring a wizard costs 125 coins and 60 gems. If the queen hired exactly 109 troops and there is only one kind of troop where a prime number of these troops were hired, then how many archers, musketeers, wizards, and witches are there?

* Figure missing

[Express the answer in the form of an ordered 4-tuple (Archers,Musketeers,Wizards,Witches)] $[(43, 35, 22, 9)]$

T6-3 *(15s)* A square is inscribed in an equilateral triangle with area $\frac{1}{4}$. Find the area of the square. $\left[7\sqrt{3} - 12\right]$

Tier 6 Clincher

T7-C1 *(30s)* A ball is dropped from a height of 10 meters. At each bounce, it loses one-third of its kinetic energy, which is given by $\frac{1}{2}mv^2$, where m is the mass and v is the velocity. How far does the ball travel before it comes to a stop if it experiences a constant downwards acceleration $10\,\text{m/s}^2$ throughout its travel? Note that the distance traveled is $D = v_i t + \frac{at^2}{2}$, where v_i is the initial velocity, t is the time taken, and a is the acceleration. $[50\,\text{m}]$

T7-C2 *(30s)* The sum of the digits of a three-digit number is 10. The hundreds digit is one more than thrice the ones digit. When the digits are reversed, the number is decreased by 594. What is the number? $[721]$

T7-C3 *(2min)* An integer is a *palindrome* if it reads the same backwards as forwards. For example, 7, 33 and 21012 are all palindromes. How many integers from 1 to 1 000 000 are palindromes? $[1998]$

Tier 7

T7-1 *(10s)* Rayjer and Maya have the $MSA - 20$, a machine that requires an input of four polynomials of the form $(x - a)$, where a is a nonnegative integer, placed one at a time, and outputs their product. Maya, in particular, wants to produce the polynomial $x^4 - 9x^3 + ax^2 - bx + c$, where a, b, and c are real numbers. To avoid wasting time, help Rayjer compute for the number of ways they can feed four polynomials of such kind to the machine to produce the desired polynomial. $[220]$

T7-2 *(20s)* Chona Mae likes multiplying consecutive integers together and Jhemherlyn likes numbers ending in 25 so she appends 25 to all of Chona Mae's products. The numbers resulting from this process are called DANK numbers. How many DANK numbers are there between 1 and 6969? $[10]$

T7-3 *(15s)* There are 55 candies, consisting of chocolate bars, lollipops, distributed in three jars M, S, A. The distribution satisfies the following conditions:

- Each pack has the same number of gummy bears inside.
- No two jars can have the same number of chocolate bars.
- There is at least 1 and at most 5 chocolate bars in each jars.
- No two jars can have the same number of lollipops.
- There is at least 1 and most 4 lollipops in each jar.
- In total, there are at least 7 and at most 17 packs of gummy bears.
- The number of gummy bears is not a perfect square.

Jar M has 5 chocolate bars, 2 lollipops, and 2 packs of gummy bears. Jar S has 4 chocolate bars, 3 lollipops, and 1 pack of gummy bears. How many packs of gummy bears are there in Jar A? $[20]$

Tier 7 Clincher

T7-C1 *(30s)* Four coins are arranged in such a way that all coins are tangent to the other three. If three of them are identical, what is the ratio of the radius of the bigger coin to the smaller coin? $\left[2\sqrt{3}+3:1\right]$

T7-C2 *(30s)* Find the inverse of the function $f(x) = \frac{1}{2}\left(\frac{2^{-x}+2^x}{2^{-x}-2^x} + \frac{2^{-x}-2^x}{2^{-x}+2^x}\right)$.
$$\left[f^{-1}(x) = \frac{1}{2}(\log_2(x-1) - \log_2(x+1))\right]$$

T7-C3 *(2min)* Judylou takes the sum of 6 consecutive powers of 2 starting from the ith power. Mary Ann takes the sum of 3 consecutive powers of 3 starting from the jth power. Sean takes the sum of consecutive integers starting from 1 to n. The minimum value that Judylou and Sean can get in common is x, and the minimum value that Mary Ann and Sean can get in common is y. Find $x - y$. $[1665]$

Tier 8

T8-1 *(20s)* Homer has a deck of eight blank cards. He writes the letters M, S, and A, on three of them and then randomly shuffles the deck. Whenever he draws the cards M, S, A in that sequence, he stores the entire deck, provided that the deck's arrangement has not occurred yet, then he repeats the entire process with a new deck of eight blank cards, Else, he shuffles the deck and repeats the process on the same deck of cards. How many cards will Homer have? $[352]$

T8-2 *(40s)* Vince is organizing a party to celebrate the ninth anniversary of `WhenInManila.com`. In his party, the following hold.

