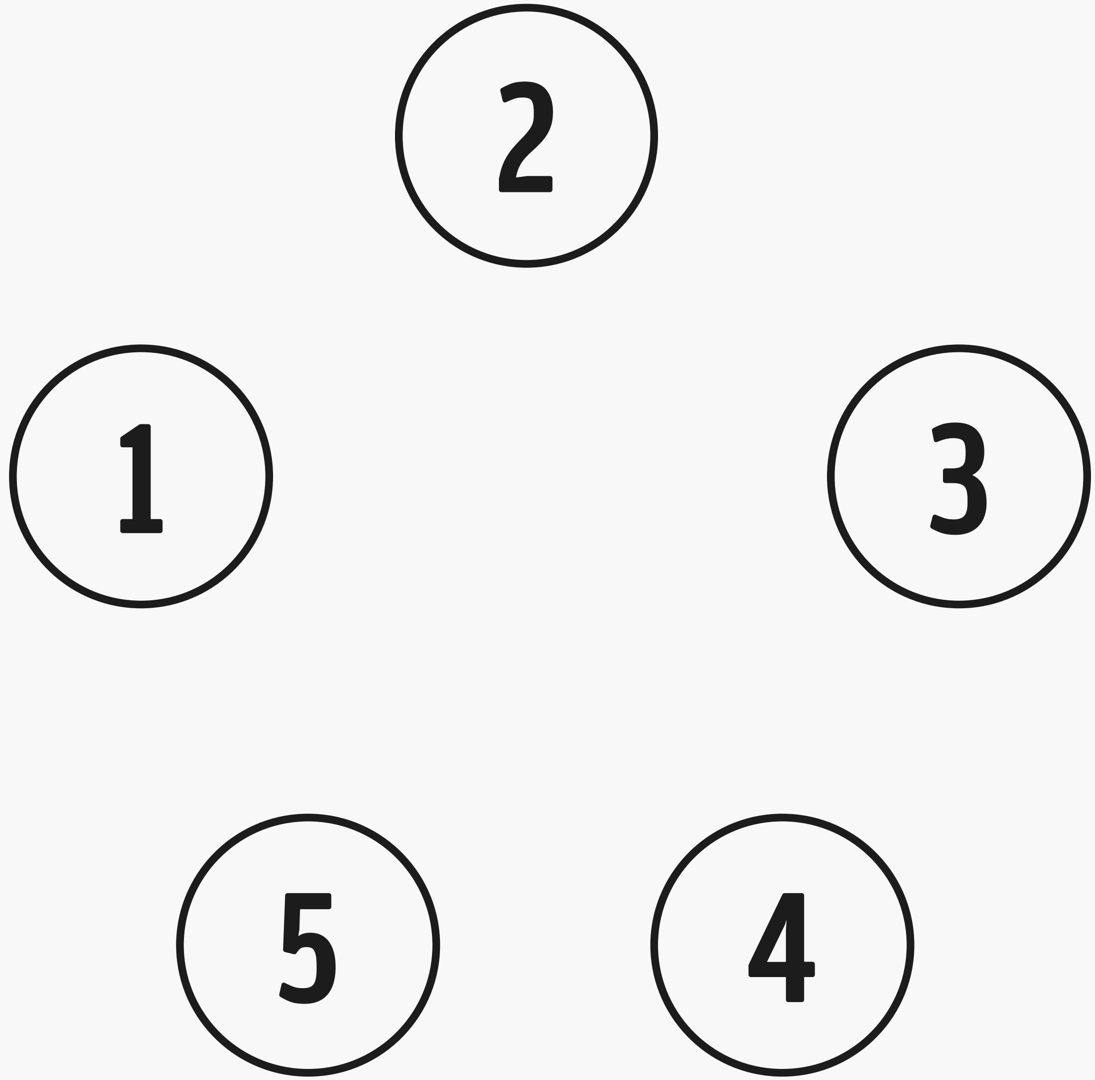
Bounds on metric dimension for families of planar graphs

