Oral Round

Tier 1

T1-1 *(10s)* Kristel decided to host a get-together party at Trampoline Park with twenty of her friends. She decides to give each of her friends a card with a number from 31 to 50, where each person holds a unique number. How many people must be selected to guarantee that at least one pair will have cards which differ by 7? [14 people]

T1-2 *(5s)* Kim received a picture containing a set of seven numbers from the Actuarial Society of the Philippines. However, four of the numbers were covered. The set has a median equal to 3 and a mode equal to 2. If the set of the numbers is let to be $\{x, y, z, m, 4, 4, 5\}$, where x, y, z, and m are natural numbers, then what is the value of the lowest number in the set? [2]

T1-3 *(10s)* Evaluate $\frac{1}{4\cdot 9}+\frac{1}{9\cdot 14}+\frac{1}{14\cdot 19}+\cdots+\frac{1}{2014\cdot 2019}$. $\left[\frac{403}{8076}\right]$

Tier 2

T2-1 *(10s)* The nth term of an arithmetic sequence is m and the mth term is n. Find the $(m+n)$th term. [0]

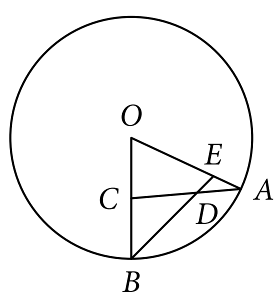
T2-2 *(10s)* Circles M and S are internally tangent to circle A. At the same time, the two circles are externally tangent to each other. The distance between the centers of the circles M and S is 10 units, while the radius of circle A is 4 units longer than that of circle Q, and 2 units longer than that of circle M. Find the sum of the measures of radii of the three circles.* [18 units]

T2-3 *(15s)* Suppose Jayne has an arithmetic sequence whose terms sum to 123. Kristel is considering another arithmetic sequence whose terms are the kth term of the first sequence added to the corresponding kth odd positive integer. The sum of the terms of the new sequence is 652. Find the number of terms in Jayne's sequence. [23]

T2-X *(15s)* Let $\overline{ABCD}$ be a four-digit number with A as its first digit, B as its second, and so on. If $\overline{ABCD}$ is the product of three consecutive prime numbers and $\frac{\overline{ABC}-2D}{7}$ is an integer, then find $\overline{DABC}$. [1100 units]

Tier 3

T3-1 *(15s)* Find the length of segment BC if the area of the circle is 64π units^2, O is the center of the circle, and the equations $\frac{OE}{EA}=3$, $\frac{CD}{DA}=\frac{3}{2}$ hold true. [4 units]



T3-2 *(10s)* Find the remainder when 17^{482} is divided by 143. [3]

* Voided, due to a clerical error. Circle Q should be circle S.

T3-3 *(15s)* Find the last three digits of the product of the squares of all odd numbers up to 2019. [625]

Tier 3 Clincher

T3-C1 *(15s)* There are 510 locker rooms numbered from 1 to 510 in a hotel. The staff opts to unlock all the doors to accommodate a group of guests, but the guests said that they want the rooms that they will occupy to be apart by one room. This prompted the staff to lock the even numbered rooms. By the time they were done, the guests requested that every room whose room number is a multiple of three, if there are no occupants inside, are to be used as a storage rooms for an event which they are to organize later. These rooms were also unlocked. How many doors are locked? [170 doors]

T3-C2 *(10s)* Find the least possible value of the expression $(x+y)(y+z)$, given that x, y, z are positive numbers satisfying the equation $xyz(x+y+z)=1$. [2]

Tier 4

T4-1 *(10s)* In triangle ABC, D is on $\overline{AC}$ such that $DC = 3(AD)$. On $\overline{BC}$, E is some point distinct from B and C. Point O is the intersection of $\overline{AE}$ and $\overline{BD}$. If $BO = BE$, find $\dfrac{EC}{OD}$. [4]

T4-2 *(15s)* Define the *Emjaeanian Number*, $E(n)$, to be the nth number in a sequence of numbers that is either divisible by 2, 3, or 5, that is, $\{2,3,4,5,6,8,9,10,12,14,15,16,18,20,\ldots\}$. For example, $E(10) = 14$ since 14 is the 10th number in the sequence above. Find p so that $E(p) = 410$. [300]

T4-3 *(10s)* If $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $a_4 = 7$, and a_n is the sum of the four previous terms for all $n \geq 5$, find the sum of the first 8 terms. [235]

Tier 5

T5-1 *(10s)* Find the set of real values satisfying $\dfrac{x+8}{x+7} - \dfrac{x+9}{x+8} = \dfrac{x+10}{x+9} - \dfrac{x+11}{x+10}$. $\left[\left\{-\dfrac{17}{2}\right\}\right]$