(a) There is a business represented in the party.

(b) Each business has exactly three associates.

(c) Not all attendees are in the same business.

(d) Any two distinct attendees are associated in exactly one business together.

(e) If an attendee A is not associated to a business B, then there exist exactly one business including A that does not include any associate of B.

Determine the sum of the number of people attending and the number of businesses represented in this party. $[21]$

T8-3 *(20s)* Let $f(x)$ be a quadratic function satisfying the following conditions:

1. $f(x+2) = f(x) + 3x + 2$
2. $f(2) = 2$

Find the coefficient of x^3 in $f(f(x))$. $\left[-\frac{9}{16}\right]$

Tier 9

T9-1 *(25s)* Homer is constructing a phoenix using sectors. For the head, he places a sector whose central angle has measure α, where $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$. Then, he used a machine to place infinitely many sectors on both sides of the head simultaneously. The machine must follow these rules:

1. A sector's central angle measure is p times the angle measure of the preceding sector, for some $p \in [0,1]$.
2. No two sectors shall overlap.

In the middle of the process, Nika placed a sector diametrically opposed to the head to serve as the tail. Luckily, the sector's angle measure β is the largest such that the sector will not violate the second rule. If $\frac{\pi}{2} < \beta < \frac{2\pi}{3}$, what is the probability that $\frac{30}{74} < p < \frac{43}{74}$?[†] $\left[\frac{143}{148}\right]$

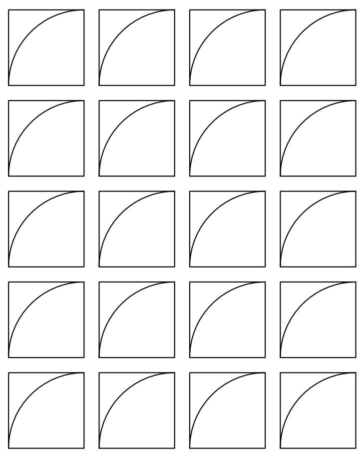
T9-2 *(30s)* Jen picks a random number $0 < p < 1$ uniformly, then flips an unfair coin so that it lands "heads" with probability p. Recording the result each time, she flips the coin 2018 times. What is the probability that exactly 69 heads are recorded? $\left[\frac{1}{2019}\right]$

T9-3 *(20s)* How many 4-digit numbers $\overline{ABCD}$ such that $A \geq B \geq C \geq D$ are there? $[495]$

Tier 10

T10-1 *(20s)* A coin which comes up heads with probability $\frac{2}{3}$ is flipped 2017 times. Find the expected number of times that a tail is followed by a head. $[448]$

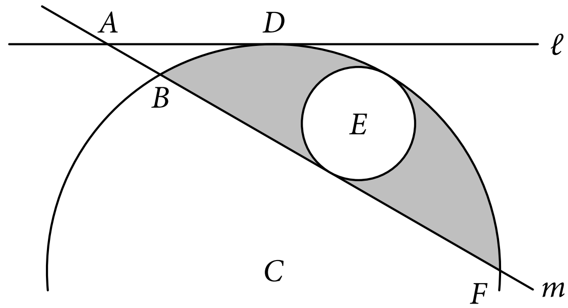
T10-2 (45s) Twenty puzzle pieces with an image of a quarter circle are arranged to form a 5×4 grid, as shown in the figure. If rotating a piece by a multiple of 90° counts as a move, then how many possible arrangements form at most two circles by doing a finite number of moves? You may write your answer in factored form.



$[2^{18} \cdot 5 \cdot 7 \cdot 293 \cdot 409]$

T10-3 (45s) As shown in the figure, ℓ is tangent to a circle, centered at C with radius 6, at point D, m meets the same circle at points B and F, and lines ℓ and m intersect at point A. If ℓ and $\overline{CF}$ are parallel lines, $\overline{AD}$ measures $6\sqrt{3} - 6$, and circle E is the largest circle contained in the area bounded by the line m and the circle C, find the area of the shaded region. (Express the answer as a single fraction.) $\left[\frac{37\pi - 36\sqrt{3}}{4}\right]$

[†] Figure missing



Tier 11

T11-1 *(20s)* Let k be a prime number greater than 2. Define N_k to be the set of all prime numbers $q \neq k$ such that q divides $k^{2q} - k$. Find the largest prime number m less than 200 such that $N_m = N_{2017}$.
[$m = 127$]

