Lightning Round (Mental)

Item	Points	Time
1 to 6	1	10 s
7 to 12	2	15 s
13 to 18	3	20 s

Prefinal Round

Item	Points	Time
1 to 5	2	15 s
6 to 9	4	30 s
10 to 12	6	45 s
13 to 14	8	60 s
15	10	75 s

Wizard Round

Item	Points	Time
1 to 6	2	15 s
7 to 11	4	30 s
12 to 15	6	45 s
16 to 18	8	60 s
19 to 20	10	75 s

Lightning Round A

LA-1 If the greatest common divisor of two positive integers m and n is 240, and their product is 30 000, what is their least common multiple? [125]

LA-2 Determine the largest integer k such that $5k - 3$ divides $625k^3$. [6]

LA-3 Find the equation of the line perpendicular to and passing through the y-intercept of the line $3x - 5 = \frac{12y}{5} + 6$. $\left[y = -\frac{4x}{5} - \frac{55}{12}\right]$

LA-4 How many permutations of the letters in the word "QUESTION" exist such that its vowels are in alphabetical order? [1680]

LA-5 Given that $x^2 - 4x - 1 = 0$, find the value of $x^2 + \frac{1}{x^2}$. [18]

LA-6 Four cubes, each with side of length 2 cm, are joined together to form a cuboid. What is the ratio of the surface area of the larger cuboid to a smaller one? $\left[\frac{1}{3}\right]$

LA-7 Thirty members of the Actuarial Society of the Philippines have an option to attend a three-day orientation from Monday to Wednesday. Eight were absent on Monday and Tuesday, 9 on Tuesday and Wednesday, and 11 were absent on Monday and Wednesday. If 8 members attended at least twice, how many people didn't go at all? [3]

LA-8 Define the operation $\heartsuit$ by $a \heartsuit b = a + ab + b$. Real numbers x, y, and z, each not equal to -1, satisfy the system $x \heartsuit y = x$ and $y \heartsuit z = y$. Determine x in terms of z. $\left[-\frac{z}{z+1}\right]$

LA-9 Consider the phrase "PIGEON PRINCIPLE". If one were to draw letters from this phrase, one by one, without replacement, how many draws at most will it take to draw at least two pairs of repeated letters? Disregard spaces. [12]

LA-X What is the sum of all integers less than 30 such that when 5 times the integer is divided by 6, the remainder is 2?* $[-\infty]$

LA-10 Find the remainder when 4^{2017} is divided by 33. [16]

LA-11 Find the angle between the two vectors $\vec{a} = 3\hat{\imath} + 2\hat{\jmath}$ and $\vec{b} = 5\hat{\imath} - \hat{\jmath}$. [45°]

LA-12 Let $f(x, y) = \frac{y^2}{2} - x^4$. Find the minimum rate of change of f at the point $\left(\frac{1}{2}, 3\right)$. $\left[-\frac{\sqrt{37}}{2}\right]$

* Voided.

LA-13 Let $a, b \in \mathbb{R}$ such that $a^2 - b^2 + 12b - 36 = 0$. Find the value of i^n if $n = 2020 - a - b$ where $i = \sqrt{-1}$. $[\pm 1]$

LA-14 Let a be the sum of the divisors of 210 and b be the sum of the divisors of 144. Find the value of $\dfrac{a+b}{25} + 1$. $[1]$

LA-15 Let $\alpha = \dfrac{5\pi}{17}$ and $\beta = \dfrac{7\pi}{34}$. Suppose γ is in the third quadrant and θ is in the fourth quadrant, such that $\tan\gamma = \dfrac{12}{5}$ and $\sin\theta = \dfrac{3}{5}$. Evaluate $\sin(\alpha + \beta + \gamma + \theta)$. Express the answer in simplified fraction form. $\left[\dfrac{56}{65}\right]$

LA-16 Evaluate $24 - \dfrac{3}{8+}\dfrac{3}{8+}\dfrac{3}{8+}\cdots$†. $\left[20 + \sqrt{19}\right]$

LA-17 Evaluate $\cos^{-1}\dfrac{1 - \sec^2 2020^\circ + \cot^2 2020^\circ}{1 + \sec^2 2020^\circ + \cot^2 2020^\circ}$. $[80^\circ]$