T5-2 *(15s)* Given 9 objects arranged in three rows and three columns, how many groups of three objects are there such that two objects of the same group are either directly above/below or on the left/right side of the third object of the group? [22]

T5-3 *(20s)* In Studyhaven, Inc., there are eight boys arranged in a straight line. The front desk asked them to insert five identical points in the line such that the boys maintain their orders with respect to each other. If more than one pole can be in between two boys and the ends of the lines have to be boys, what is the probability that the arrangement yields only one pole being in between the fourth and fifth boys from the left? $\left[\dfrac{3}{11}\right]$

Tier 6

T6-1 *(15s)* Find the Cartesian equation of the set of points whose sum of distances from two points $(14,3)$ and $(6,3)$ is 11. $\left[\dfrac{4(x-10)^2}{121} + \dfrac{4(y-3)^2}{57} = 1 \text{ or } 228x^2 + 484y^2 - 4560x - 2904y + 20\,259\right]$

T6-2 *(15s)* MSA Academic Advancement Institute plans to give a mock examination on the subjects covered by the UPCAT using four Saturdays. The morning session is dedicated to answering the exam while the afternoon session is used to discuss the answers. Prial, a curious student, decided to list down several questions before the first Saturday comes. He wants to ask the teacher the same number of questions during each discussion period, and be left with none in his list after the fourth Saturday. However, each time he attends the morning session, the number of questions in his list doubles. If

Prial initially has at least one question, what is the least number of questions he must have before the first Saturday so that he is left with no questions after fourth Saturday? [15]

T6-3 *(20s)* Let $g(x) = x^3 + ax^2 + bx - 6\binom{n}{3}$ have consecutive solutions in $\mathbb{Z}^+$. If $6\binom{n}{3}$ is divisible by 180, find b.† [242]

T6-X *(10s)* Determine the sum of the reciprocals of the roots of the equation $\frac{69}{31}x + 1 + \frac{1}{x} = 0$. [−1]

Tier 7

T7-1 *(20s)* Find the value of $\sqrt{1 + 2019\sqrt{1 + 2020\sqrt{1 + 2021\sqrt{1 + \cdots}}}}$. [2020]

T7-2 *(20s)* Find the value of $\cos\left(\frac{\pi}{2019}\right)\cos\left(\frac{2\pi}{2019}\right)\cdots\cos\left(\frac{1009\pi}{2019}\right)$. $\left[\frac{1}{2^{2019}}\right]$

T7-3 *(20s)* Jayne and Kristel are friends who regularly visit a particular orphanage. This time, Jayne brought 10 cupcakes while Kristel brought 20 marshmallows. There are 6 kids at the orphanage and the way of distributing Jayne's cupcakes is independent from the way of distributing Kristel's marshmallows. If there are n ways to distribute the marshmallows and cupcakes such that a kid may be left empty-handed or a kid may be given all, what is the sum of all prime factors of n? [64]

Tier 8

T8-1 *(25s)* How many integers k exist such that 5^k has at least 1000 digits but at most 2000? (Note: $\log_5 2 \approx 0.430\,67$) [1431]

T8-2 *(25s)* Define a sequence $P = \{p_n\}$ using the following: $p_1 = 2$, $p_2 = 2^2$, $p_n = p_{n-1} \cdot p_{n-2}$ for all natural numbers n. What is the units digit of p_{2019}? [8]

T8-3 *(25s)* Given $A = \sqrt{26} - 5$ and $B = \frac{1}{A}$, find $y = \frac{A^4 + AB^3}{A^2 + 1}$. [101]

Tier 9

T9-1 *(20s)* Find the remainder when $(32P3)(31^{24} + 1)(31^{12} + 1)(31^6 + 1)(31^4 + 31^2 + 1)$ is divided by 47. [9]

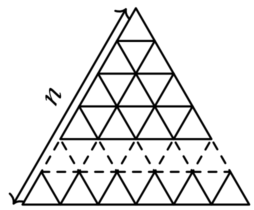
T9-2 *(20s)* Find all positive integers k less than 100 such that dividing $32k$ by 105 leaves a remainder of 5. [10]

T9-3 *(25s)* The sequence $122\,333\,444\,455\,555\,666\,666\,777\,777\,788\,888\,888\,999\,999\,999\cdots$ is formed by writing the positive integers in order in such a way that each integer n is written n times. Give an ordered pair (x, y) where x and y are the 2018th and 2019th digits in the sequence respectively. [(4, 5)]

Tier 10

T10-1 *(10s)* The diagram below shows a grid with n rows, with the kth row being composed of $2k - 1$ identical equilateral triangles for all $k \in \{1, \ldots, n\}$. If there are 513 different rhombi each made up of two adjacent smaller triangles in the grid, what must be the value of n? [19]

† Voided due to a technical error.