Five houses problem

- There are five houses: 1, 2, 3, 4, and 5.
- Connect each house to each other house with lines.
- None of the lines can cross each other.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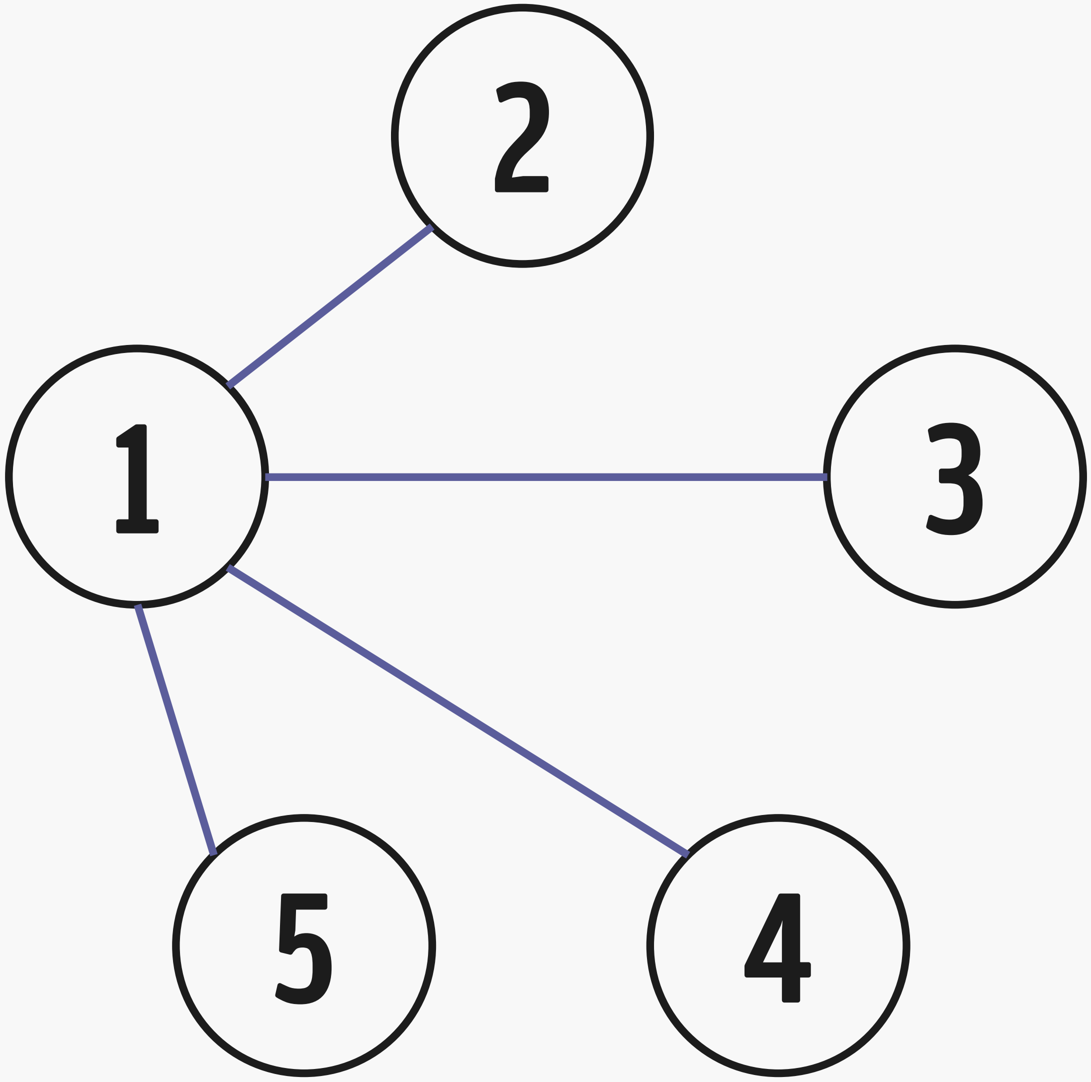
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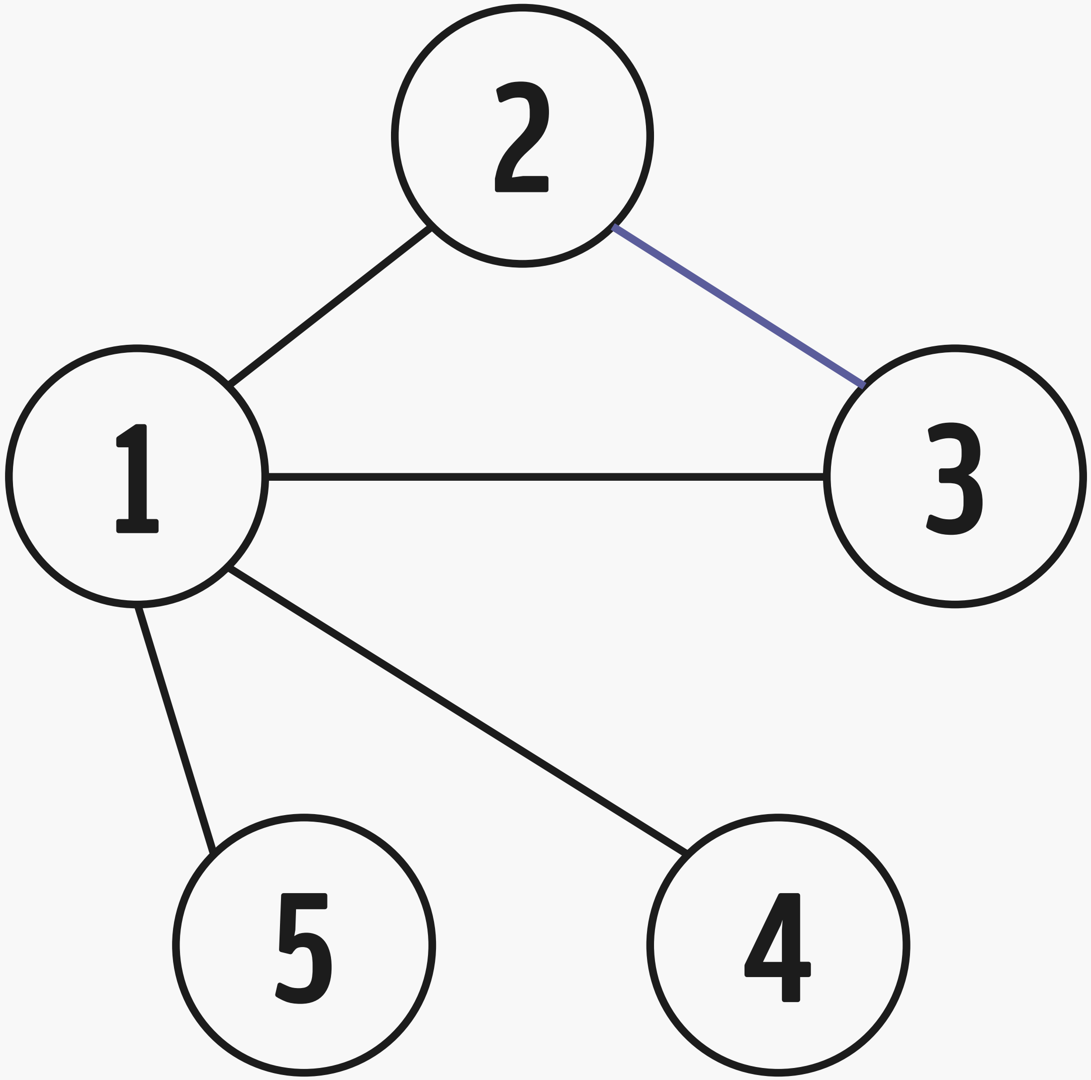
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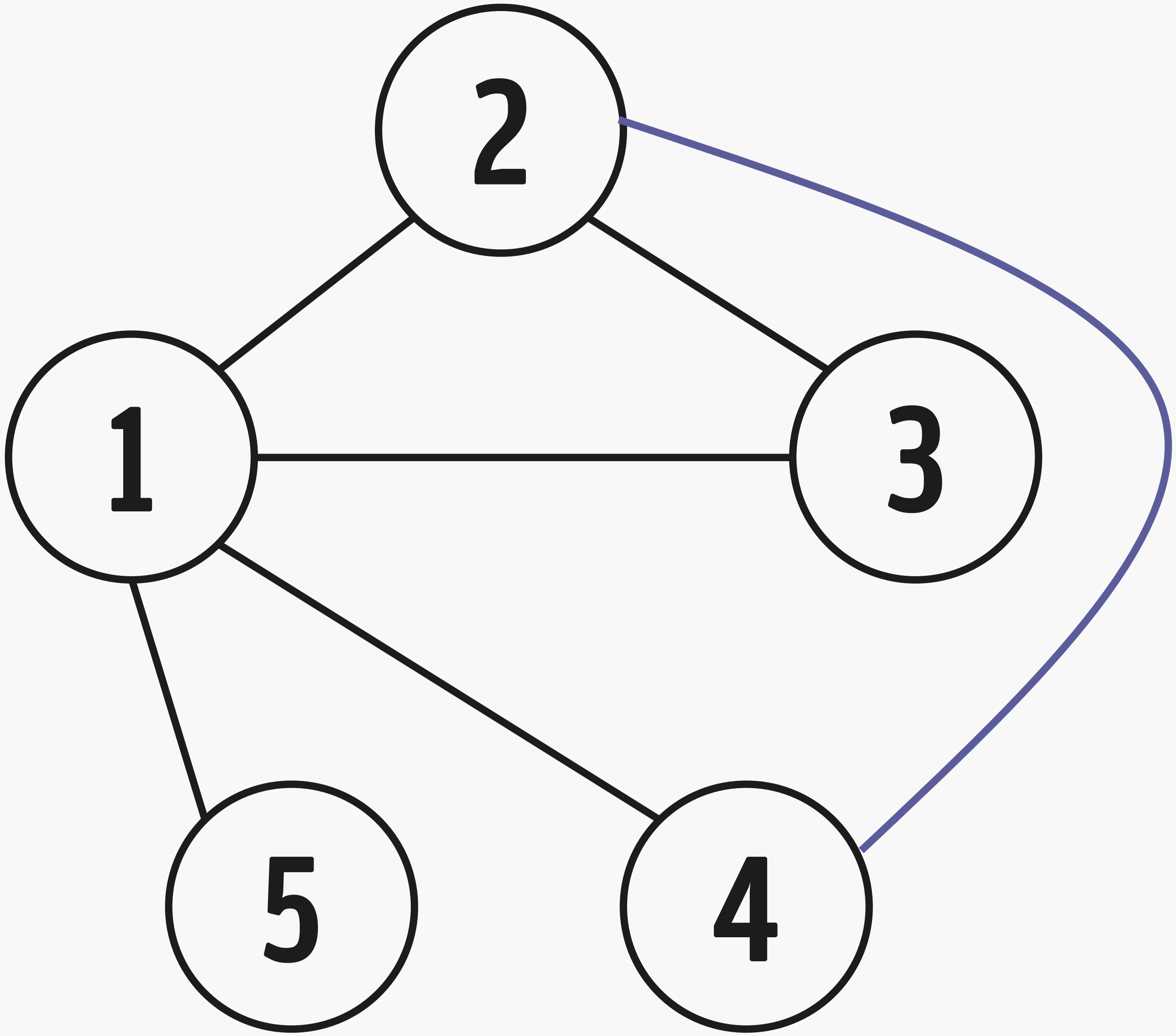
