Pre-final, Wildcard Round

Item	Points	Time
1 to 5	2	15 s
6 to 9	4	30 s
10 to 12	6	45 s
13 to 14	8	60 s
15	10	75 s

Wizard Round

Item	Points	Time
1 to 8	2	15 s
9 to 12	4	30 s
13 to 15	6	45 s
16 to 18	8	60 s
19 to 20	10	75 s

Pre-final Round 1

P1-1 Today is Saturday, February 15, 2014, and we are celebrating the 41st Annual Search for the Math Wizard. This event is always held on a Saturday and if possible on Math Club's Anniversary Day. What was the year that this event was first held on an anniversary day? [1976]

P1-2 What is the greatest 4-digit number composed of 6, 7, 8, and 9 with each number only appearing once, that is divisible by 7? [9786]

P1-3 Given $f(x) = 4x - 5$ and $g^{-1}\left(\frac{1}{x}\right) = 4x - 5$, evaluate $(f \circ g)(-3)$. $\left[\frac{7}{4}\right]$

P1-4 Kel wanted to measure the height of a tree beside his dormitory. He went to the nearest window in the third floor of his dormitory and found that the angle of elevation to the top of the tree is 30° and the angle of depression to the bottom of the tree is 60°. If the dormitory is 30 m from the tree, how tall is the tree? $[40\sqrt{3}\,\text{m}]$

P1-5 Evaluate $\displaystyle\lim_{x\to\pi/2} \frac{\tan 2x - e^{2x-\pi} - 1}{\cos x}$. [4]

P1-6 What are the possible values of x if $\sqrt{x + 8\sqrt{3}}$ is equal to a certain number $A + B\sqrt{3}$, where A and B are both positive integers? [16, 19, 49]

P1-7 Compute for the determinant of the matrix $M = \begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 1 & 4 & 2 \\ 1 & 4 & 2 & 0 \\ 4 & 2 & 0 & 1 \end{bmatrix}$. [119]

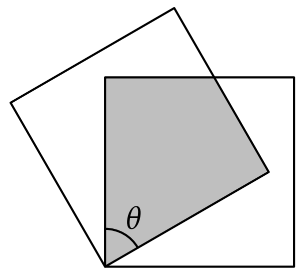
P1-8 What is the volume of the torus formed by revolving the circle $(x-5)^2 + (y-12)^2 = 9$ about $y = 2$? $[180\pi^2\ \text{units}^3]$

P1-9 Arrange the following numbers such that their units digits are in increasing order: 2012^{2013}, 2013^{2014}, 2014^{2015}, 2017^{2019}. $[2012^{2013}, 2017^{2019}, 2014^{2015}, 2013^{2014}]$

P1-10 A game of chance in numbers is played as follows. After each play, according to the outcomes, the player either receive p or q points and the scores accumulate from play to play. It has been noticed that there are 21 non-attainable scores, and one of these is 42. Find p and q. $[(p,q) = (15,4), (4,15)]$

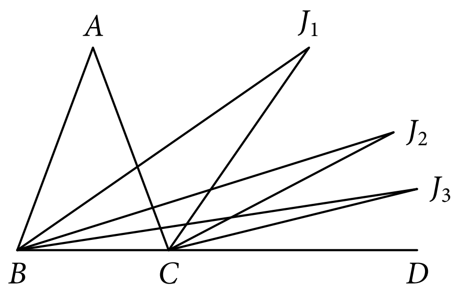
P1-11 If $x + \frac{1}{x} = 2\cos\theta$, find the value of $x^n + \frac{1}{x^n}$ in terms of n and θ. $[2\cos n\theta]$

P1-12 The figure shows two squares each with side one unit, arranged so that they overlap as represented by the shaded region. If $\theta = 60°$ and is gradually increasing at the rate of 1°/sec, find the rate at which the shaded area is changing. $\left[\dfrac{\pi}{270}\text{ units}^2/\text{sec}\right]$



