Pre-final, Wildcard Round

Item	Points	Time
1 to 4	2	15 s
5 to 7	4	30 s
8 to 9	6	45 s
10 to 11	8	60 s
12	10	75 s
Clincher		30 s

Wizard Round

Item	Points	Time
1 to 5	2	15 s
6 to 9	4	30 s
10 to 12	6	45 s
13 to 15	8	60 s
16 to 17	10	75 s

Pre-final Round 1

P1-1 Evaluate $\displaystyle\lim_{x\to 0^+} \frac{\sin x - x}{x^3}$. $\left[-\dfrac{1}{6}\right]$

P1-2 Evaluate $\tan(\sec^{-1} 4) - \csc(\cot^{-1} 3)$. $\left[\sqrt{15}-\sqrt{10}\right]$

P1-3 In a school fair, a total of 41 people received prizes from AXA Philippines after winning in 3 game booths. Seven people won in booths A and B, 8 won in booths A and C, while 9 won in booths B and C. If each booth has 20 winners, how many people won in all 3 game booths? [5]

P1-4 Find the value of $\left(1+\frac{3}{4}\right)\left(1-\frac{3}{5}\right)\left(1+\frac{3}{6}\right)\left(1-\frac{3}{7}\right)\cdots\left(1+\frac{3}{20}\right)\left(1-\frac{3}{21}\right)$. $\left[\dfrac{23}{50}\right]$

P1-5 Find the integer term of the rationalized and simplified form of $\dfrac{2}{1+\sqrt{2}+\sqrt{3}+\sqrt{5}}$. $[-2]$

P1-6 Find the greatest common divisor of 119 121 and 2 866 980. [2019]

P1-7 Three solids A, B, and C have faces that are made up of the same figure and no two faces merge when another solid is formed by gluing one solid onto another. When one face of A is glued to one face of B the solid formed has 12 faces. When one face of B is glued to one face of C the solid formed has 26 faces. When one face of A is glued to one face of C the solid formed has 24 faces. Find the number of faces of the solid formed when one face of A is glued to one face of B and one face of B is glued to one face of C. [30]

P1-8 If a is a constant greater than 4 and it satisfies the equation $\displaystyle\int_4^a \frac{x-4}{x+2\sqrt{x}}\,\mathrm{d}x = 16$, find the value of $a^2 - 16a$. [720]

P1-9 Two identical circles of radius r intersect, forming a region bounded by 60° arcs of each circle. If the circles both move along the perpendicular bisector to the line segment containing their intersection points and approach each other at a rate of $\frac{1}{2}r$ per second, what is the rate of increase per second of (the area of) the region in terms of r? $\left[\dfrac{2\pi r}{3}-\sqrt{3}r\right]$

P1-10 Assuming $t \in (0,1)$, solve for the value of t in the given equation: $\cos^{-1}\left(\sqrt{2(t+1)-(t+1)^2}\right) = 2\cos\tan^{-1} t - \cos^{-1}\left(\dfrac{1}{t^{-1}-t^{-2}} - \dfrac{t}{t-1}\right)$ $\left[\sqrt{\dfrac{16}{\pi^2}-1}\right]$

P1-11 Find all points $(x,y)\in\mathbb{R}^2$ in the conic $x^2-10xy+y^2=1$ such that for all suitable $k\in\mathbb{R}$, the line $-3y = k-3x-2$ intersects the cubic $-x^3+y^3-2x^2-4xy+3x^2y-3xy^2-2y^2=k$. $[(-1,0),(0,1)]$

P1-12 A member of the Actuarial Society of the Philippines had previously chosen 12 cards from a standard deck of cards such that each suit has 3 cards. These cards are then randomly and equally distributed to 4 people. Find the probability that at least 1 and at most 2 persons receive 2 cards of the same suit and all people receive cards from at least 2 suits. $\left[\dfrac{162}{1925}\right]$

Pre-final Round 1 Clincher (for Wizard Round)

P1-C1 Find the remainder when 4^{2017} is divided by 33. $[16]$

Pre-final Round 1 Clincher (for Wildcard Round)

P1-C2 If $a^x = b^y = c^z$ and $b^2 = ac$, express $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$ in terms of y only. $\left[\dfrac{3}{y}\right]$

Pre-final Round 2

P2-1 Find the perimeter of the right triangle with hypotenuse of length 20 m and whose area is maximum. $\left[(20 + 20\sqrt{2})\text{ m}\right]$

