The Annual Nationwide Search for the Math Wizard 2019

Oral Rounds

February 23, 2019

Thanks to Nathanael Joshua Balete for the projective plane figure and some corrections. Square brackets denote inexact wording. If you have any corrections or additions, please contact me at cj@cjquines.com.

Prefinal Round 1

1. (15s) Evaluate $\lim_{x\to 0^+} \frac{\sin x - x}{x^3}$. $\left[-\frac{1}{6}\right]$

2. (15s) Evaluate $\tan\left(\sec^{-1}(4)\right) - \csc\left(\cot^{-1}(3)\right)$. $\left[\sqrt{15} - \sqrt{10}\right]$

3. (15s) In a school fair, a total of 41 people won prizes from Axa Philippines after winning in 3 game booths. There are 7 people who won in booths A and B, 8 people who won in booths A and C, and 9 people who won in booths B and C. If each booth had 20 winners, how many people won in all three game booths? [5]

4. (15s) Find the value of

$$\left(1+\frac{3}{4}\right)\left(1-\frac{3}{5}\right)\left(1+\frac{3}{6}\right)\left(1-\frac{3}{7}\right)\cdots\left(1+\frac{3}{20}\right)\left(1-\frac{3}{21}\right).$$

$\left[\frac{23}{50}\right]$

5. (30s) Find the value of the integer term of the rationalized and simplified form of

$$\frac{2}{1+\sqrt{2}+\sqrt{3}+\sqrt{5}}.$$

$[-5]$

6. (30s) Find the greatest common divisor of 119 121 and 2 866 980. [2019]

7. (30s) Three solids A, B, and C, have faces that are made up of the same figure, and no two faces merge when another solid is formed by gluing the face of one solid to another. When one face of A is glued to one face of B, the solid formed has 12 faces. When one face of B is glued to one face of C, the solid formed has 26 faces. When one face of A is glued to one face of C, the solid formed has 24 faces. Find the number of faces of the solid formed by gluing one face of A to one face of B, and one face of B is glued to one face of C. [30]

8. (45s) If a is a constant greater than 4 that satisfies the equation

$$\int_a^4 \frac{x-4}{x+2\sqrt{x}}\,dx = 16,$$

find the value of $a^2 - 16a$. [720]

9. (45s) Two identical circles with radius r intersect, forming a region bounded by $60°$ arcs along each circle. If the two circles both move along the perpendicular bisector to the line segment connecting their intersection points, and approach each other at a rate of $\frac{1}{2}r$ per second, what is the rate of increase per second of the region in terms of r? $\left[\dfrac{2\pi r}{3} - \sqrt{3}r\right]$

10. (60s) Assuming that $t \in (0,1)$, solve for the value of t in the given equation:

$$\cos^{-1}\left(\sqrt{2(t+1)-(t+1)^2}\right) = 2\cos\tan^{-1} t - \cos^{-1}\left(\frac{1}{t^{-1}-t^{-2}} - \frac{1}{t-1}\right).$$

$\left[\sqrt{\dfrac{16}{\pi^2} - 1}\right]$

11. (60s) Find all points $(x,y) \in \mathbb{R}^2$ in the conic $x^2 - 10xy + y^2 = 1$ such that for all suitable $K \in \mathbb{R}$ the line $-3y = K - 3x - 2$ intersects the cubic

$$-x^3 + y^3 - 2x^2 - 4xy + 3x^2y - 3xy^2 - 2y^2 = K.$$

$[(-1,0),(0,1)]$

12. (75s) A member of the Actuarial Society of the Philippines had previously chosen twelve cards from a standard deck such that each suit has 3 cards. These cards are then randomly and equally distributed to 4 people. Find the probability that at least one and at most two persons receive two cards of the same suit and all people received cards form at least two suits. $\left[\dfrac{162}{1925}\right]$

C1. (30s) Find the remainder when 4^{2017} is divided by 33. [16]

C2. (30s) If $a^x = b^y = c^z$ and $b^2 = ac$, express $\dfrac{1}{x} + \dfrac{1}{z} + \dfrac{1}{y}$ in terms of y only. $\left[\dfrac{3}{y}\right]$

Prefinal Round 2

1. (15s) Find the perimeter of the right triangle with hypotenuse of length 20 meters and whose area is maximum. $\left[\left(20 + 20\sqrt{2}\right)\ \text{m}\right]$