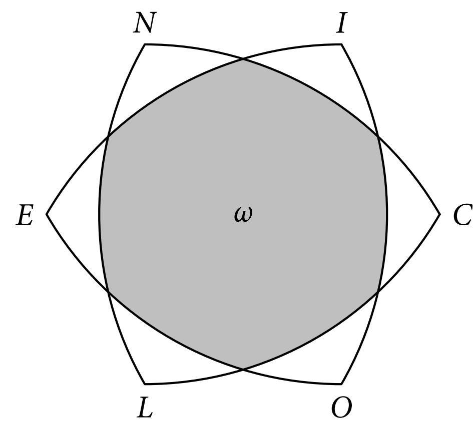
T11-2 *(25s)* Let $\frac{p}{q}$ be a ratio of positive integers where $q < 2018$ such that $\frac{p}{q}$ is the closest number to but not equal to $\frac{17}{55}$. Find $p + q$. [2609]

T11-3 *(20s)* Rayah is playing a game with a biased coin, denoted by $C[h, t]$, where h is the probability of heads and $t \neq 0$ is the probability of tails. If Rayah rolls a head, she gets one point and rolls the coin again. If she rolls a tail, she gets nothing and the game ends. Determine the absolute difference between the expected number of points she gets when a $C[2-\sqrt{2}, \sqrt{2}-1]$ coin is used and when a $C[\sqrt{2}-1, 2-\sqrt{2}]$ coin is used.
$\left[\frac{\sqrt{2}}{2}\right]$

Tier 12

T12-1 *(25s)* Let $f(x) = \dfrac{(2x+1)^{4/3} + (2x+1)(2x-1)^{1/3}}{(2x+1)^{2/3} + (4x^2-1)^{1/3} + (2x-1)^{2/3}} + (2x-1)^{2/3}$. Evaluate $\sum_{n=1}^{2017} 2f(n)$.
$[4035^{5/3} - 1]$

T12-2 A Reuleaux triangle is a figure formed by constructing identical circular segments with central angle $\phi = \frac{\pi}{3}$ from each vertex of an equilateral triangle to its other two vertices. Two identical Reuleaux triangles NCL and IOE with circular arc of length 2π are drawn such that the distances of each vertex from point ω are all equal and $NICOLE$ is a regular hexagon, as shown in the figure below. Find the perimeter of the shaded region. (Express the answer as a sum or product of cos and arccos.)
$\left[36 \arccos \frac{5+\sqrt{33}}{12}\right]$



T12-3 (45s) Find the solution set contained in $[0, 2\pi]$ of $2\cos^2 3x - 14\cos^2 2x - 2\cos 5x + 24\cos 3x - 89\cos 2x + 50\cos x > 43$. $\left[\left(\frac{\pi}{3}, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right)\right]$

Final Round

Wave 1

W1-1 Suppose that there are six brown chips, twelve black chips, and eighteen white chips. How many ways are there to choose from a set of nineteen chips such that twelve of them are of the same color and six of the remaining are also of the same color? [309 758 922]

W1-2 Jeremiah bought a temperature-dependent clock with radius 30 cm. He discovered that the second hand of the clock is moving at a rate of $2^{x-24}\pi$ cm/s where x is the measure room temperature (in $^\circ C$) at instant. If at $15:00:00$ on a regular clock, it is also $15:00:00$ on the temperature-dependent clock, the measured room temperature is $24^\circ C$, and the room temperature starts to increase by $1^\circ C$ and decrease by $1^\circ C$, for every j and k minutes that have passed on a regular clock, respectively, given that j is an odd number and k is an even number from the set $S = \{a_0 = 1, \ldots, a_n = n + a_{n-1}, n \text{ is a natural number}\}$, determine the time on the temperature-dependent clock on the following day at $15:00:00$ on a regular clock. [$18:22:30$]

W1-3 Let $\{L_n\}$ be the sequence whose nth term is given by $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$, and $\{a_n\}$ be another sequence such that $a_1 = 1$, $a_2 = 3$, and $a_{n+2} = a_n + 2018$ for $n \geq 1$. Find the value of $\sum_{n=1}^{2018} \tan^{-1}\left(\frac{L_{1009}}{L_{a_n+1009}}\right)$. $\left[\frac{\pi}{4}\right]$

Wave 2

W2-1 Find the sum of the last two digits of $6969, 6969^2, \ldots, 6969^{2018}$. [90 870]