P2-2 If a biased coin turns up with a head thrice as likely as with a tail, what is the standard deviation of turning up a head with 1200 coin tosses? $[15]$

P2-3 Given that $\dfrac{x^2}{4} + \dfrac{8}{x^2} = 10$, find the value of $\dfrac{x^6}{64} + \dfrac{512}{x^6}$. $[940]$

P2-4 Given that $\vec{a} = (5, -5, 6)$, $\vec{b} = (2, 0, -3)$, what is $\vec{a} \times \vec{b} \cdot 2\vec{a}$? $[0]$

P2-5 Find the length of the curve parametrized by $x = t - \sin t$, $y = 1 - \cos t$, for $t \in \left(\dfrac{\pi}{2}, \pi\right)$. $[2\sqrt{2}]$

P2-6 Given that $\alpha \in \left[0, \dfrac{\pi}{2}\right]$, find the smallest value of α such that $\dfrac{\sec 5\alpha + \cos 3\alpha \tan 5\alpha}{\csc 3\alpha + \cot 3\alpha \sec 5\alpha} = \cot 4\alpha$. $\left[\dfrac{\pi}{16}\right]$

P2-7 Kyth and her 8 friends went to a wedding reception. They learned they all cannot sit together at once to eat. If only 5 seats can be taken at a time, how many different ways can Kyth and her friends sit together if Kyth takes as long as the whole event to finish eating? $[840]$

P2-8 Let x, u, v be differentiable functions defined on $\mathbb{R}^2$ and r, s be differentiable functions defined on $\mathbb{R}$. If $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 4$, $u_s = v_r = 1$, $u_r + v_s = 2$, and $\dfrac{\mathrm{d}(s,r)}{\mathrm{d}t} = \nabla x(u,v)$, find $|(2019x)_u + (2019x_v)|$. $[4038]$

P2-9 Find the number of ordered pairs (x, y) of integers in the interval $[-2019, 2019]$ that satisfy $\dfrac{51}{x} + \dfrac{85}{y} = \dfrac{119}{xy}$. $[808]$

P2-10 Evaluate $\sec^2 1°(2-\csc^2 1°)+\sec^2 3°(2-\csc^2 3°)+\sec^2 5°(2-\csc^2 5°)+\cdots+\sec^2 2069°(2-\csc^2 2069°)$. $[0]$

P2-11 Let $a_1, a_2, \ldots, a_n$ be real numbers such that for each $i \in \{1, 2, \ldots, n\}$, $\log a_i$ is a root of the polynomial function $P(x)$ with even degree n. If $\left(1000a_1^{\log 100a_1}\right)^{(\log 1000a_2^{\log 100a_2})(\log 1000a_3^{\log 100a_3})\cdots(\log 1000a_n^{\log 100a_n})} = 10$ and $(\log 100a_2)(\log 100a_3)\cdots(\log 100a_n) = \dfrac{6}{4 + \ln a_1^2}$, find $P(-3)$. $\left[\dfrac{\ln 10}{3}\right]$

P2-12 Find the number of quadratic polynomials $Q(x)$ such that all of its roots are integers and $Q(0) = Q'(0) + 98$. $[6]$

Pre-final Round 3

P3-1 Evaluate the expression $(\sin 0.6\pi - i \sin 1.1\pi)^{1000}$. [1]

P3-2 Determine the prime factorization of A given $A = \dfrac{2 \cdot 4 \cdot 6 \cdots 2018 \cdot 2020 \cdot 5 \cdot 10 \cdot 15 \cdots 2015 \cdot 2020}{404 \cdot 403 \cdot 402 \cdots 2 \cdot 1 \cdot 1010 \cdot 1009 \cdot 1008 \cdots 2 \cdot 1}$. [$2^{1010}5^{404}$]

P3-3 Simplify the expression $\dfrac{(y^2+y+1)^2-(y+1)^2+2y^2}{y^2+2y+4}$. [$y^2$]

P3-4 What is the last digit of $(3^5+2^5)(3^5-2^5)(3^{20}+2^{20}+6^{10})$? [5]

P3-5 The sum of two numbers is 326, while the product of the two numbers is 4665. Find the sum of the cubes of the two numbers. [30 083 606]