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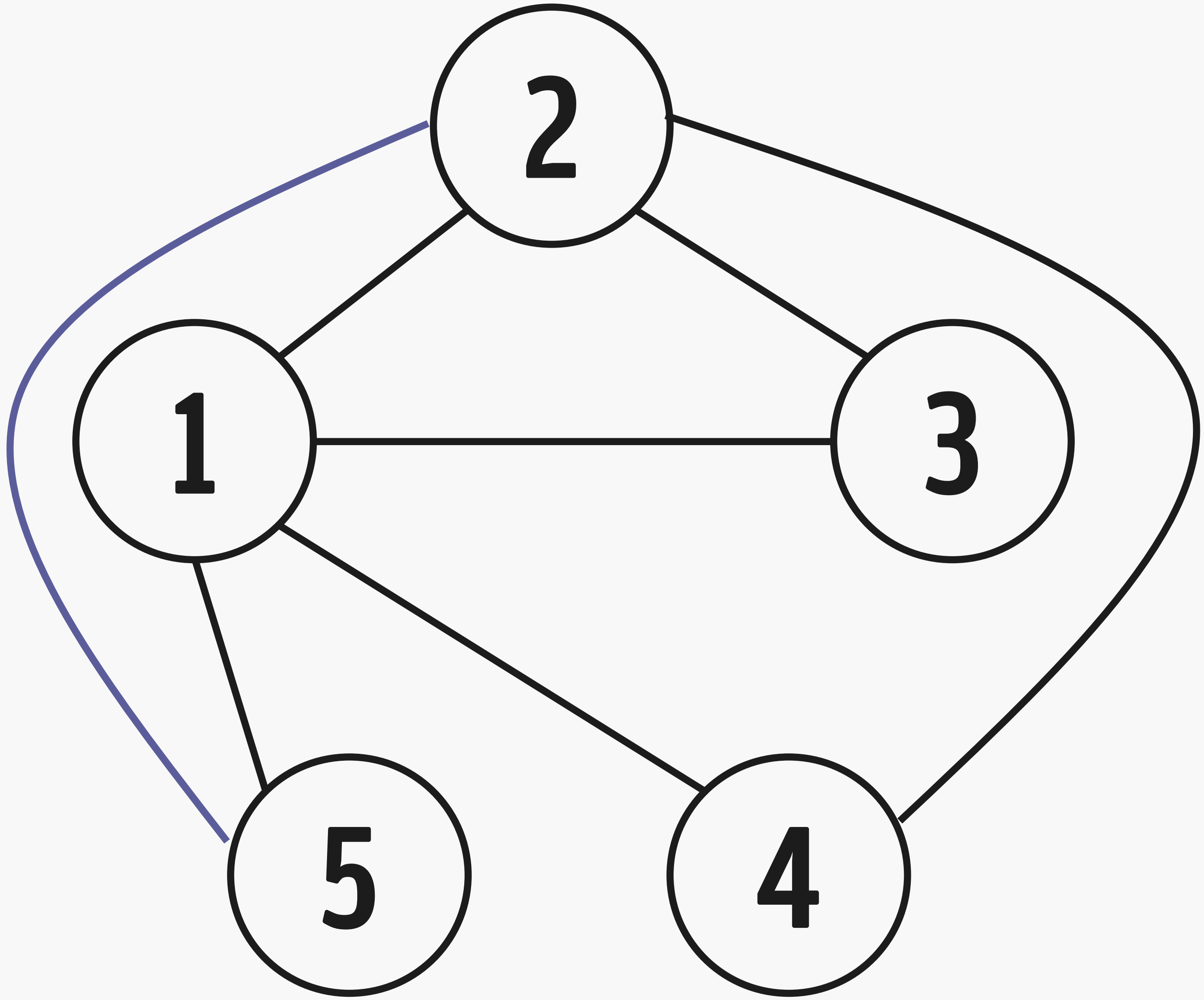
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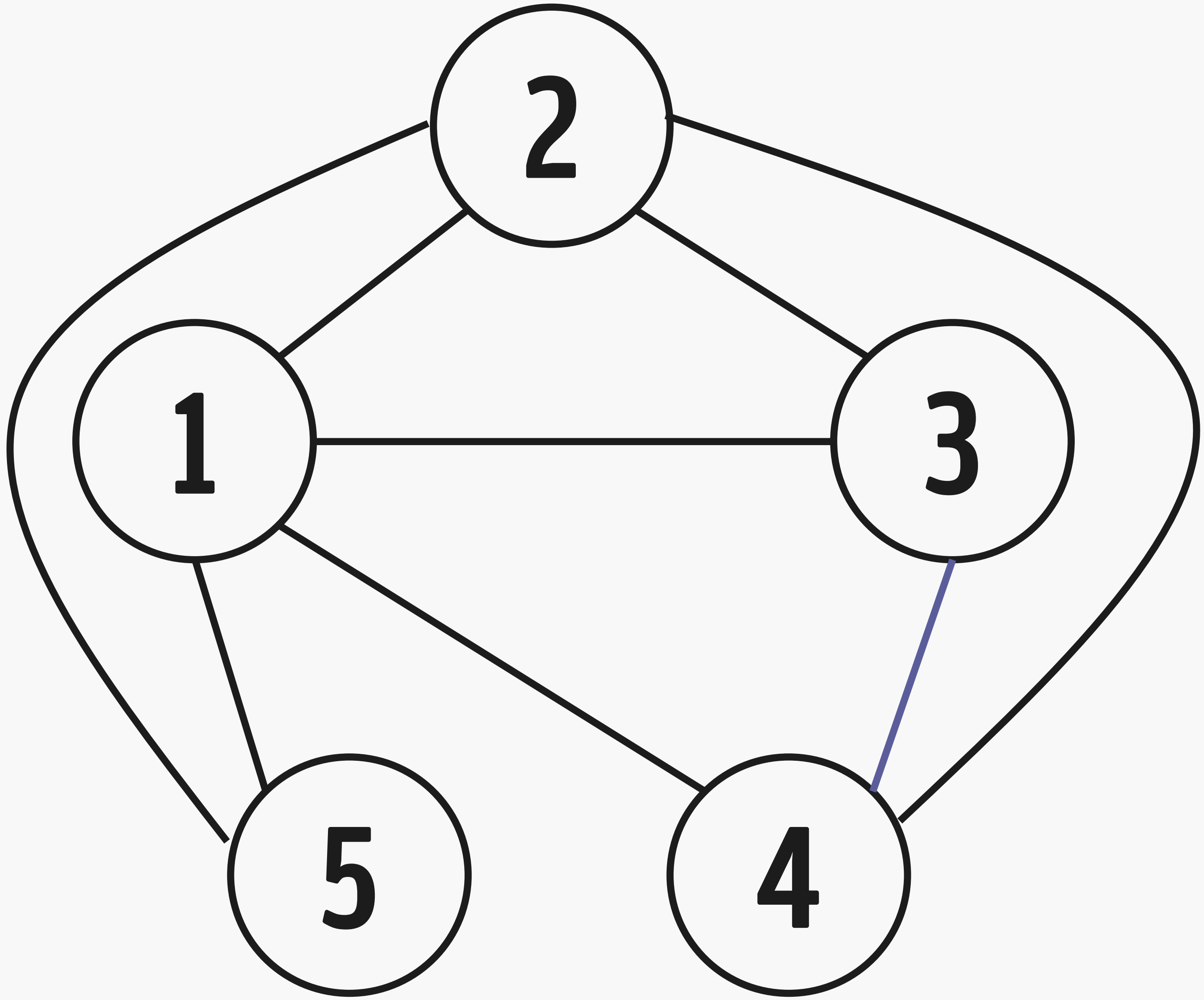
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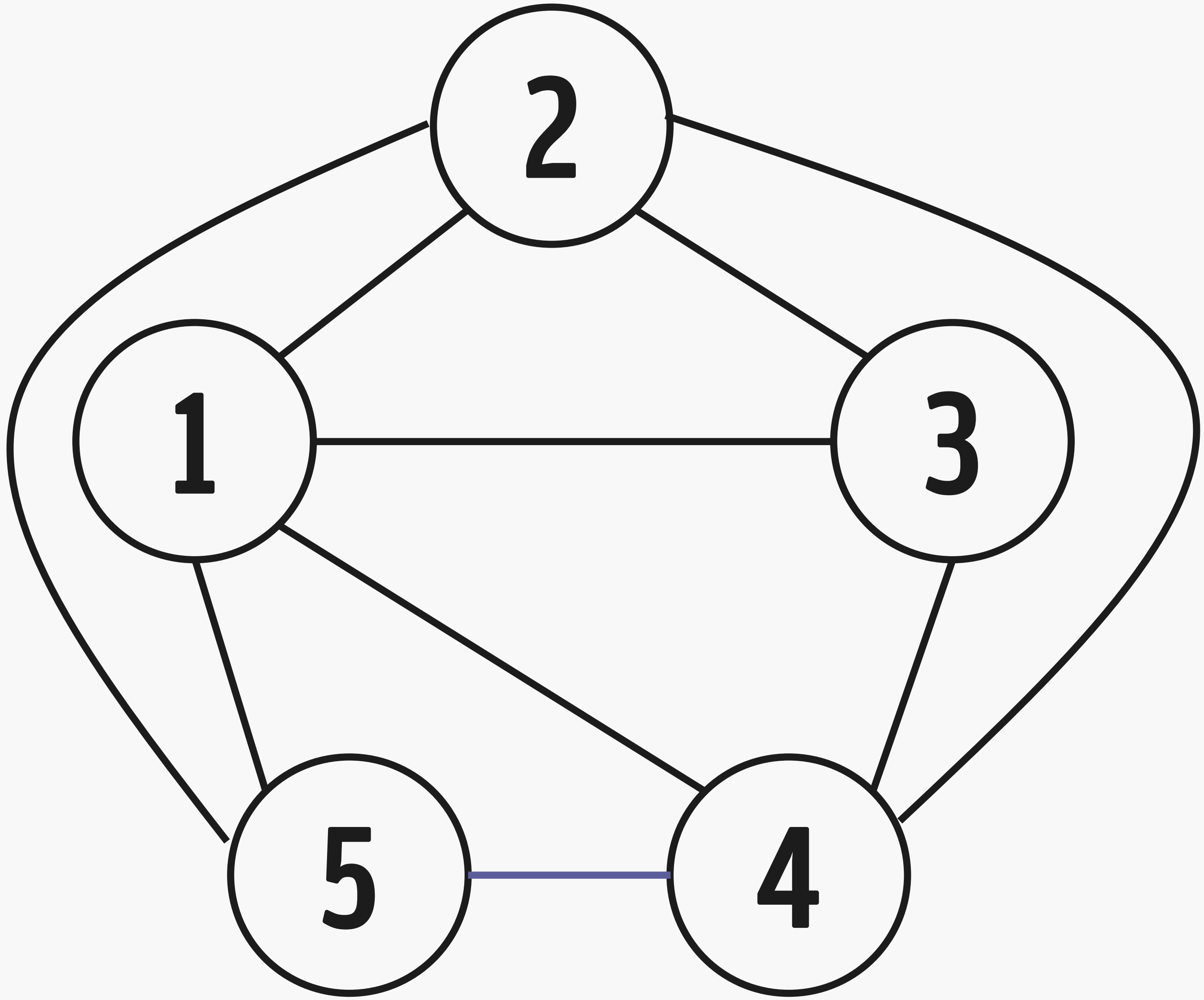
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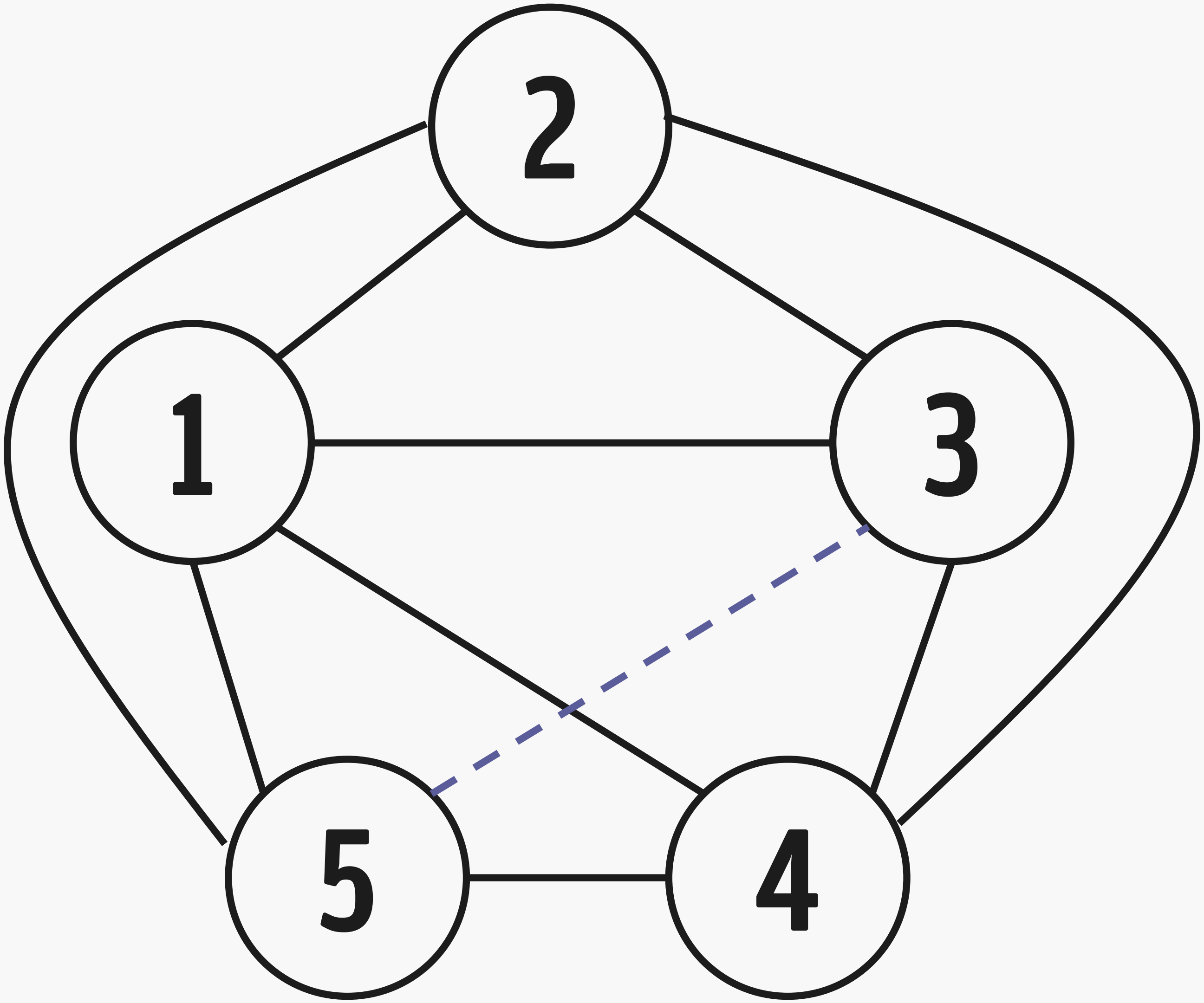
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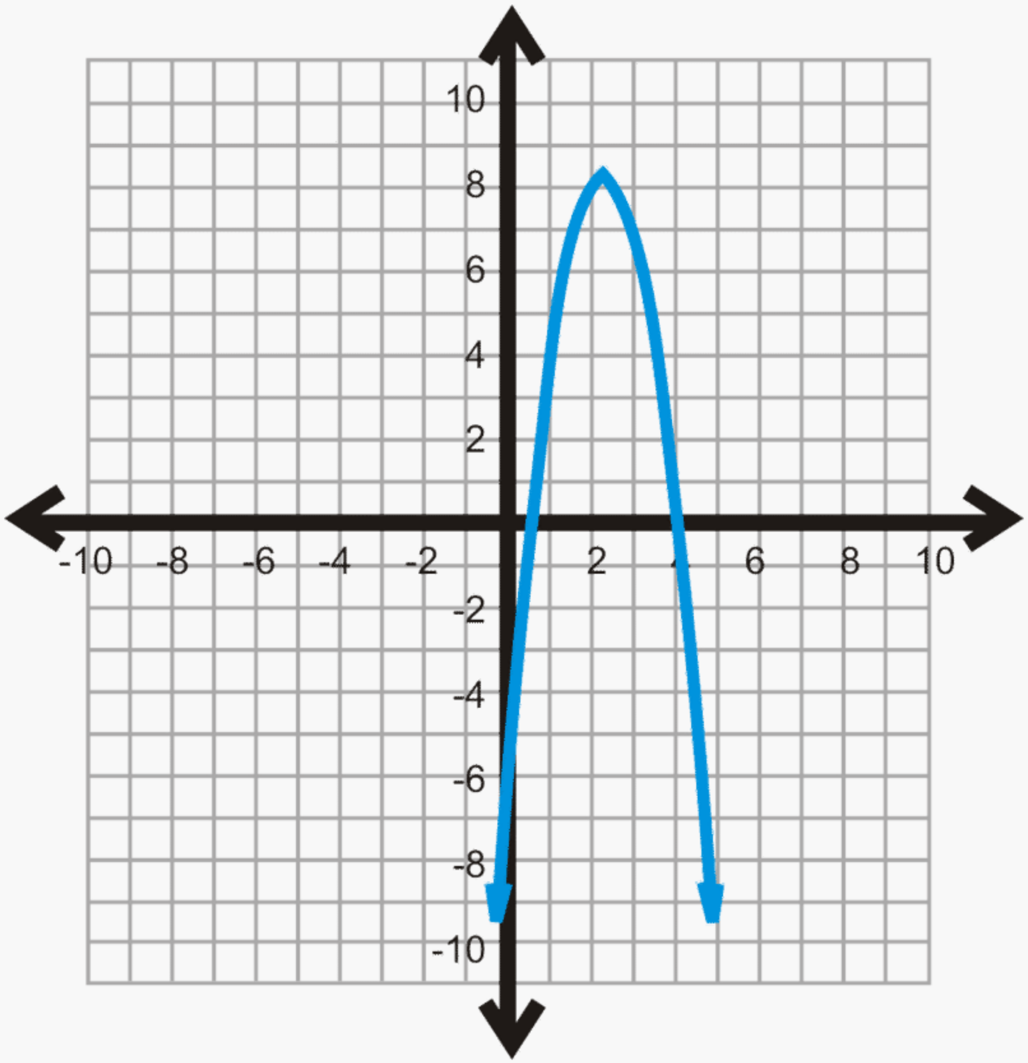
November 2016