T10-2 *(15s)* The numerical value of $2^{16} - 2^6 \cdot 2019^2(1995) - 2^{14} \cdot 2019 + 2019^4$ may be expressed as ms^a where m, s, and a are prime numbers. Find the value of the sum $m + s + a$. [4009]

T10-3 *(30s)* Given $N = \dfrac{2018^2 - 1298^2 - 80^3}{64}$, find the number of positive integral factors of $N - 100$. [24 factors]

Tier 11

T11-1 *(30s)* Find the exact value of $\cos 20^\circ - 4\sin^2 20^\circ \cos 20^\circ - \cos^4 7.5^\circ + 4\cos^4 3.75^\circ - 4\cos^2 3.75^\circ + 2$. $\left[\dfrac{26 - \sqrt{3}}{16}\right]$

T11-2 *(30s)* Find three prime numbers such that their sum is 122 and their product is 6862. [2, 47, 73]

T11-3 *(60s)* Find the radius of the smaller circle that is tangent to the lines $x - 3y = -6$ and $x + 2y = 2$ and passes through the point $(0, 1)$. $\left[\dfrac{\sqrt{135 - 90\sqrt{2}}}{5} \text{ units}\right]$

Tier 12

T12-1 *(45s)* Let $h(x) = ax^2 + bx + c$ such that a and c are real numbers; $b \geq -2019$; $a \neq 0$; and $b^2 - 4ac = 0$. If $h(h(\xi)) = 0$ for some rational number ξ, then find the sum of the possible integer values of $a\xi$. [20 832]

T12-2 *(60s)* Given real numbers m and s satisfying $m^3 - 3m^2 + 6m - 10 = 0$ and $s^3 - 3s^2 + 6s + 2 = 0$, respectively, such that $ms = a$, find the range of positive values for a. [$a \leq 1$]

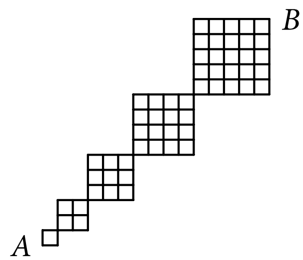
T12-3 *(60s)* Let $w = a + b + c + d$, $x = d - c + b - a$, $y = a^2 + b^2 + c^2 + d^2$, and $z = \dfrac{a + c}{b + d}$, where a, b, c, and d are rational numbers. If the set $S = \{w, x, y, z\}$ is arranged in increasing order, then the resulting set is $T = \left\{\dfrac{29}{68}, \dfrac{13}{7}, \dfrac{97}{21}, \dfrac{2735}{441}\right\}$. Find the value of $ab + ac + ad + bc + bd + cd$. $\left[\dfrac{3337}{441}\right]$

Final Round

Wave 1

W1-1 Define an *n-Floresian Series* to be a series of n squares wherein each ith square in the series, $1 \leq i \leq n$, has dimension $i \times i$, and if $i \neq 1$ or $i \neq n$, then the ith square is connected to the $(i - 1)$th square by its lower-left corner, and is connected to the $(i + 1)$th square by its upper-right corner.

Consider a 5-Floresian Series. Place this series of squares on a grid as illustrated in the figure below. Let A and B be points on the grid such that A is the lower-left corner of the first square in the series and B is the upper-right corner of the fifth square in the series. Find the number of shortest routes from A to B. *(Express the answer in prime power factorization.)* [$2^7 3^3 5^2 7^2$]

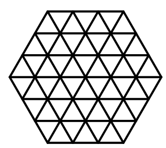


W1-2 When 369, 489, and 592 are divided by integer n, remainders r_1, r_2, and r_3 are obtained, with $r_1 + 2 = r_2 + 1 = r_3$. Find the largest such integer n. [17]

W1-3 Suppose that a unit square is inscribed in the first circle, which is inscribed in an octagon; the octagon is inscribed in the second circle which is inscribed in a 16-gon; the 16-gon is inscribed in the third circle which is inscribed in a 32-gon; and so on. If all of the polygons are regular, all of the circles are concentric, and the outermost figure is a circle, then what is the value which the area of the nth circle approaches as n approaches infinity? $\left[\dfrac{\pi^3}{16} \text{ units}^2\right]$