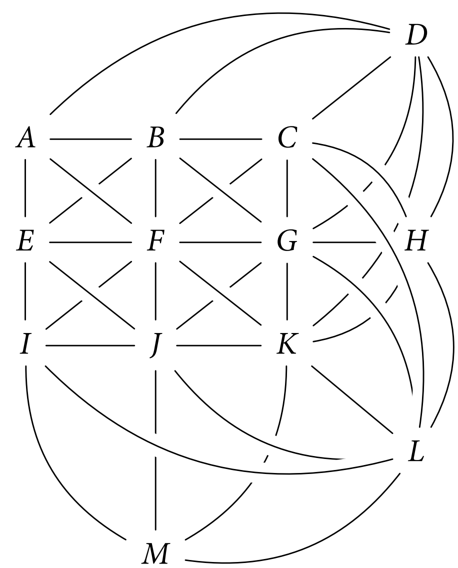
P3-6 Suppose that A, A' are fixed points, equidistant from the origin, and P is the set of all points p such that $|pA| + |pA'| = 4$. Furthermore, let B, B' be points in P such that BB' passes through the origin, is perpendicular to AA', and $|BB'| = 2$. Find the equation for the set of all points for which the distance from the midpoint of AA' to A is constant.* [$x^2 + y^2 = 3$]

P3-7 Let $A = \{35, x, 21\}$ and $B = \{x, y, 6\}$. Suppose x is the smallest possible integer such that the LCM of the elements of A is 420 and y is the smallest possible integer such that the LCM of the elements of B is 2100. Find the LCM of x and y. [700]

P3-X Let $\triangle ABC$ be a right triangle with right angle at B. Points D and E divide $\overline{BC}$ into equal parts. Find $m\angle ADB + m\angle AEB + m\angle ACB$ in radians. [$\dfrac{\pi}{2}$]

P3-8 Evaluate $\sin 1 + \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{\lfloor (n+1)/2 \rfloor}}{n!}$. [$\cos 1$]

P3-9 In the figure is a model of the finite projective plane of order 3. A projectivity from $IJKH$ to $AEIM$ is the product of the perspectivity of $IJKH$ to $EBDK$ with center M and perspectivity of $EBDK$ to $AEIM$ with center H. Give the images of I, J, K, and H under this projectivity. [$I \mapsto E, J \mapsto A, K \mapsto M, H \mapsto I$]



* Voided, due to a technical error. Replaced by problem **P3-X**.

P3-10 Suppose $\sec x = \sqrt{5}$ and $\tan 3y = \dfrac{\sqrt{5}}{2}$. Let T be the expression $\dfrac{\sum_{i=0}^{3} \sin(x+iy)}{\sum_{i=0}^{3} \cos(x+iy)}$. Assuming $2x \in [0, \pi]$, $y = \left[\dfrac{\pi}{3}, \dfrac{\pi}{2}\right]$, find $19T + 12$ such that its denominator is rationalized. $[5\sqrt{5}]$

P3-11 Let X, Y be independent random variables with $E(X) = \dfrac{2}{1+\frac{1}{2}}$, $E(Y) = \dfrac{3\sqrt{2}-3}{2}$, $V(X) = 0$, $V(Y) = 2^{-4}$. Solve for $E(E(V(E(3X+2Y) - V(4Y)Y)X)X)$. $[2]$

P3-12 Evaluate the integral $\displaystyle\iiint_{\Omega} \frac{(3x+y-4)^2}{9x^2+y^2+z^2-18x-2y+10}\, dz\, dy\, dx$, where $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid 9(x-1)^2 + (y-1)^2 + z^2 \le 1\}$. $\left[\dfrac{8\pi}{27}\right]$

Pre-final Round 3 Clincher (for Wizard Round)

P3-C1 Solve the real-valued function $f(x)$ satisfying $f\left(\dfrac{x+1}{x-1}\right) = \dfrac{x^2+1}{x^2-1}$. $\left[f(x) = \dfrac{x^2+1}{2x}\right]$

P3-C2 If $\sin 2x = \dfrac{1}{3}$, find the exact value of $\sin^6 x + \cos^6 x$. $\left[\dfrac{11}{12}\right]$

P3-C3 Suppose $\tan x = \dfrac{1}{2}$. Determine the value of $\tan 4x$. $\left[-\dfrac{24}{7}\right]$

Wildcard Round

WC-1 Shawn and Seth are observing a blimp in the sky. Shawn is on the ground while Seth is on top of a building that is built on the ground. If the angle of elevation of the blimp from Seth is 30° and Shawn is 15 m away from the nearest side of the building, how high is the blimp from the ground assuming that Shawn's and Seth's lines of sight coincide and their heights are negligible? $[10\sqrt{3}\,\text{m}]$

WC-2 Mr. Krazy and his friends, John, Louie, Josh, and Eric went on a seminar headed by Actuarial Society of the Philippines. After some time, Mr. Krazy went to the comfort room and left his bag on the table. Having mischievous friends, Mr. Krazy found out that his bag went missing went he returned. He asked his friends where his bag is and these are their answers:

John: The bag is with Josh.