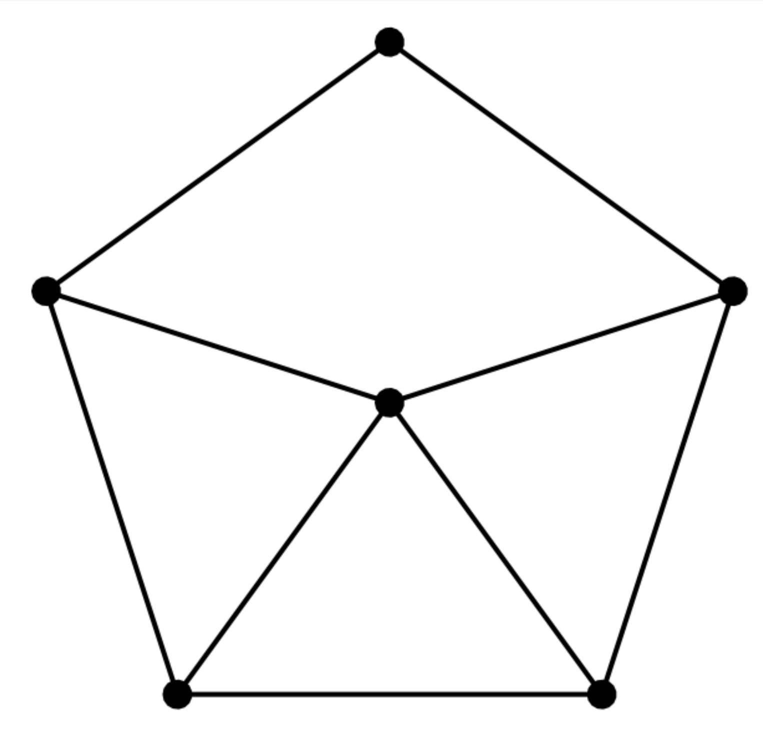
- Michael wanted to do a project with me for a science fair.