Wave 2

W2-1 As shown in the figure, a regular hexagon of side length 3 is divided into equilateral triangles of side length 1. Find the number of trapezoids of different sizes lengths on the given figure. [522]



W2-2 Judylou takes the sum of 6 consecutive powers of 2 starting from the ith power. Mary Ann takes the sum of 3 consecutive powers of 3 starting from the jth power. Sean takes the sum of consecutive integers from 1 to n. The minimum value that Judylou and Sean can get in common is x, and the minimum value that Mary Ann and Sean can get in common is y. Find $x - y$. [1665]

W2-3 Jonell and Vincent are playing a cooperative game called *Hi-Lo 2*. An integer m between 2^{2p-1} and 2^{2q+1}, where p and q are prime numbers and $q > p$, is randomly chosen. Jonell gives an integer n_1 between 2^{2p-1} and 2^{2q+1} and Vincent either says *Hi*, if $m > n_1$, or says *Lo*, if $m < n_1$. The process repeats until Jonell gives the integer $n_i = m$ and their score will be $44 - r$, where r is the total number of guesses. If they want an optimal play, what is their lowest possible score? $[43 - 2q]$

Wave 3

W3-1 Given $x = 3 - \sqrt{2}$, find $y = (x-3)^2\left(\dfrac{x^4 - 6x^3 + 43x - 79}{x^2 - 6x + 13}\right)$. [10]

W3-2 Let n be a positive even integer greater than 2. An *n-valley permutation* is a permutation $(a_1, \ldots, a_n)$ of the ordered n-tuple $(1, \ldots, n)$ such that $a_1 > \cdots > a_{n/2}$ and $a_{n/2} > \cdots > a_1$. Determine the number of 44-valley permutations. *(You may express your answer in factorial notation.)* $\left[\dfrac{43!}{21!22!}\right]$

W3-3 Suppose that $\sum_{k=0}^{n-m-1}(k+m+1) = \alpha$, where m and n are positive integers greater than or equal to 2, with $m < n$. If x and y are real numbers, write $\sum_{i=1}^{n}(x+i) + \sum_{j=1}^{m}(y-j)$ in terms of n, m, and α.

$[nx + my + \alpha]$

Wave 4

W4-1 Four coins are arranged in such a way such that all coins are tangent to the other three. If three of them are identical, then what is the ratio of the radius of the bigger coin to the smaller coin? $\left[2\sqrt{3}+3\right]$

W4-2 Let a_1, a_2, b_1, and b_2 be real numbers. The graph of a cubic polynomial function $P(x) = x^3 + 43x^2 + a_1x + b_1$ with (complex) zeros p, q, r intersects the graph of a quadratic polynomial function $Q(x) = x^2 + a_2x + b_2$ with (complex) zeros r, s exactly once. Find the value of $p+q+pq+s$. $[167]$

W4-3 Let a, b, c be positive real numbers. If $a+b+c = abc$, and $\dfrac{1}{\sqrt{1+a^2}} + \dfrac{1}{\sqrt{1+b^2}} + \dfrac{1}{\sqrt{1+c^2}} > k$, find the (greatest) value of k. $[1]$

Wave 5

W5-1 How many six digit numbers are there with a digit whose square root is irrational in the ten thousands position, a Fibonacci number (1) in the hundreds position, a square-free number (2) in the thousands position, a prime power (3) in the hundred thousands position, and whose last two numbers combine to make a prime number? $[31\,500]$

> (1) The *Fibonacci sequence* F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ where $F_0 = 0$ and $F_1 = 1$. A *Fibonacci number* is an element of the Fibonacci sequence.
>
> (2) A *square-free number* is an integer which is divisible by no other perfect square than 1.
>
> (3) A *prime power* is a positive integer power of a single prime number.

W5-2 Four identical circles A, B, C, and D are drawn such that circle A is externally tangent to circle C, circle B is externally tangent to circle D, and circles A, B, C, and D intersect at point E. If r is the radius of A, find the maximum area bounded by the four circles. $\left[(2\pi r^2 + 4r^2)\text{ units}^2\right]$

W5-3 Define a figure to be a *circular square* if it is formed by constructing identical circular segments with central angle $\theta = \dfrac{\pi}{4}$ from each midpoint of two adjacent vertices of a square to its other two adjacent vertices. Two identical circular squares $JRCB$ and $EIKA$ with circular arcs of length 2π inches are drawn such that the distances of each vertex from point α are all equal and $JERICKBA$ is a regular octagon, as shown in the figure below. Find the area of the largest octagon that can be inscribed in the shaded region. $\left[(160\sqrt{2} - 32\sqrt{17+8\sqrt{2}})\text{ inches}^2\right]$