Louie: I don't have the bag.

Josh: I did not take the bag, Eric and Louie did.

Eric: John has the bag.

If there is only one person telling the truth, who took the bag? [Eric and Louie]

WC-3 Gwyneth is tossing an unfair regular 7-sided die. The first side appears twice as likely while the other sides with numbers that divide 6 appear thrice as likely. What is the probability that a side with a prime number will appear? $\left[\dfrac{4}{7}\right]$

WC-4 Find all ordered pairs (x, y) of positive integers such that the difference between the area of a square with side $(x + y)$ units and the sum of the areas of squares with side lengths x units and y units is

22. $[(11, 1), (1, 11)]$

WC-5 Integrate $\int \cos \ln x \, dx$. $\left[\dfrac{x \cos \ln x + x \sin \ln x}{2} + C\right]$

WC-6 What are the points closest to the origin of the inversion of $36x^2 + 25y^2 = 1$ in the unit circle? $[(0, 5), (0, -5)]$

WC-7 Find the 2019th smallest real number α such that $\alpha > 1$ and $\sin \ln \alpha + 2 \cos(4 \ln \alpha) \sin(3 \ln \alpha) = 0$. $[e^{2019\pi/7}]$

WC-8 Find a quadratic polynomial with a leading coefficient of 3, whose roots are 2 less than the reciprocals of the fourth powers of the roots of $2x^2 + x + 1$. $[3x^2 + 9x + 54]$

WC-9 Solve the system $\sin^2 2x + \cos^2 y + 8 \sin x \cos x = 2 \cos y - 1$ and $\cos^2 y + 3 \sin 2x = 2 \cos y - 3$ where $x \in [0, 2\pi]$ and $y \in [0, 2\pi]$. [no real solutions]

WC-10 Find the number of ordered 98-tuples $(x_1, x_2, x_3, \dots, x_{98})$ of positive integers greater than 1 such that $x_1 x_2 x_3 \cdots x_{98} = 2^{101}$. [161 700]

WC-11 Find the probability that a positive odd number less than 2019 has an odd sum of divisors. $\left[\dfrac{22}{1009}\right]$

WC-12 Find the equation of the plane such that its intersection with the surface $7x^2 - 48xz - 7z^2 + 25y^2 = 0$ is a single point. $[3x + 4z = 0]$

Wizard Round

W-1 Let f be a differentiable function having a differentiable inverse. If $f(0) = 1$, $f'(0) = \dfrac{1}{2}$, find $(f^{-1})'(1)$. $[2]$

W-2 A particle moves along the curve $x = e^t$, $y = \cos 2t$ where t represents time. Determine the magnitude of the initial acceleration of the particle. $[\sqrt{17}]$

W-3 Suppose $\sin \theta = \dfrac{2}{5}$, $\tan \theta > 0$, and $\dfrac{(\cos \theta + \cot \theta)^2}{\cos \theta \cot \theta} - 2 = \dfrac{a}{b}$, where a, b are positive coprime integers. Find the value of $a + b$. $[39]$

W-4 Adarna House sold books for different grade levels – Grades 1, 2, 3, and 4 – this month. These books were sold for ₱200, ₱250, ₱275, and ₱300, respectively. On that day, the manager told that 35% of the books were for Grade 1, 20% for Grade 2, 15% for Grade 4, and the rest were for Grade 3. Find the average cost of a book that day. [₱247.50]

W-5 Give the name of the surface that is the graph of the equation $4y^2 - 8y - 9x^2 = 9z - 13$ in $\mathbb{R}^3$. [hyperbolic paraboloid]

W-6 A surface bounded by the trapezoid $ABCD$ is such that there is a right angle at C and $BC = 1$, $AB = 3$, $CD = 2$. Find the mass of this surface if the density at a point is equal to the sum of the lengths of its perpendiculars to the sides BC and CD. $\left[\dfrac{9}{2}\right]$

W-7 Amy, Rose, Jen, Elle, Ysa, Lindy, and Fern went on a financial consultation conducted by Accenture. To be able to avail a free consultation, they would need to fall in line. However, not all of them are in good terms:

- Amy doesn't want to be beside Fern or Rose;
- Lindy wants Fern to be exactly behind her and doesn't want Rose beside;
- Elle can't take it when Jen or Ysa is beside her;
- Ysa is just okay with any order as long as she is not the last;
- while Rose really wants to be first in line and doesn't want Elle to be the next.