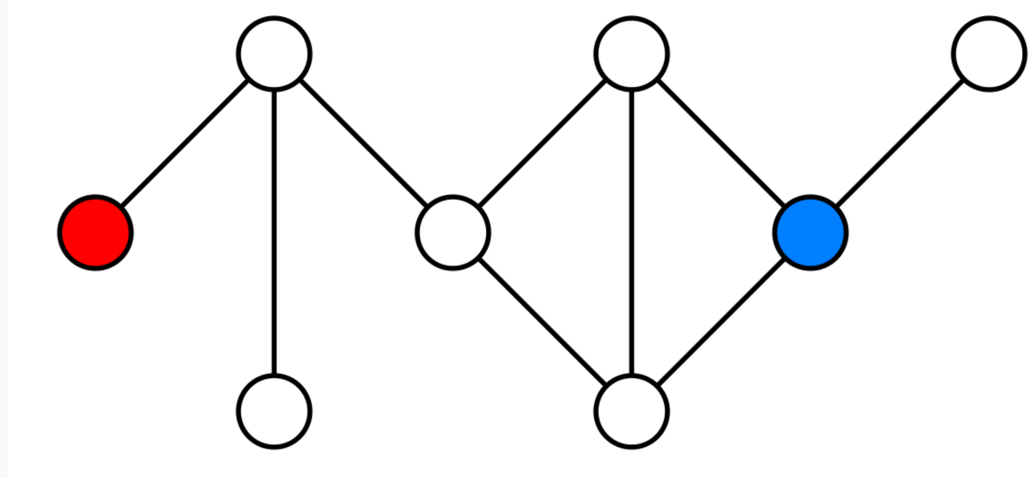
November 2016

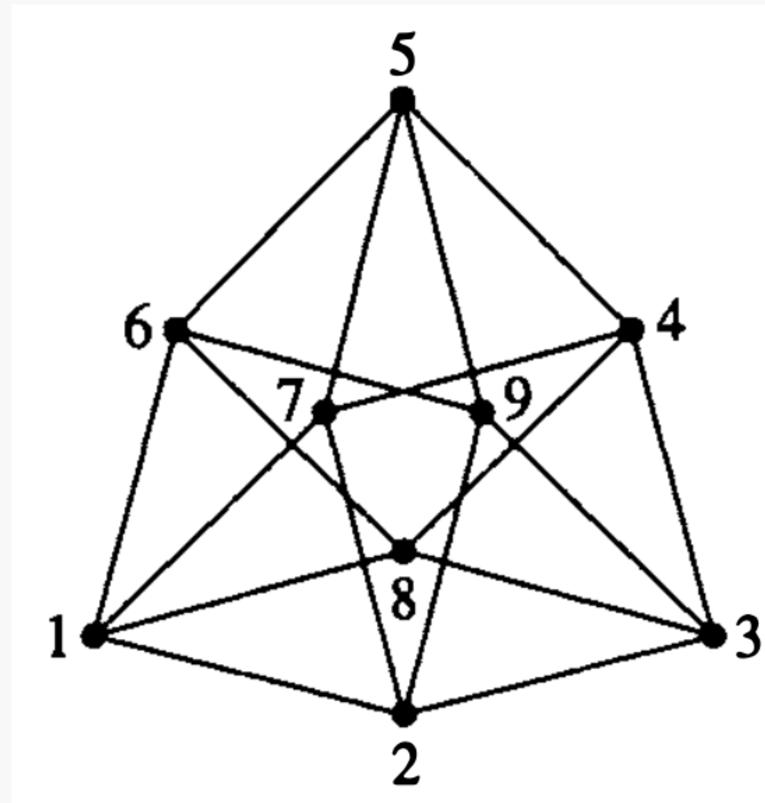
- Michael wanted to do a project with me for a science fair.
- We both decided to do **graph theory**.



(Not these graphs.)