P1-13 Determine the remainder when 200^{200} is divided by 312. $[40]$

P1-14 In $\triangle ABC$, $m\angle A = 41°$. Extend $\overline{BC}$ to an arbitrary point D. The angle bisectors of $\angle ABC$ and $\angle ACD$ intersect at point J_1, and the angle bisectors of $\angle J_1BC$ and $\angle J_1CD$ intersect at point J_2, and so on. Find the sum of the measures of $\angle A$ and $\angle J_i$ for all integers i, in degrees. $[82°]$



P1-15 Solve for the value of $\left\lceil \sum_{c=1}^{10^6} \dfrac{1}{\sqrt{c}} \right\rceil + 2014$. $[4013]$

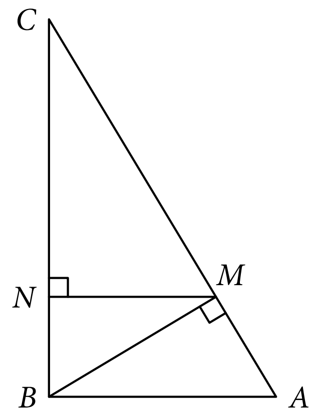
Pre-final Round 1

P2-1 How many positive integers less than 100 share a common factor with 100 other than 1? $[59]$

P2-2 Let $f(x) = ax^3 + bx^2 + c$. If the tangent lines at $x = 1$ and $x = 2$ of the graph of the function are parallel, find $\dfrac{a}{b}$. $\left[-\dfrac{2}{9}\right]$

P2-3 Suppose that $\log_9 a + \log_9 b = \log_9 a \cdot \log_9 b$ and $\log_a b = 2$. What is the value of a? $[27]$

P2-4 Triangle ABC has sides $AB = 3$, $BC = 4$, and $AC = 5$. Point M is chosen along AC such that $AC \perp BM$, and point N is chosen along BC such that $MN \perp BC$. What is the ratio of the area of $\triangle MNC$ to the area of $\triangle ABC$? $\left[\dfrac{256}{625}\right]$



P2-5 Fenina the fairy takes charge of granting wishes to 85 children every year. What is the probability that every year there is a month when she grants wishes of at least 8 children? [100%]

P2-6 In how many ways can the integers from 1 to 9 be arranged such that the sum of any three consecutive numbers is divisible by 3? [1296]

P2-7 Solve for all values of x such that $3\sec^{-1}\left(\sqrt{x^2+9}-3\right)+3\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)=\csc^{-1}(1)$. [$\pm 4$]

P2-8 Suppose you are running on the edge of a circular field of radius 10 m. You marked your starting place as point A and then you ran in a clockwise manner. After n meters, you marked your position as point B and continued running. After another n meters, you marked your position as point C, and n meters more and marked your final position as point D. If $m\angle BAD = 20^\circ$ and you didn't pass your starting position, what is your distance from your starting position? [10 meters]

P2-9 A very hungry Jovy thought of using all his ₱300 allowance on food. He wanted to buy monay and ice tea, which cost ₱8 and ₱27, respectively. How many of each will he buy to spend exactly ₱300? [24 monays and 4 ice teas]

P2-10 Find the sum of the infinite series $\frac{1}{3}+\frac{4}{9}+\frac{7}{27}+\frac{10}{81}+\frac{13}{243}+\cdots$. $\left[\frac{5}{4}\right]$

P2-11 Determine the greatest value of n such that 7^n can divide 1000! and leave a remainder of zero. [164]

P2-12 A health-fanatic Maynard wants to measure the distance of the road he travelled. He did this by calculating of the equation of the road, which he found to be $y=\frac{2}{3}(x+1)^{3/2}$. Find the distance travelled as he went from $x=2$ to $x=7$. $\left[\frac{38}{3}\right]$

P2-13 Let M be the set of all 3×3 matrices whose entries are the first nine prime numbers, each appearing only once. Determine the least possible determinant of an element in M. [-4868]

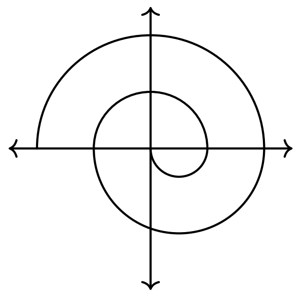
P2-14 In a game of ping-pong, a deuce happens when both players reach a match point. A player will only win the match if he manages to have a two-point advantage against the other. Each set merits one point. Sebastian and King were playing ping-pong until they reach a match point. If Sebastian has a 40% chance of winning a set, what is the probability that he will win the match?* $\left[\frac{4}{13}\right]$

* Some of the ping-pong terminologies in this question are wrongly used.