If Lindy wants to be third in line and their preferences are to be met, then who could change their positions in the line? [Jen, Yssa, Elle]

W-8 Determine the set of all $a \in \mathbb{R}$ such that $\begin{bmatrix} 3a & 0 & 0 \\ 2017 & a & -1 \\ 2018 & 1 & 2a \end{bmatrix}$ has real eigenvalues. $[(-\infty, -2] \cup [2, \infty)]$

W-9 Find the inverse Laplace transformation of $\dfrac{6s}{(s^2+9)^2} + \dfrac{2s+1}{(s-4)(s^2+2s-15)}$.

$\left[\sin 3t + 3t\cos 3t - \dfrac{1}{4}e^{t} - \dfrac{3}{16}e^{-5t} + \dfrac{7}{16}e^{3t}\right]$

W-10 If $\cos^2\alpha + \cos^2\beta = 2$ and $\cos(\alpha-\beta) + \cos(\alpha+\beta) = -1$, evaluate $(\csc\beta)\left(\tan\dfrac{\alpha+\beta}{2} - \tan\dfrac{\alpha-\beta}{2}\right)$. Assume $\alpha, \beta \in$ QI. [2]

W-11 The electric potential energy, in joules, between two point charges A and B with charges Q_A and Q_B, respectively, is given by $E = \dfrac{kQ_AQ_B}{|AB|}$ where $|AB|$ denotes the distance between A and B in meters, k is a fixed number, and the charges are in coulombs. A point charge Q is placed 1 m above a charged circular disk of radius $\sqrt{3}$ meters whose charge is evenly distributed. If the charge of Q is 3 coulombs and the total charge of the disk is $\dfrac{1}{k}$ coulombs, find the total electric potential energy experienced by the charge Q. [2 joules]

W-12 Find the ordered pairs (x, y) of positive integers less than 2019 such that for some $k \in \mathbb{N}$, $x = 2k$ and $y + 2x = 4^k - xy$. [5]

W-13 If U and X are 3×3 matrices with entries that are real numbers, U is upper triangular with all diagonal entries equal to 1, and $U = X\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 3 & 4 & 5 \end{bmatrix}$. Find the sum of the third-row entries of X. $\left[\dfrac{19}{2}\right]$

W-14 Find the general solution of $(1-\cos 2x)y'' + (\sin x)y' - 2y = 0$. $[y = 4C_1\sin^2 x + 4C_2\csc^2 x]$

W-15 Let f be a function defined on the positive integers such that:

- If a, b are positive integers such that $\gcd(a,b) = 1$ then $f(ab) = f(a)f(b)$.
- If $p = 5$ or $p = 7$ then $f(p^k) = k$ for all positive integers k.
- If p is a prime neither equal to 5 nor equal to 7, then $f(p^k) = 0$ for all positive integers k.
- $f(1) = 1$

Evaluate $\displaystyle\sum_{k=1}^{\infty}\frac{f(k)}{k}$. $\left[\dfrac{215}{144}\right]$

W-16 Let $f(x) = xe^x$. If $f^{(2019)}(x) = \sum_{n=1}^{\infty} A_n \cos(nx) + \sum_{n=1}^{\infty} B_n \sin(nx)$, find A_1. $\left[\cosh \pi + \dfrac{\sinh \pi}{\pi}\right]$

W-17 For each $n \in \mathbb{N}$ and $z \in \mathbb{C}$ we have $z^n - 1 = \prod_{\substack{d \in \mathbb{N} \\ d \mid n}} \Phi_d(z)$ and the Φ_d's are polynomials. How many integer values of k, where $0 \le k < 2000$, are such that $\Phi_{2000}\left(\operatorname{cis} \dfrac{k\pi}{1000}\right) = 0$? $[800]$