LA-18 Evaluate $\displaystyle\lim_{(a,b)\to(0,2)} \tan^{-1}\frac{8 - 4a - 4b}{4 - (a+b)^2}$. $\left[\dfrac{\pi}{4}\right]$

Lightning Round B

LB-1 In a list of 5 natural numbers, the mean and mode are 5. Find the largest possible population variance. $[14.8]$

LB-2 Let p, q, and r be propositions. Determine the number of possibilities for the truth values for p, q, and r so that the statement $((r \wedge q) \vee \neg q) \Rightarrow (p \vee \neg p)$ is true. $[8]$

LB-3 If r and s are the nonzero roots of $7x^2 - 10x + 15 = 0$, find the quadratic equation in x whose roots are $\dfrac{1}{r}$ and $\dfrac{1}{s}$. $[15x^2 - 10x + 7 = 0]$

LB-4 What is the remainder when 7^{2020} is divided by 18? $[7]$

LB-5 Define $\varphi(x)$ as the number of elements in the set $\{a \in \mathbb{Z} \mid a \text{ and } x \text{ are relatively prime}\}$. Evaluate $\varphi(289) + \varphi(13) + \varphi(4)$. $[286]$

LB-6 Let θ be a real number such that $\cos\theta - \cot\theta = 3$. Evaluate $\dfrac{1 + \cos\theta}{\sin\theta}$. $\left[\dfrac{1}{3}\right]$

LB-7 Let a, b, and c be nonzero real numbers such that $\displaystyle\int_0^{2020} (5ax^4 + 3bx^2 + c)\,dx = \int_1^{2020} (5ax^4 + 3bx^2 + c)\,dx$. What is the value of $a + b + c$? $[0]$

LB-8 The members of the Actuarial Society of the Philippines have brought 3 kinds of desserts for their meeting. Ten members chose biko and ube, 29 chose yema or ube, 33 chose yema or biko, and each dessert was chosen by 20 members. At least how many people were in the reunion? $[33]$

LB-9 Evaluate $\log_{\sqrt[3]{x}} \sqrt[3]{x\sqrt[3]{x\sqrt[3]{x}}}$. $\left[\dfrac{13}{9}\right]$

† To save space, I used this notation for the continued fraction $3/(8 + 3/(8 + 3/(8 + \cdots)))$.

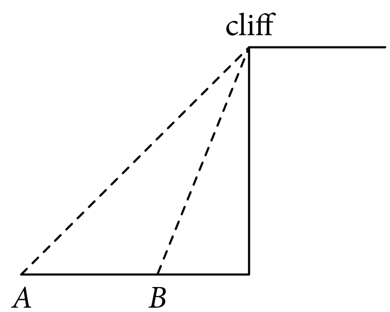
LB-10 [FIGURE[‡]] Suppose $CD = 8$ and E is the midpoint of AC. If $\overline{BE}$ and $\overline{CD}$ are congruent angle bisectors and $AB = 3CD$, what is the perimeter of $\triangle ADE$? [44 units]

LB-11 Given the circle with the equation $x^2 + y^2 + 14x - 48y = 0$, find the area of the sector whose central angle is $12°$. $\left[\frac{125\pi}{6}\text{ units}^2\right]$

LB-12 Consider the hyperbola given by $16x^2 - 25y^2 - 160x = 0$. In the standard form, find the equation of one of its asymptotes having a negative slope. $[4x + 5y = 20]$

LB-13 Angel wants to list down all angles coterminal to $\frac{61\pi}{90}$ that lie on the closed interval $[2000°, 3500°]$. What is the average of the largest and the smallest angles in the interval, in degrees? [2822°]

LB-14 From the top of the cliff, the angles of the depression of Gonzo's ship at points A and B are 45° and 60°, respectively, where A and B are on the same horizontal line with the foot of the cliff. If the distance between A and B is 36 km, how far is Gonzo's ship from the foot of the cliff when it is at point B? $\left[(18 + 18\sqrt{3})\text{ km}\right]$



LB-15 Three friends won a prize of ₱420 and decided to buy gifts for their significant others. They decided that each gift must not exceed ₱200. What is the probability that they have to chip in from their own money after buying gifts? $\left[\frac{243}{1000}\right]$