P2-15 Let $\triangle FIX$ be an isosceles triangle with $\overline{FI} \cong \overline{FX}$. If the angle bisector of $\angle I$ meets $\overline{FX}$ at Y, and $IX = IY + FY$, find the measure of $\angle F$ in degrees. [100°]

Pre-final Round 3

P3-1 Drunk Flick is travelling on the Cartesian plane. Starting from the origin, he travels a semicircle with diameter 1 unit with upward concavity and travels another semicircle with diameter 2 units with downward concavity, and so on, as shown in the figure. If he travelled a total of 25π units, at what point did he stop? [$(5, 0)$]



P3-2 Jake, Mike, Lyka, Cheska, Mika, and Floyd went to eat at Gerry's Grill. In how many ways can they sit at a round table if Floyd wants to sit across Mika? [24]

P3-3 Let S be the sum of n positive real numbers. If the product of these numbers is n, find the minimum value of S^n in terms of n. [n^{n+1}]

P3-4 $\overline{ZYX}$ is a 3-digit number. If $\overline{505\,XYZ}$ is the least possible 6-digit number of this form that is divisible by 7, 8, and 9, determine the value of $\overline{ZYX}$. [800]

P3-5 If the radius of the ball is measured with an error of at most 5%, estimate using linear approximation the percentage error in the volume of the ball. [15%]

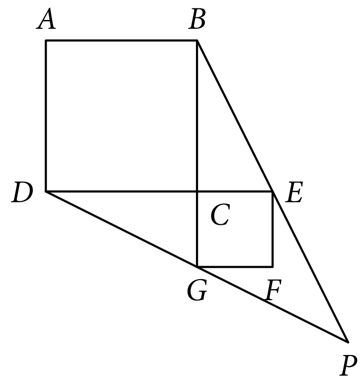
P3-6 An ant is moving along the curve $y = x^2 + 4x$ at a constant speed. At what point(s) in the Cartesian plane will the absolute value of the x-component and the absolute value of the y-component of his velocity be the same? $\left[\left(-\frac{3}{2}, -\frac{15}{4}\right) \text{ and } \left(-\frac{5}{2}, -\frac{15}{4}\right)\right]$

P3-7 A regular polygon was cut from a piece of cardboard and a pin was put through its centroid such that it can rotate. What is the least number of sides the polygon must have so that rotating it by 2.008° will make it coincide with its original position? [180]

P3-8 At 8:00 AM, Mae's yacht departed the pier and travelled in the direction S20°E at a speed of 25 mph. At the same time, Anjo's yacht left the same pier towards the direction N80°W at a speed of 15 mph. At what time will they be 105 miles apart? [11:00 AM]

P3-9 Anthony owns a s'mores factory. He determines the cost of production of making s'mores is given by $C(x, y) = 3x^2 + 2xy + y^2$, where x is the amount of cookie and y is the amount of marshmallow in kilograms that the factory produces each day. In order to lessen their expenses, they are cutting the production by 1 kg/day for cookies and 2 kg/day for marshmallows. If the factory is currently producing 5 kg of cookies and 7 kg of marshmallows, at what rate is their production cost decreasing? [₱92 /day]

P3-10 A kite can be formed by first connecting two squares, $ABCD$ and $CEFG$, having area of 2 and 1 square units, respectively. Points B, C, and G are collinear and point F is outside square $ABCD$. Lines are drawn from B to E and from D to G and they intersect at point P. Find the area of kite $ABPD$. $\left[(4+\sqrt{2})\ \text{units}^2\right]$



P3-11 Anna the ant started walking on the lower-leftmost square of a 4×4 chessboard going to the upper-rightmost square. If a trap was randomly hidden in one of the inner squares on the board (i.e. it is not in an edge square of the board) and Anna only travels one unit upward or one unit to the right at every instance, what is the probability that she avoided the trap? $\left[\frac{21}{40}\right]$