5
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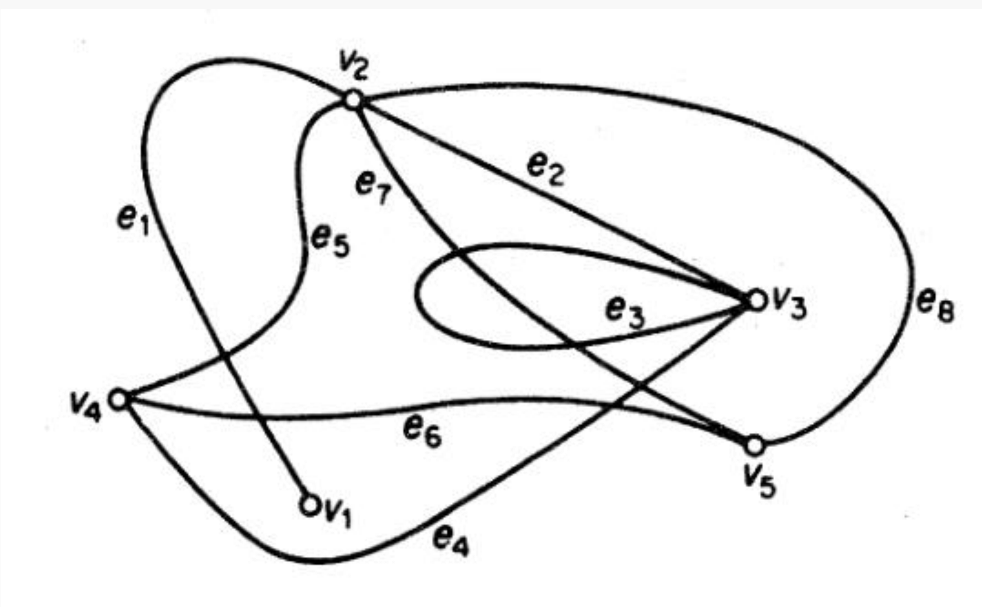
v_2
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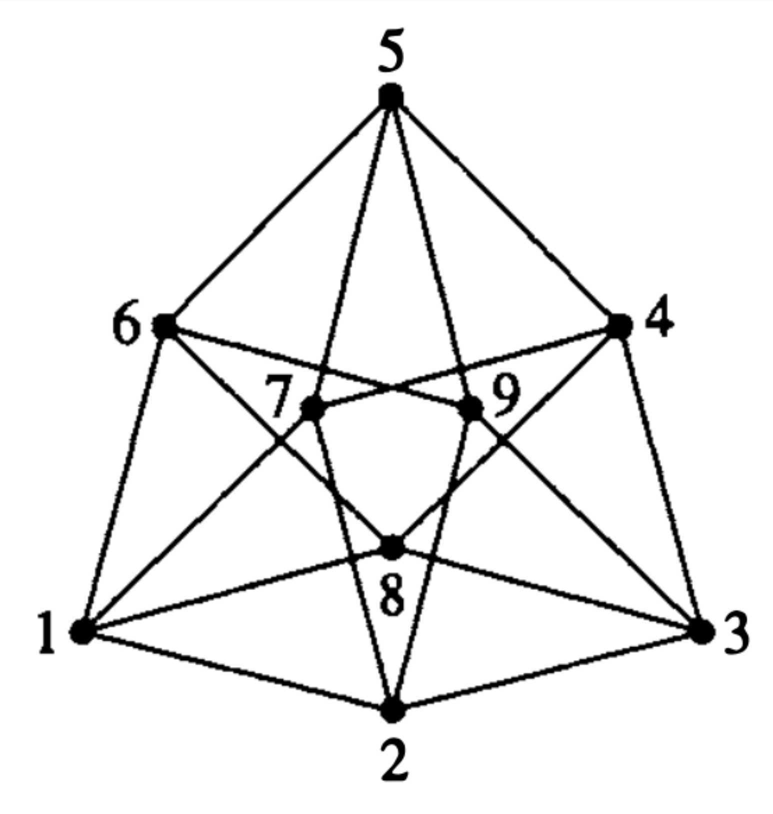
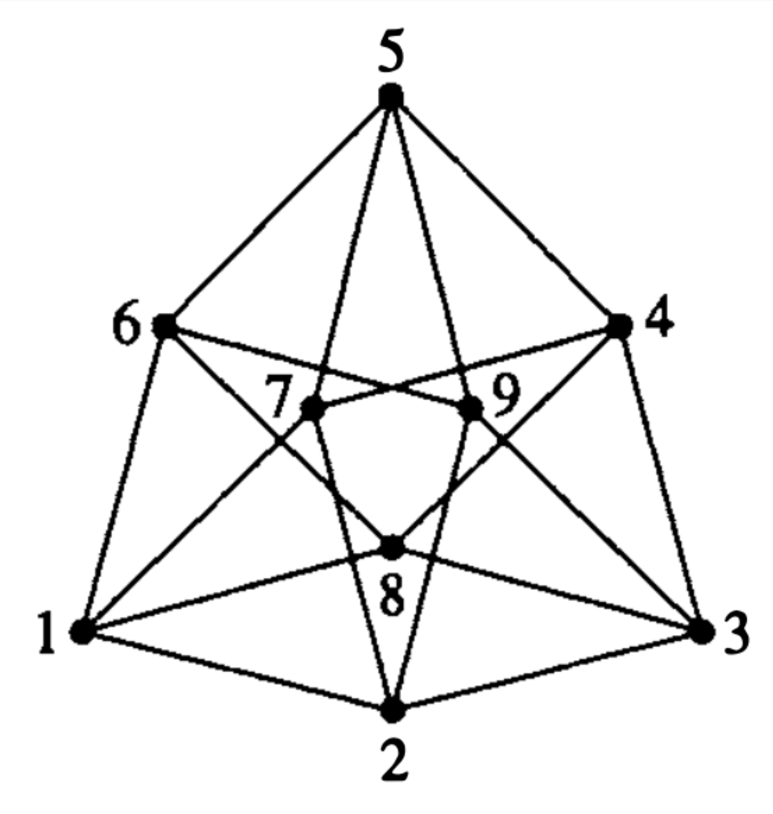
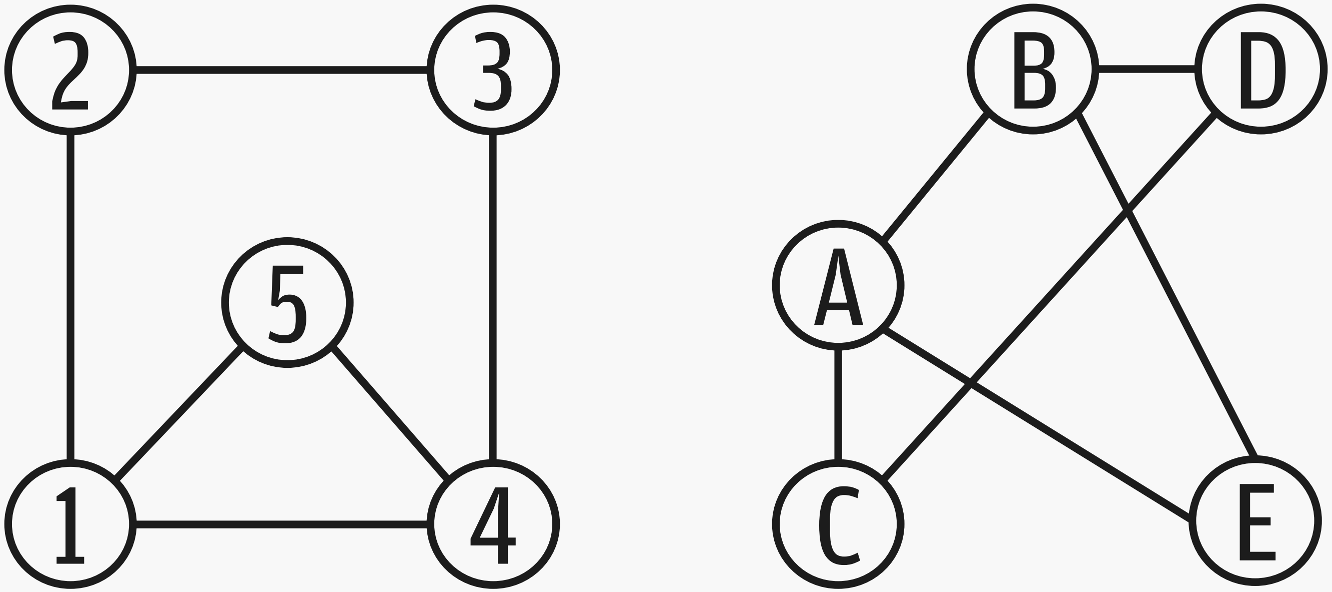


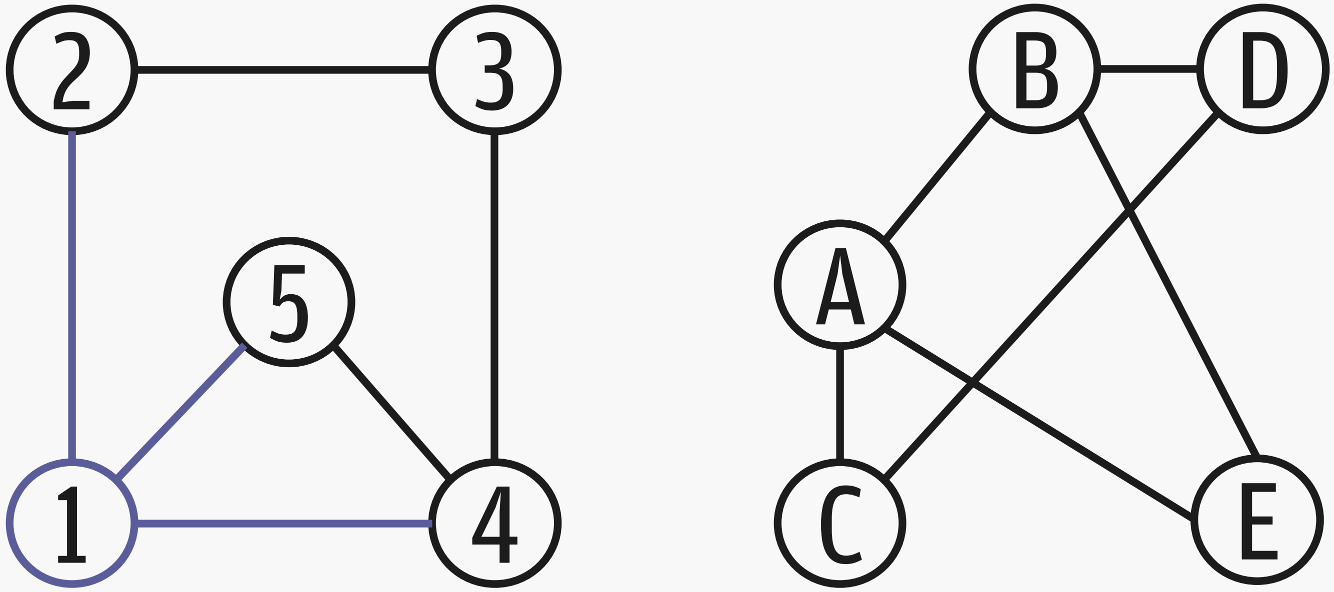
- A graph has **vertices**.