LB-16 Let α be in the first quadrant and 2β be in the second quadrant. If $\tan\alpha = \frac{1}{2}$ and $\sin\beta = \frac{1}{\sqrt{10}}$, evaluate $\cot(2a + 2b)$. Express the answer in simplified fraction form. $\left[\frac{13}{9}\right]$

LB-17 A particular round-robin tournament has 100 contestants. Each contestant will play every other contestant exactly two times. Knowing this, what is the minimum number of games that must be played to ensure that each contestant has played at least two games? [9706]

LB-18 Find $\sum_{n=0}^{\infty}\left(\left(\frac{1}{4}\right)^{n+2} + 3^{2n}10^{3-n}\right)$. $\left[\frac{120\,001}{12}\right]$

Lightning Round B Clincher

LB-C1 Evaluate $\int_0^1 \frac{dx}{x^2 + 3x + 2}$. $\left[\ln\frac{4}{3}\right]$

[‡] The figure is impossible to make, so perhaps I might have encoded it wrong. If X is the intersection of BE and CD then CBX is a degenerate triangle.

LB-C2 Find all ordered pairs of integer solutions (x, y) to the equation $x^2 + y^2 + 2 = (x-1)(y-1)$. $[(-1,-1)]$

LB-C3 Evaluate $\lim_{x\to 0} \dfrac{\sec(x) + x^2}{x^2}$. $[+\infty]$

Pre-final Round A

PA-1 Mr. Dave bought a Christmas gift for his son Pudding that is a quadric defined by the equation $3x^2 + 4y^2 - 6z^2 - 24 = 0$. What type of quadric is this? [hyperboloid of one sheet]

PA-2 Given the matrix $A = \begin{bmatrix} 8 & 6 & 8 \\ 2 & 3 & 3 \\ 4 & 9 & a^2 \end{bmatrix}$, what are the possible values of a such that A is a rank-2 matrix? $\left[a = \pm 2\sqrt{2}\right]$

PA-3 A right circular cylinder with radius 5 m is cut by a plane diagonally such that $h_1 h_2$ is equal to $12\,\mathrm{m}^2$ and $h_1 - h_2 = 4\,\mathrm{m}$ where $h_1 > h_2$. What is the volume of the original cylinder if the volume of the cut cylinder is 40% of the original volume?[§] $[375\pi\,\mathrm{m}^3]$

PA-4 Evaluate $\int_C (2 + x^2 y)\,\mathrm{d}s$, where C is the upper half of the unit circle $x^2 + y^2 = 1$. $\left[2\pi + \dfrac{2}{3}\right]$

PA-5 Find the greatest common divisor of 2431 and 3120. [13]

PA-6 The year 2020 is a leap year. Leap years occur in every year divisible by 4, with the exception of years divisible by 100, unless they are also divisible by 400. In the time period between 3000 to 3300 (inclusive), how many years are going to be leap years or divisible by 5, but not divisible by 3? [78]

PA-7 Consider worker n. The ratio between the rate at which workers n and $n-1$ complete a job is constant for all n. Worker 1 can complete the job in 1 hour, and workers 3 and 5, working together, can complete the job in 8.5 hours. If we have an infinite number of workers, how many hours would it take for them to finish the job? $\left[\dfrac{2}{3}\text{ hours}\right]$

PA-8 Joshua has prepared a game for his friend. He has brought three boxes, one of which contains chocolates, while the other two are empty. His friend needs to pick the correct box containing the chocolates. On each box, there is a statement, exactly one of which is true. The statements are as follows. On the first box, "The chocolates are in this box." On the second box, "The chocolates are not in this box." On the third box, "The chocolates are not in the first box." [second box]

PA-9 Evaluate $\iint_{[1,2]\times[-4,1]} (4xy + x - y)\,\mathrm{d}A$. $[-4]$

PA-10 Let $\rho(t)$ be the radius of curvature of $f(t) = t^2$. Determine the value $\displaystyle\int_{-\infty}^{\infty} \frac{\rho(t)}{\sec^6 \tan^{-1}(2t)}\,\mathrm{d}t$. $\left[\dfrac{1}{2}\right]$

PA-11 Find the value of $\sin^6\theta + \cos^6\theta$ if $\sin(2\theta) = \dfrac{2}{5}$. $\left[\dfrac{22}{25}\right]$