P3-12 Compute the number of ordered pairs of integers (a, b) such that $1 \le a \le 143$, $1 \le b \le 143$ and $\log_b a$ is rational. $[210]$

P3-13 Compute the binormal vector $\vec{B} = \vec{T} \times \vec{N}$ (the cross product of the unit tangent and unit normal vectors) for the vector-valued function $\vec{R}(t) = \{5\cos 3t, 5\sin 3t, 20t\}$. $\left[\frac{1}{5}(4\sin 3t, 4\cos 3t, 3)\right]$

P3-14 Evaluate the expression $(1-\cot 1^\circ)(1-\cot 2^\circ)\cdots(1-\cot 44^\circ)$. $[4\,194\,304]$

P3-15 The function g, defined by $g(x) = \dfrac{ax+b}{cx+d}$, where a, b, c, and d are nonzero real numbers, has the following properties: $g(51) = 31$, $g(31) = 51$, and $g(g(x)) = x$ for all values of x except $-\dfrac{d}{c}$. Express $g(x)$ as a sum of a positive integer and a fraction in x. $\left[41 + \dfrac{100}{x-41}\right]$

Wildcard Round

WC-1 Find the zeroes of $g(x) = 2\sin^2 x + 3\cos x - 3$ in the closed interval $[0, \pi]$ $\left[0, \frac{\pi}{3}\right]$

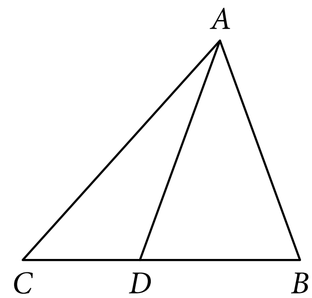
WC-2 Puff, Pica and Pat are in a game with each of them in a different team. The heart team always tells the truth. The square team always tells a lie. The triangle team tells the truth after telling a lie or tells a lie after telling the truth. These were the statements from the girls:

- Pica said, "I am from the triangle team."
- "Pat is the member of the triangle team."
- Puff said, "Pat is in the triangle team."

What team does Puff belong to? [heart team]

WC-3 If a random divisor of 10^{20} is chosen, what is the probability that it is a multiple of 20^{14}? $\frac{1}{9}$

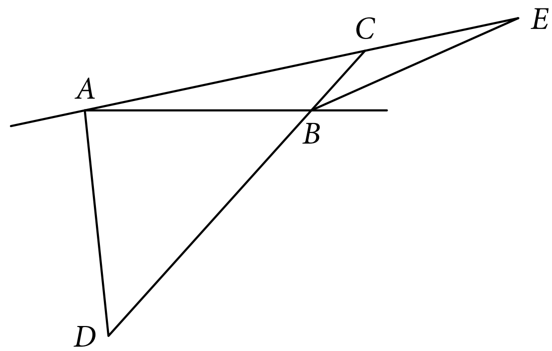
WC-4 In the figure, $\angle CAB$ is bisected by segment AD, $m\angle CAD$ is equal to 30 degrees, and $AC \cong AD$. If BD is equal to $\sqrt{2}$, find CD. $\left[(\sqrt{6}-\sqrt{2})\text{ units}\right]$



WC-5 Find a positive real number x such that the areas under the curve $y = \dfrac{1}{x}$ and the x-axis from 2 to x, and from x to 4, are equal. $\left[2\sqrt{2}\right]$

WC-6 Suppose that the function $f(x)$ satisfies $f(a+b) = \dfrac{(f(a)+f(b))^2}{4}$, $\forall a, b \in \mathbb{R}$. How many times does the graph of $f(x)$ intersect the x-axis if it doesn't pass the border? $[0]$

WC-7 In $\triangle ABC$, the angle bisectors of the exterior angles of $\angle A$ and $\angle B$ intersect opposite sides at D and E, respectively, and $AD = AB = BE$. Find $m\angle CAB + m\angle ABD$. $[60^\circ]$