W2-2 Thirty people are seated around a circle. A person is removed. Then the 2nd person to the right of him/her is removed, and the 3rd person to the right of the last removed person is removed, and so on. This is repeated until one person is left. What is his position to the right of the first removed person?[‡] [16th]

W2-3 Let $f(x) = ax^2 + bx + c$ be a quadratic function such that $|f(x)| \leq 1$ for all $0 \leq x \leq 1$. Determine the maximum value of $6|a| + |b| + 9|c|$. [65]

[‡] Not verbatim

Wave 3

W3-1 Rodolfo and Vincent are playing *Spectrum Sempra*. There are five gems in the game: a Red Ruby, a White Diamond, a Blue Sapphire, a Yellow Topaz and a Brown Garnet. The game also includes a pile of cards, each with a picture featuring two different kinds of gems and two different colors, where one or both of the gems have the wrong color. One pair is represented in exactly one card. The cards are revealed one at a time. The game is as follows:

- If a card shows a correctly colored gem, the person who grabs the said gem first gets the card.
- If a card shows two incorrectly colored gems, the person who grabs the gem whose color or kind is not represented in the card gets the card.
- Flip another card, and repeat the process until all cards are played.

The game ends when the pile of cards is exhausted. The player with the highest number of cards wins. So far, all of Vincent's cards contain the color red, while Rodolfo's cards are all of those that contain a diamond. At this point of the game, what is the minimum number of cards left in the pile? [60]

W3-2 A circles is inscribed in a 1×1 square. In two opposing corners of a 1×1 square, a circle is placed in the area bounded by the perimeters of the square and the inscribed circle such that its diameter is the largest to coincide with the diagonal formed by the two corners and no two circles intersect. This process continues infinitely, yielding to a string of circles where one of their diameters lie on the diagonal defined by the two opposing corners. What is the total area of the circles? $\left[\dfrac{3\pi\sqrt{2}}{16}\right]$

W3-3 Determine the number of distinct numbers that appear in the sequence $\left[\dfrac{1^2}{2020}, \dfrac{2^2}{2020}, \ldots, \dfrac{2020^2}{2020}\right]$ where $[x]$ denotes the greatest integer less than or equal to x. [1516]

Wave 4

W4-1 While diving to the headquarters of `WhenInManila.com`, Steve has passed by a lamp post and has observed that there is a lamp post every time he traveled a distance of thirty feet. If he stops at a distance of $\sum_{k=1}^{2018} k^{43}$ feet from the first lamp post, then how far is he from the closest lamp post?

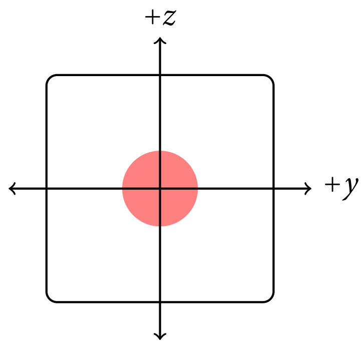
[9 ft]

W4-2 Determine the last two digits of $2^{2018}(2^{2019} - 1)$. [28]

W4-3 Find the last two digits of the least common multiple of $2018!^{1024} - 1$ and $(2019!^{1009} + 2019^{1009})(2018!^{1008} + 2018!^{1007} + 2018!^{1006} + \cdots + 2018!^2 + 2018! + 1)$. [59]

Wave 5

W5-1 As shown in the figure, a die is placed in the Cartesian space with its center at the origin such that the positive x-axis intersects the face with one circle. If there are only four possible moves: rotation of 90° clockwise or counterclockwise around the $+y$ or $+z$-axis, what is the probability that the positive x-axis intersects the face with six circles by doing exactly five random moves? $\left[\dfrac{5}{32}\right]$



W5-2 Let $x_1, x_2, \ldots, x_{2018}$ be positive numbers whose sum is 2. Find the minimum value of

$$\frac{x_1}{1 + x_2 + x_3 + \cdots + x_{2018}} + \frac{x_2}{1 + x_1 + x_3 + \cdots + x_{2018}} + \cdots + \frac{x_{2018}}{1 + x_1 + x_2 + \cdots + x_{2017}}.$$

$$\left[\frac{1009}{1513}\right]$$

W5-3 Evaluate the sum $\dfrac{1}{1^2 \cdot 3^2 \cdot 5^2} - \dfrac{1}{3^2 \cdot 5^2 \cdot 7^2} + \dfrac{1}{5^2 \cdot 7^2 \cdot 9^2} - \cdots$. $\left[\dfrac{1}{9} - \dfrac{\pi}{2^6} - \dfrac{\pi^3}{2^9}\right]$