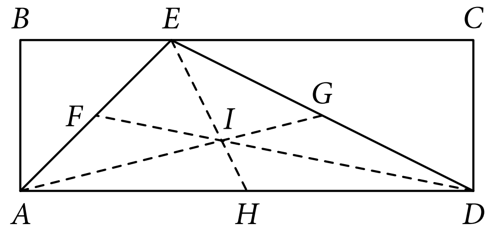
PA-12 There is a 3-in-8 chance that an online business is going bankrupt; a 5-in-6 chance for a corporate business; and a 1-in-2 chance for a service business. A third of thriving businesses are service-type

[§] I think h_1 and h_2 refer to the maximum and minimum height of one of the cut pieces (that is actually not a cylinder) that is 40% of the original volume.

and is equal to the number of failing online businesses. If there are 100 businesses, how many businesses are going bankrupt? [55]

PA-13 Find the sum of all the possible integers k such that $20 \le k \le 25$ and for some integer x, $x^2 \equiv k \pmod{41}$. [89]

PA-14 Let the quadrilateral $ABCD$ be a rectangle, where $AB = 12$ and $BC = 36$. Furthermore, let F, G, and H be the midpoints of $\overline{AE}$, $\overline{ED}$, and $\overline{AD}$, respectively, and let I be the point of intersection of $\overline{AG}$, $\overline{EH}$, and $\overline{EF}$. If $\overline{AB} \cong \overline{BE}$, find the perimeter of $\triangle AIH$. $\left[4\sqrt{17} + 2\sqrt{5} + \frac{18}{5}\right]$



PA-15 Let $z = 720x^4$, where $x \neq 0$. Express $\sin(\tan^{-1} 60x^2 + \cos^{-1} 24x^2) - \sin(\cos^{-1} 24x^2 - \tan^{-1} 60x^2)$ in terms of z, such that the denominator is rationalized. $\left[\frac{4z\sqrt{1+5z}}{1+5z}\right]$

Pre-final Round A Clincher

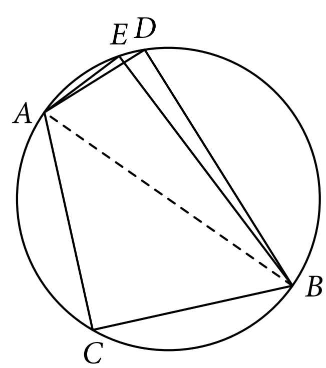
PA-C1 Suppose $\tan x = \frac{1}{2}$. Determine the value of $\tan(4x)$. $\left[-\frac{24}{7}\right]$

Pre-final Round B

PB-1 Determine the arclength of the curve given by $y = \ln \csc x$, where $\frac{\pi}{6} \le x \le \frac{\pi}{3}$. $\left[\frac{2\sqrt{3}}{3}\right]$

PB-2 Evaluate $\ln \sum_{k=1}^{1921} e^2 \left(\cos\left(\frac{\pi}{2k}\right) + i \sin\left(\frac{\pi}{2k}\right)\right)^{2k}$. [2]

PB-3 Suppose that AB is a diameter of the circle with $m\angle DBC = 73°$ and $m\angle CBE = 78°$. Find $m\angle DAE$. [5°]

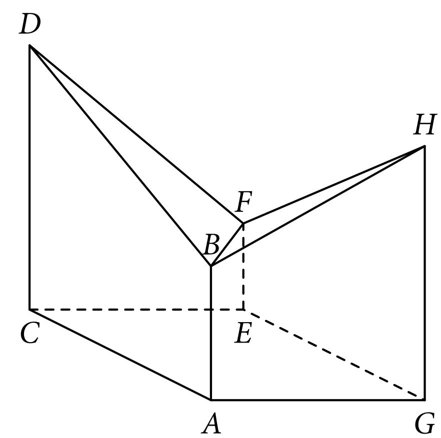


PB-4 Find the coefficient of the x^{-3} term in the expression $\left(3x^3 - \frac{1}{x^2}\right)^9$? [972]

PB-5 What combinations of values of p and q will make the negation of the statement $p \Rightarrow (p \wedge q)$ true? [p is true, q is false]

PB-6 In a coin factory, a batch of 10 000 coins had a 2% decrease from the usual standard deviation for getting a side, making them biased. What is the probability of getting the biased side? Answer in decimals to the nearest tenths. [0.6]