- A graph has **vertices**.
- A graph has **edges** connecting two vertices.



These two graphs are the same.



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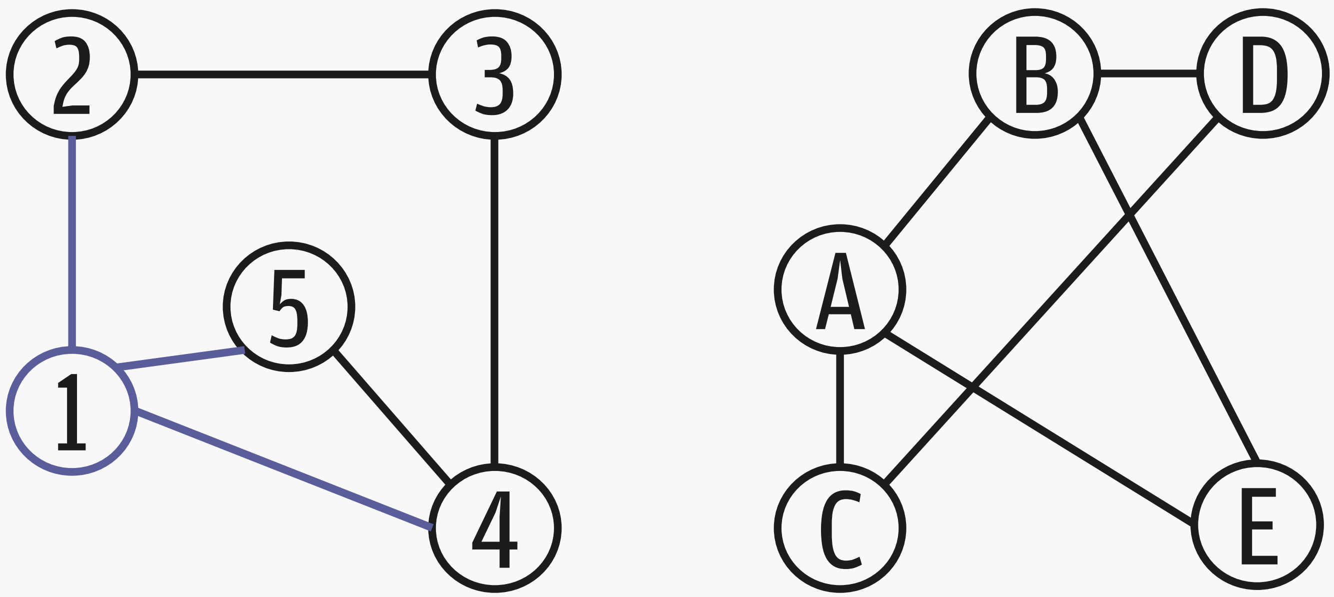

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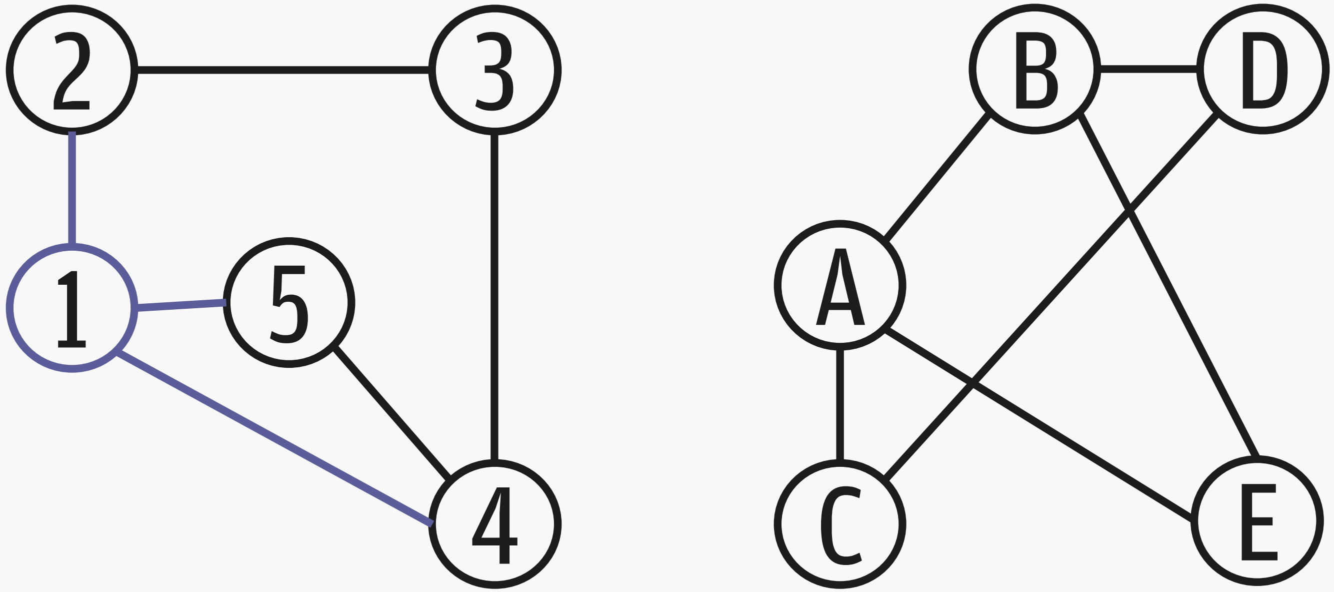

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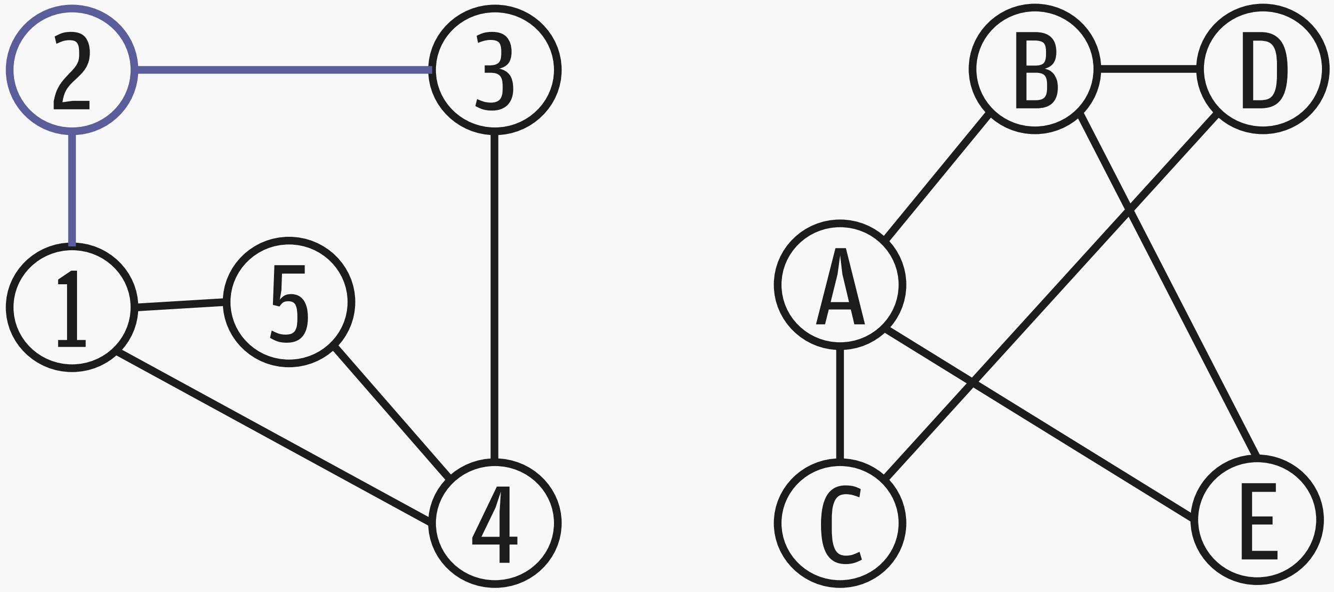