WC-8 Evaluate $\int_0^{\pi/2} \frac{1}{\theta^2}(\theta\cos\theta - \sin\theta)\, d\theta$. $\left[\dfrac{2-\pi}{\pi}\right]$

WC-9 A mathematician's conjecture states that if p is a prime number of the form $\delta k - 1$, where k is a positive integer, then $p^2 - p + 1$ is a composite number. What must be the least value of δ so that the conjecture can be proven true? $[6]$

WC-10 Given the following system of equations

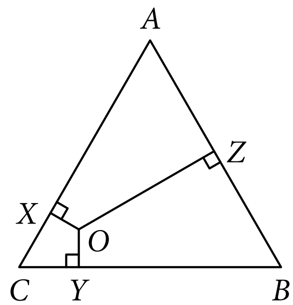
$$\begin{cases} (x+y)^2 + 3(x+z)^2 - (y+z)^2 = -2 \\ 2(x+y)^2 - (x+z)^2 - (y+z)^2 = -2 \\ 3(x+y)^2 + 4(x+z)^2 - (y+z)^2 = -2 \end{cases},$$

solve for $(x+y+z)^2$, if $(x+y), (x+z), (y+z) > 0$. $[9]$

WC-11 In how many ways can you connect four points A, B, C, and D if each point must be connected to at least one other point? $[41]$

WC-12 Determine all roots of $f(x) = x^6 - 8x^5 - 67x^4 + 1040x^3 - 4797x^2 + 9720x - 7425$. $[3, 5, -11]$

WC-13 OX, OY, and OZ are all perpendicular to the sides of the equilateral $\triangle ABC$, which has an area of $12\sqrt{3}$ square units. If OX and OY are equal to 1 unit, and 2 units, respectively, find BZ. $\left[\dfrac{7\sqrt{3}}{3} \text{ units}\right]$



WC-14 Determine the interval of convergence of the power series given by $\sum_{n=0}^{\infty} \dfrac{x^n}{7+n^3}$. $[[-1,1]]$

WC-15 Let n be a positive integer. Suppose there are 2014 positive integers less than n, which are relatively prime to n. Find the sum of all positive integers less than n^2 which are relatively prime to n, in terms of n. $[1007n^3]$

Wizard Round

W-1 Dora wants to go to Diego's house. On her way there, she encountered four paths with the following inscriptions:

Path 1: "This is the way to Diego's house."

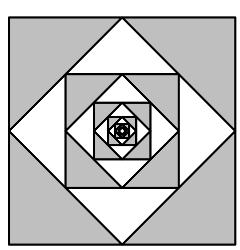
Path 2: "Path 3 does not lead to Diego's house."

Path 3: "Exactly 2 of the path inscriptions are lying."

Path 4: "Path 2 leads to Diego's house."

If exactly one of the inscriptions is telling the truth, which path should Dora take? [Path 4]

W-2 Smaller squares are made by connecting the midpoints of the sides of the bigger square. This process is done infinitely many times to produce the figure shown. If the area of the biggest square is 1 square unit, find the area of the shaded region. $\left[\dfrac{2}{3} \text{ units}^2\right]$



W-3 Find the sum of a and b in the differential equation $\dfrac{d^2x}{dt^2} + a\dfrac{dx}{dt} + b = 0$, if $x = e^t$ is a solution. $[-1]$

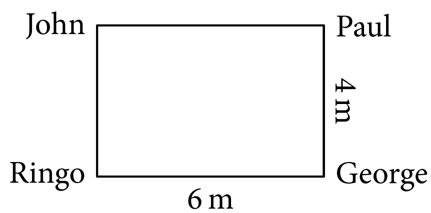
W-4 Evaluate $\lim_{x\to 0} \dfrac{1+x-e^x}{\sin^2 x}$. $\left[-\dfrac{1}{2}\right]$

W-5 What are the possible ordered pairs (x, y) where $x > y > 0$, and x and y are integers, if the difference of the product of x and y and the sum of x and y is equal to 20? $[(22,2),(8,4)]$

W-6 Two real numbers a and b satisfy the equation $\sqrt{7+2\sqrt{2}} = \sqrt{a} + \sqrt{b}$. What is the sum of the squares of the two numbers? $[45]$