PB-7 Suppose $CD = 7\,\text{m}$, $AB = 3.55\,\text{m}$, $EF = 2.28\,\text{m}$, and $GH = 6.73\,\text{m}$. If the area of $CEGA$ is $24\sqrt{2}\,\text{m}^2$, find the volume of the solid in simplified radical form. $\left[\dfrac{5357\sqrt{2}}{50}\,\text{m}^3\right]$



PB-8 Find the equation of the plane containing the point $P(1,4,3)$ and the line $\ell : x = 2-3t, y = 5+2t, z = 4+t$. $[x + 4y - 5z - 2 = 0]$

PB-9 Find all positive integers n for which $n! + 5$ is a perfect cube. $[n = 5]$

PB-10 Let I_1 be the interval of convergence of the power series $\sum\limits_{n=0}^{\infty} \dfrac{(x+\pi)^n}{\sqrt{n}}$ and let I_2 be the interval of convergence of the power series $\sum\limits_{n=0}^{\infty} \dfrac{(x-2)^{2n+1}}{2^n}$. Find $I_2 \setminus I_1$. $[(-1-\pi, 2-\sqrt{2})]$

PB-11 Give the last two digits of 109^{109}. [89]

PB-12 Find all potential functions of $f(x,y,z) = (2xy+1, x^2+2yz, y^2-2)$. $[x^2y + x + y^2z - 2z + k, k \in \mathbb{R}]$

PB-X Let a, b, and c be positive real numbers and suppose $a+b+c = 1$. Find the minimum value of $a^2 + 2b^2 + c + \dfrac{8}{5}$.[¶] [2]

PB-13 How many positive integers are there such that $\dfrac{n^2-4}{n+10}$ is also an integer? [6]

PB-14 Let $A = \begin{bmatrix} 2 & 22 & 20 \\ 0 & 20 & 19 \\ 0 & 21 & 19 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -3 & 4 & 2 \\ 3 & 1 & 2 \end{bmatrix}$. Find the determinant of A. $[-570]$

PB-15 Find all solutions to the equation $\sin^{-1}\sqrt[3]{x} - \cos^{-1}(2\sqrt[3]{x}) = \dfrac{\pi}{6}$. $\left[x = \dfrac{1}{8}\right]$

Wizard Round

[¶] Voided.

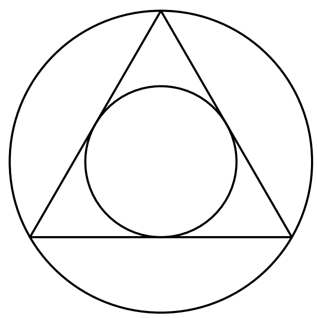
W-1 How many 4-digit positive integers are there such that there is at least one digit that appears at least twice? [4464]

W-2 Find the equation of the tangent plane to the surface defined by the vector equation $R(u,v) = \sin u\hat{\imath} + \cos u \sin v\,\hat{\jmath} + \sin v\hat{k}$ at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$. $\left[\frac{3\sqrt{3}}{8}x + \frac{3}{4}y - \sqrt{3}z + \frac{\sqrt{3}}{8} = 0\right]$

W-3 Evaluate $\log_{\sin 1^\circ} \cot 1^\circ + \log_{\sin 2^\circ} \cot 2^\circ + \log_{\sin 3^\circ} \cot 3^\circ + \cdots + \log_{\sin 89^\circ} \cot 89^\circ + \log_{\cos 1^\circ} \cot 1^\circ + \log_{\cos 2^\circ} \cot 2^\circ + \log_{\cos 3^\circ} \cot 3^\circ + \cdots + \log_{\cos 89^\circ} \cot 89^\circ$. [1]

W-4 Determine the set of all $a \in \mathbb{R}$ such that the matrix $A = \begin{bmatrix} a+3 & -1 \\ 1 & 2a-3 \end{bmatrix}$ has imaginary eigenvalues. $[(4,8)]$

W-5 Suppose a regular triangle of length ℓ is inscribed in a circle A, and circle B is inscribed in the said triangle. If circle A has radius r_A and circle B has radius r_B, find the area inside circle A but outside circle B, in terms of ℓ. $\left[\frac{\pi\ell^2}{4}\right]$



W-6 Suppose nonzero numbers a, b, c, d, and e form an arithmetic progression. If $\frac{b+d}{2} + \frac{a+e}{4} = kc$, then what is k? $\left[\frac{3}{2}\right]$