2. (15s) If a biased coin with a head thrice as likely than a tail, what is the standard deviation of turning up a head with 1 200 tosses? [15]

3. (15s) Given $\dfrac{x^2}{4} + \dfrac{8}{x^2} = 10$, find the value of $\dfrac{x^6}{64} + \dfrac{512}{x^6}$. [940]

4. (15s) Given that $\vec{a} = (5,-5,6)$ and $\vec{b} = (2,0,-3)$, what is $\vec{a} \times \vec{b} \cdot 2\vec{a}$? [0]

5. (30s) Find the length of the curve parametrized by the system $x = t - \sin t$ and $y = 1 - \cos t$, where t ranges from $\dfrac{\pi}{2}$ to π. $\left[2\sqrt{2}\ \text{units}\right]$

6. (30s) Given that $\alpha \in \left[0, \dfrac{\pi}{2}\right]$, find the smallest value of α such that

$$\frac{\sec(5\alpha) + \csc(3\alpha)\tan(5\alpha)}{\csc(3\alpha) + \cot(3\alpha)\sec(5\alpha)} = \cot(4\alpha).$$

$\left[\dfrac{\pi}{16}\right]$

7. (30s) Kyth and her 8 friends went to a wedding reception. They learned they all cannot sit together at once to eat. If only 5 seats can be taken at a time, how many different ways can Kyth and her friends sit together if Kyth takes as long as the whole event to finish eating? [840]

8. (45s) Let x, u, and v be differentiable functions defined on $\mathbb{R}^2$, and r and s be differentiable functions defined on $\mathbb{R}$. If $\dfrac{dx}{dt} = 4$, $u_s = v_r = 1$, $u_r + v_s = 2$, and $d\langle s, r\rangle/dt = \nabla x(u, v)$, find $\left|(2019x)_u + (2019x_v)\right|$. [4038]

9. (45s) Find the number of ordered pairs (x, y) of integers in $[-2019, 2019]$ that satisfy the equation $\dfrac{51}{x} + \dfrac{85}{y} = \dfrac{119}{xy}$. [808]

10. (60s) Evaluate

$$\sec^2 1^\circ \left(2 - \csc^2 1^\circ\right) + \sec^2 3^\circ \left(2 - \csc^2 3^\circ\right) + \sec^2 5^\circ \left(2 - \csc^2 5^\circ\right) + \cdots + \sec^2 2019^\circ \left(2 - \csc^2 2019^\circ\right).$$

[0]

11. (60s) Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers such that for each $i \in \{1, 2, \ldots, n\}$, $\log \alpha_i$ is a root of the polynomial function $P(x)$ with even degree n. If

$$\left(1000\alpha_1^{\log 100\alpha_1}\right)^{\left(\log 1000\alpha_2^{\log 100\alpha_2}\right)\left(\log 1000\alpha_3^{\log 100\alpha_3}\right)\cdots\left(\log 1000\alpha_n^{\log 100\alpha_n}\right)} = 10$$

and

$$(\log 100\alpha_2)(\log 100\alpha_3)\cdots(\log 100\alpha_n) = \frac{6}{4 + \ln \alpha_1^2},$$

find $P(-3)$. $\left[\dfrac{\ln 10}{3}\right]$

12. (75s) Find the number of quadratic polynomials $Q(x)$ such that all of its roots are integers and $Q(0) = Q'(0) + 98$. [6]

Prefinal Round 3

1. (15s) Evaluate $(\sin(0.6\pi) - i\sin(1.1\pi))^{1000}$. [1]

2. (15s) Determine the prime factorization of A, given

$$A = \frac{(2)(4)(6)\cdots(2018)(2020)(5)(10)(15)\cdots(2015)(2020)}{(404)(403)(402)\cdots(2)(1)(1010)(1009)(1008)\cdots(2)(1)}.$$

$[2^{1010} \cdot 5^{404}]$

3. (15s) Simplify the expression $\dfrac{(y^2+y+1)^2 - (y+1)^2 + 2y^2}{y^2 + 2y + 4}$. $[y^2]$

4. (15s) What is the last digit of $\left(3^5 + 2^5\right)\left(3^5 - 2^5\right)\left(3^{20} + 2^{20} + 6^{10}\right)$? [5]

5. (30s) The sum of two numbers is 326, while the product of the two numbers is 4 665. Find the sum of the cubes of the two numbers. [30 083 606]