W-7 Find the value of the expression $(\sinh 1 - \cosh 1) + (\sinh 2 - \cosh 2) + (\sinh 3 - \cosh 3) + \cdots$ in terms of e. $\left[\dfrac{1}{1-e}\right]$

W-8 John, Paul, George, and Ringo formed the 4 m by 6 m rectangle as shown in the figure. George got bored so he moved 2 m downward. In order for the figure they formed to still be cyclic (can be inscribed in a circle), John moved to the right until the figure is cyclic. What is the area of the circle that circumscribes the quadrilateral they formed? $\left[\dfrac{65\pi}{4}\text{ units}^2\right]$



W-9 How many positive integers x are there, such that $x < 100$ and simultaneously satisfies $x \equiv 3 \pmod 5$ and $x \equiv 2 \pmod 3$? $[7]$

W-10 A jar containing 15 coins amounts to ₱100. The coins are either 25-centavo, 1-peso, 5-peso, or 10-peso. If we get a coin in the jar, what is the probability that it is a 5-peso coin? $\left[\dfrac{11}{30}\right]$

W-11 Let $x \in \mathbb{R}$ such that $\csc x + \tan x = 2$. Evaluate $\sin 2x$. $\left[\dfrac{24}{25}\right]$

W-12 Find the value of b such that the family of curves $y = \dfrac{6}{b}(x+k)^{1/3}$ are the orthogonal trajectories of the family of curves given by $y = \dfrac{1}{x+c}$. $[2\sqrt[3]{9}]$

W-13 If having the roots of $x^3 - ax^2 + bx - c = 0$ be in geometric progression implies $b = 2$, express c in terms of a. $\left[c = \dfrac{8}{a^3}\right]$

W-14 What is the least number of people that will ensure the presence of either three mutual friends or three mutual strangers? $[6]$

W-15 Evaluate the expression $\tan^{-1}\dfrac{1}{1+1(2)} + \tan^{-1}\dfrac{1}{1+2(3)} + \tan^{-1}\dfrac{1}{1+3(4)} + \cdots$. $\left[\dfrac{\pi}{4}\right]$

W-16 Let α be a convex quadrilateral with diagonals of lengths $\sqrt{2}$ and $4\sqrt{5}$. Form the parallelogram β by joining the midpoints of the sides of α. If the lengths of the diagonals of β are x and y, find the value of $x^2 + y^2$. $[41]$

W-17 Yesterday was Valentine's Day and a lot of guys bought a gift for their girlfriends. A certain store that sells flowers, chocolates and stuffed toys tracks their customers on Valentine's Day. This store does not tolerate "two-timing boyfriends" so they only sell one item to each customer. According to their inventory, 50% of its customers bought flowers, 35% of its customers bought chocolate, and 15% of its customers bought stuffed toys. The result of their tracking study were as follows: 75% of the girls who got flowers loved their boyfriend more, 90% of the girls who got chocolates loved their boyfriend more, and 95% of the girls who got stuffed toys loved their boyfriend more. What is the probability that the girl received a stuffed toy given that she loved her boyfriend more? $\left[\dfrac{19}{111}\right]$

W-18 A rectangle is inscribed in a circle. The ratio of the area of the rectangle to the area of the circle is $1 : r$, where r is the radius of the circle. Find the area of the smallest such circle. $\left[\dfrac{\pi^3}{4}\text{ units}^2\right]$

W-19 Yesterday was Valentine's Day, and Cupid organized a speed-dating event for single ladies. Cupid allowed every participant to date as many guys as they want, and he promised that if a participant dated a total of n guys, she will be guaranteed that the kth guy she dated, where k is relatively prime to n, will fall in love with her. Vanessa attended the event yesterday and she felt that 32 guys have fallen in love with her. What is the greatest number of guys that Vanessa dated? [120]

W-20 Find the value of $h(2\sqrt{2})$ if the function $h : \mathbb{R} \to \mathbb{R}$ satisfies $x - h(y) = h(h(y)) + xh(y) + h(x) - 1$ $\forall x, y \in \mathbb{R}$. $[-3]$