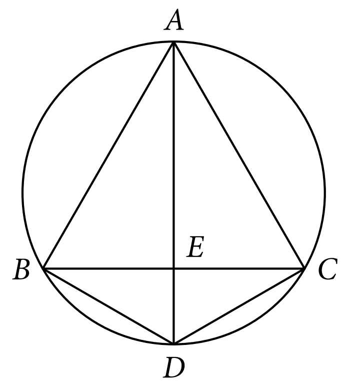
W-7 Let X be a random variable with moment-generating function $M_X(t) = \frac{t + e^t + e^{2t}}{5}$. Find the variance of X. $\left[\frac{9}{25}\right]$

W-8 Determine the general solution of the differential equation $(4x^2 - 8x + 4)y'' + ((1 - 4\sqrt{3})x + 4\sqrt{3}y)y' + 3y = 0$ that is valid in any interval not including the singular point. $\left[y(x) = (\cos c_1 + \cos c_2 \ln|x-1|)|x+1|^{\sqrt{3}/2}\right]$

W-9 Water runs into a conical tank at a rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep? $\left[\frac{1}{2\pi}\ \text{ft/min}\right]$

W-10 Find the radius of the osculating circle of the curve $y = \sin x$ at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$. $\left[\frac{7\sqrt{7}}{4}\right]$

W-11 Let A, B, C, and D be points in a circle, E be the intersection of $\overline{BC}$ and $\overline{AD}$, $\overline{AE}$ an angle bisector of $\angle A$, and $\angle BDC = 120^\circ$. Suppose $\overline{BD}$, $\overline{DC}$, and $\overline{BE}$ have integer lengths, and $BC = 2\sqrt{3}$. What would be the perimeter of $\triangle BDC$ if the length of $\overline{BE}$ is minimized? $[4 + 2\sqrt{3}]$



W-12 Determine the number of unique solutions to the system $2\sin^2 x - \sin x \cos y - \cos^2 y = 0$, $\sin(x+y) + \sin(x-y) = 0$ over $[0, 2\pi] \times [0, 2\pi]$. [8]

W-13 Let $s(n)$ be a function denoting the sum of the base-10 digits of an integer n. How many integers $n < 10^4$ have the property that $s(n) = 12$? [415]

W-14 Patrick woke up trapped in a room with 4 doors, labelled North, South, East, West. Only one door is the true exit; opening the other doors would leak toxic gas. Each door has a statement written on it. However, only one of these statements tells the truth, but not necessarily written on the exit door, while the other ones are false. To which door should Patrick exit if he wants to leave the room safely, given the following statements on each of the doors:

- North: "Either the statement on the West door is true or the East door is not the exit."
- East: "The exit is not on the South, West, or East doors."
- South: "Both the East and North doors are not exits."
- West: "Either the North door or the South door is the exit." [West]

W-15 Let m be a positive integer. Find the smallest[‖] (x, y) of positive integers such that $x^2(x^2 + y) = y^{m+1}$. [$(6, 12)$]

W-16 A function $f(x, y, z)$ is said to be *harmonic* in a region D if it satisfies the Laplace equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$. If S is the surface $x^2 + y^2 + z^2 - 6x + 8z = 0$, and f is harmonic on some region D, with $|\nabla f(x, y, z)| = \frac{3}{5}$, calculate $\iint_S f \nabla f \cdot \vec{n}\, \mathrm{d}S$. [$60\pi$]

W-17 Evaluate $\frac{1}{4} + \frac{1}{28} + \frac{1}{70} + \cdots + \frac{1}{9700}$. $\left[\frac{33}{100}\right]$

W-18 Let z be an integer such that $\sum_{i=1}^{215} \frac{1}{i} = \frac{z}{215!}$. What is the remainder when z is divided by 109? [55]

W-19 Let $g(x)$ be a polynomial function such that $g(g(x)) = x^2 + xg(x)$. Find $g(-99)$. [100]

W-20 If $a, b, c \in \mathbb{R}^+$ such that $a + b + c = 39$, what is the minimum value of $\frac{6a^3}{b^2 + c^2} + \frac{6b^3}{c^2 + a^2} + \frac{6c^3}{a^2 + b^2}$? [117]

[‖] Using which ordering?