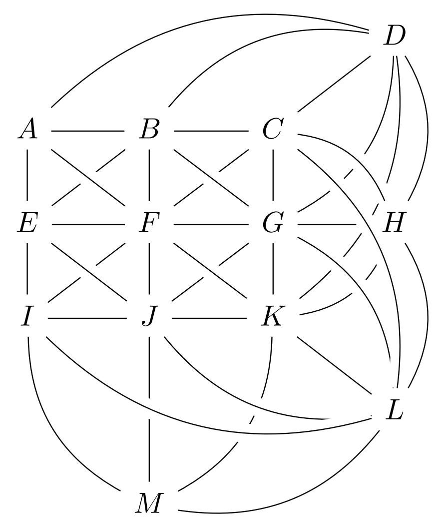
6. (30s) Suppose that A and A' are fixed points equidistant from the origin and P is the set of all points p such that $|pA| + |pA'| = 4$. Furthermore, let B and B' be points on P such that BB' passes through the origin, is perpendicular to AA', and $|BB'| = 2$. Find the equation of the set of all points for which the distance of the midpoint of AA' to A is constant.[1] $\left[x^2 + y^2 = 3\right]$

6. (30s) Let $\triangle ABC$ be a right trianlge with right angle at B. Points D and E divide BC into equal parts. Find $m\angle ADB + m\angle AEB + m\angle ACB$ in radians. $\left[\dfrac{\pi}{2}\right]$

7. (30s) Let $A = \{35, x, 21\}$ and $B = \{x, y, 6\}$. Suppose that x is the smallest positive integer such that the LCM of the elements of A is 420, and y is the smallest possible integer such that the LCM of the elements of B is 2 100. Find the LCM of x and y. [700]

8. (45s) Evaluate $\sin(1) + \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{\left\lfloor \frac{n+1}{2} \right\rfloor}}{n!}$. [cos(1)]

9. (45s) In the figure is a model of a finite projective plane of order 3. A projectivity from $IJKH$ to $AEIM$ is the product of a perspectivity from $IJKH$ to $EBDK$ with center M, and a perspectivity from $EBDK$ to $AEIM$ with center H. Get the images of I, J, K and H under this projection.



$[E, A, M, I]$

10. (60s) Suppose that $\sec x = \sqrt{5}$ and $\tan 3y = \dfrac{\sqrt{5}}{2}$. Let T be the expression below:

$$\frac{\displaystyle\sum_{i=0}^{3} \sin(x + iy)}{\displaystyle\sum_{i=0}^{3} \cos(x + iy)}.$$

[1] voided; the answer was printed on the question slips.

Assuming that $2x \in [0, \pi]$ and $y \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$, find the value of $19T + 12$ such that its denominator is rationalized. $\left[5\sqrt{5}\right]$

11. (60s) Let X and Y be independent random variables with $E(X) = \dfrac{2}{1+\frac{1}{2}}$ and $E(Y) = \dfrac{3\sqrt{2}-3}{2}$, $V(X) = 0$ and $V(Y) = 2^{-4}$. Solve
$$E\left[\left(E\left[\left(V\left[E(3X+2Y) - V(4Y)\right]\right)Y\right]X\right)X\right].$$
[2]

12. (75s) Evaluate the integral
$$\iiint_{\Omega} \frac{(3x+y-4)^2}{9x^2+y^2+z^2-18x-2y+10}\, dz\, dy\, dx,$$
where $\Omega = \left\{(x, y, z) \in \mathbb{R}^3 \mid 9(x-1)^2 + (y-1)^2 + z^2 \leq 1\right\}$. $\left[\frac{8\pi}{27}\right]$

C1. (30s) Solve for the real-valued function $f(x)$ satisfying $f\left(\frac{x+1}{x-1}\right) = \frac{x^2+1}{x^2-1}$. $\left[f(x) = \frac{x^2+1}{2x}\right]$

C2. (30s) If $\sin(2x) = \frac{1}{3}$, find the exact value of $\sin^6 x + \cos^6 x$. $\left[\frac{11}{12}\right]$

C3. (30s) Suppose $\tan x = \frac{1}{2}$. Determine the value of $\tan 4x$. $\left[-\frac{24}{7}\right]$

Wildcard Round

1. (15s) Shawn and Seth are observing a blimp in the sky. Shawn is on the ground while Seth is on the top of a building that is built on the ground. If the angle of elevation of the blimp from Seth is 30° and Shawn is 15 meters away from the nearest side of the building, how high is the blimp from the ground, assuming that Shawn and Seth's line of sight coincide and their heights are negligible? $\left[10\sqrt{3}\text{ m}\right]$

2. (15s) Mr. Krazy with his friends John, Louie, Josh and Eric went on a seminar headed by the Actuarial Society of the Philippines. After some time, Mr. Krazy went to the comfort room and left his bag on their table. Having mischievous friends, Mr. Krazy found out that his bag was missing when he returned. He asked his friends where his bag is and these are their answers:
   - John: The bag is with Josh.
   - Louie: I don't have the bag.
   - Josh: I did not take the bag. Eric and Louie did.
   - Eric: John has the bag.

   If there is only one person who told the truth, who took the bag? [Eric and Louie]

3. (15s) Gwyneth is tossing an unfair regular 7-sided die. The first side appears twice as likely, while the other sides with numbers that divide 6 appear thrice as likely. What is the probability that a side of prime number will appear? $\left[\frac{4}{7}\right]$

4. (15s) Find all ordered pairs (x, y) of positive integers such that the difference between the area of a square with sides $x + y$ units, and the area of a square with sides $x - y$ units, is 22. $[(11, 1), (1, 11)]$

5. (30s) Integrate $\int \cos(\ln x)\, dx$. $\left[\dfrac{x\cos(\ln x) + x\sin(\ln x)}{2} + C\right]$

6. (30s) What are the points closest to the origin of the inversion of $36x^2 + 25y^2 = 1$ in the unit circle? $[(0, 5), (0, -5)]$

7. (30s) Find the 2019th smallest real number α such that $\alpha > 1$ and

$$\sin(\ln \alpha) + 2\cos(4\ln\alpha)\sin(3\ln\alpha) = 0.$$

$\left[e^{2019\pi/7}\right]$

8. (45s) Find a quadratic polynomial with a leading coefficient of 3 whose roots are two less than the reciprocals of the fourth powers of the roots of $2x^2 + x + 1$. $\left[3x^2 + 9x + 54\right]$

9. (45s) Solve the system

$$\begin{aligned} \sin^2 2x + \cos^2 y + 8\sin x\cos x &= 2\cos y - 1 \\ \cos^2 y + 3\sin 2x &= 2\cos y - 3. \end{aligned}$$

where $x \in [0, 2\pi]$ and $y \in [0, 2\pi]$. [no real solutions]

10. (60s) Find the number of ordered 98-tuples $(x_1, x_2, \ldots, x_{98})$ of positive integers greater than 1 such that $x_1x_2\cdots x_{98} = 2^{101}$. [161 700]

11. (60s) Find the probability that a positive odd number less than 2019 has an odd sum of divisors. $\left[\dfrac{22}{1009}\right]$

12. (75s) Find the equation of the plane such that its intersection with the surface

$$7x^2 - 48xz - 7z^2 + 25y^2 = 0$$

is a single point. $[3x + 4z = 0]$

Wizard Round

1. (15s) Let f be a differentiable function having a differentiable inverse. If $f(0) = 1$ and $f'(0) = \dfrac{1}{2}$, find $\left(f^{-1}\right)'(1)$. [2]

2. (15s) A particle moves along the curve $x = e^t$, $y = \cos 2t$ where t represents the time. Determine the magnitude of the initial acceleration of the particle. $\left[\sqrt{17}\right]$

3. (15s) Suppose $\sin\theta = \dfrac{2}{5}$ and $\tan\theta > 0$ and

$$\frac{(\cos\theta + \cot\theta)^2}{\cos\theta\cot\theta} - 2 = \frac{a}{b},$$

where a and b are positive coprime integers. Find the value of $a + b$. [39]

4. (15s) Adarna House sells books for different grade levels – Grades 1, 2, 3 and 4 – this month. These books were sold for PhP 200, PhP 250, PhP 275, and PhP 300, respectively. On that day, the manager said that 35% of the books sold were for Grade 1, 30% for Grade 2, 15% for Grade 4, and the rest was for Grade 3. Find the average cost of a book sold that day. [PhP 247.50]

5. (15s) Give the name of the surface that is the graph of the equation $4y^2 - 8y - 9z^2 = 9z - 13$ in $\mathbb{R}^3$. [hyperbolic paraboloid]

6. (30s) A surface bounded by the trapezoid $ABCD$ is such that there exists a right angle at point C and $BC = 1$, $AB = 3$, and $CD = 2$. Find the mass of this surface if the mass of a point is equal to the sum of the lengths of its perpendiculars to the sides BC and CD. $\left[\frac{9}{2}\right]$

7. (30s) Anna, Rose, Jan, Elle, Ysa, Lindy, and Fern went on a financial consultation conducted by Accenture. To have a free consultation, they need to fall in line. Anna doesn't want to be beside Fern or Rose; Lindy wants Fern to be exactly behind her and doesn't want Rose beside; Elle can't take it when Jan or Ysa is beside her; Ysa is just okay with any order as long as she's not in the last; while Rose really wants to be the first in the line and doesn't want Elle to be next. If Lindy wants to be third in line and all their preferences are met, [who are able to switch positions in the line?] [Jan, Ysa, and Elle]

8. (30s) Determine the set of all $a \in \mathbb{R}$ such that the matrix

$$\begin{bmatrix} 3a & 0 & 0 \\ 2017 & a & -1 \\ 2018 & 1 & 2a \end{bmatrix}$$

has real eigenvalues. $[(-\infty, -2] \cup [2, \infty)]$

9. (30s) Find the inverse Laplace transformation of

$$\frac{6s^2}{(s^2+9)^2} + \frac{2s+1}{(s-4)(s^2+2s-15)}.$$

$$\left[\sin(3t) + 3t\cos(3t) - \frac{1}{4}e^t - \frac{3}{16}e^{-5t} + \frac{7}{16}e^{3t}\right]$$

10. (45s) If $\cos^2\alpha + \cos^2\beta = 2$ and $\cos(\alpha - \beta) + \cos(\alpha + \beta) = -1$, evaluate

$$(\csc b)\left(\tan\left(\frac{a+b}{2}\right) - \tan\left(\frac{a-b}{2}\right)\right).$$

Assume $\alpha, \beta \in$ QI. [2]

11. (45s) The electric potential energy in joules between two points A and B with charges Q_A and Q_B is given by

$$\xi = \frac{kQ_AQ_B}{|AB|},$$

where $|AB|$ denotes the distance between A and B in meters, k is a fixed number, and the charges are in coulombs. A point charge Q is placed 1 meter above a charged circular disk of radius $\sqrt{3}$ meters whose charge is evenly distributed. If the charge of Q is 3 coulombs and the total charge of the disk is $\frac{1}{k}$ coulombs, find the total electric potential energy experienced by the charge Q. [2 joules]

12. (45s) Find the number of ordered pairs (x, y) of positive integers less than 2019 such that for some $k \in \mathbb{N}$, $x = 2^k$ and $y + 2x = 4^k - xy$. [5]

13. (60s) If U and X are 3-by-3 matrices with entries that are real numbers, U is upper triangular with all diagonal entries equal to 1, and

$$U = X \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 3 & 4 & 5 \end{bmatrix},$$

find the sum of the third row entries of X. $\left[\frac{19}{2}\right]$

14. (60s) Find the general solution of

$$(1 - \cos 2x)\, y'' + \sin x y' - 2y = 0.$$

$\left[y = 4C_1 \sin^2 x + 4C_2 \csc^2 x\right]$

15. (60s) Let f be a function defined on positive integers such that:
    - if a and b are positive integers such that $\gcd(a, b) = 1$, then $f(ab) = f(a)f(b)$,
    - if $p = 5$ or $p = 7$, then $f(p^k) = k$ for all positive integers k,
    - if p is a prime neither equal to 5 nor equal to 7, then $f(p^k) = 0$ for all positive integers k,
    - $f(1) = 1$.

    Evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{f(k)}{k}$. $\left[\frac{215}{144}\right]$

16. (75s) Let $f(x) = xe^x$. If

$$f^{(2019)}(x) = \sum_{n=1}^{\infty} A_n \cos(nx) + \sum_{n=1}^{\infty} B_n \sin(nx),$$

find A_1. $\left[\cosh(\pi) + \frac{\sinh(\pi)}{\pi}\right]$

17. (75s) For each $n \in \mathbb{N}$ and $z \in \mathbb{C}$, we have

$$z^n - 1 = \prod_{d \in \mathbb{N},\, d \mid n} \Phi_d(z),$$

and the Φ_d's are polynomials. How many integer values of k where $0 \leq k < 2000$, are such that $\Phi_{2000}\left(\operatorname{cis} \frac{k\pi}{1000}\right) = 0$? [800]