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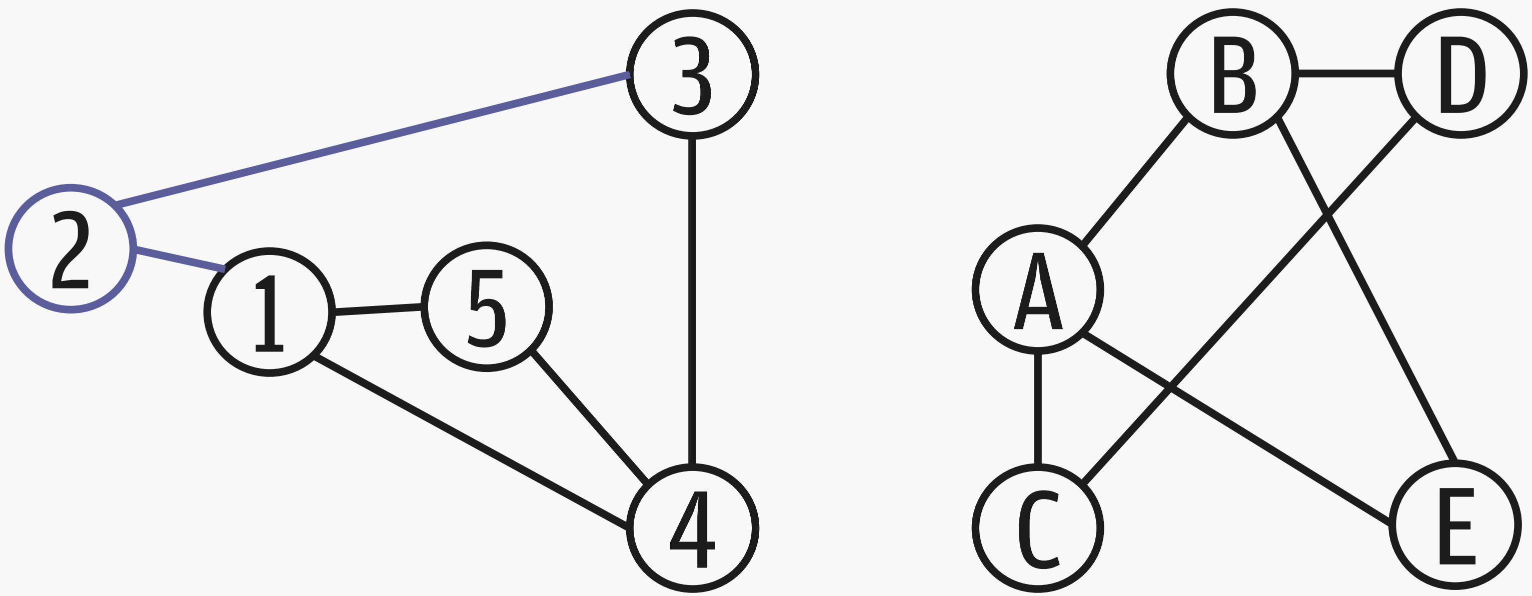

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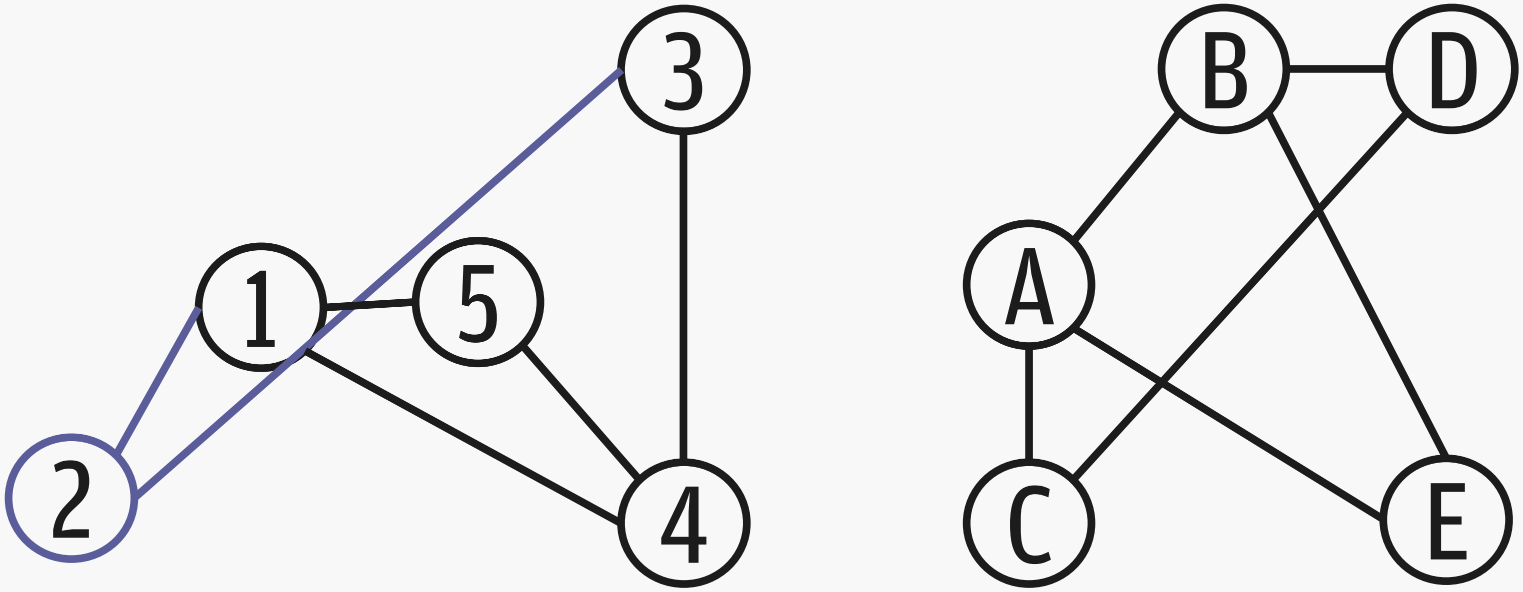

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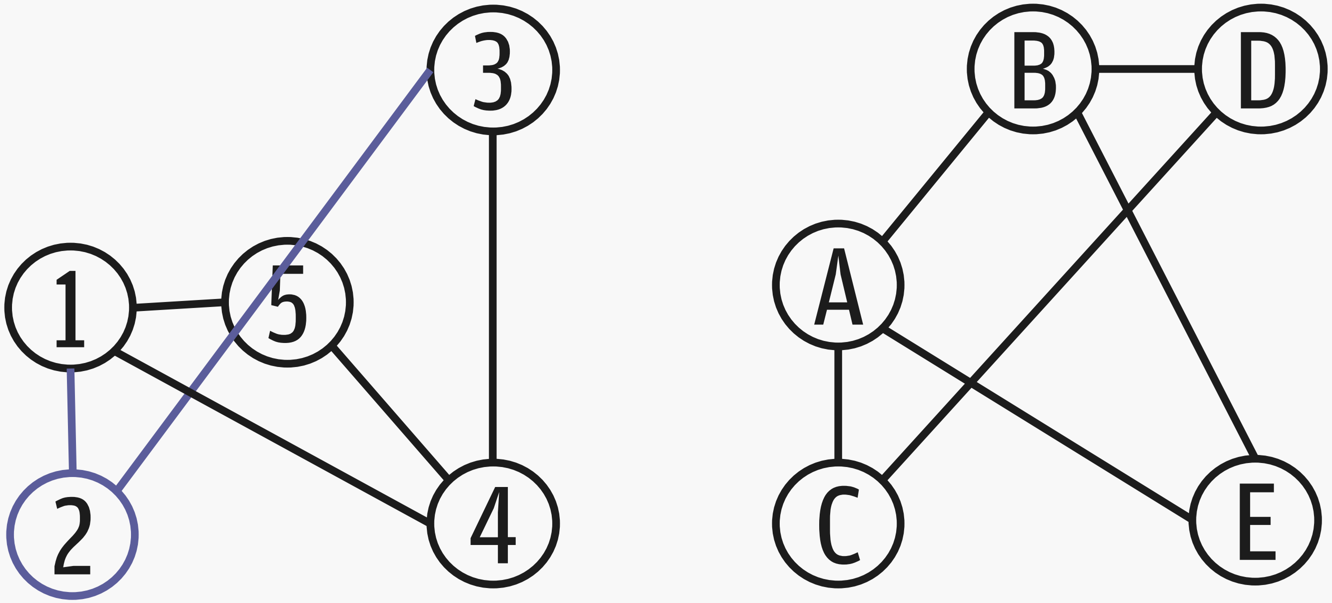

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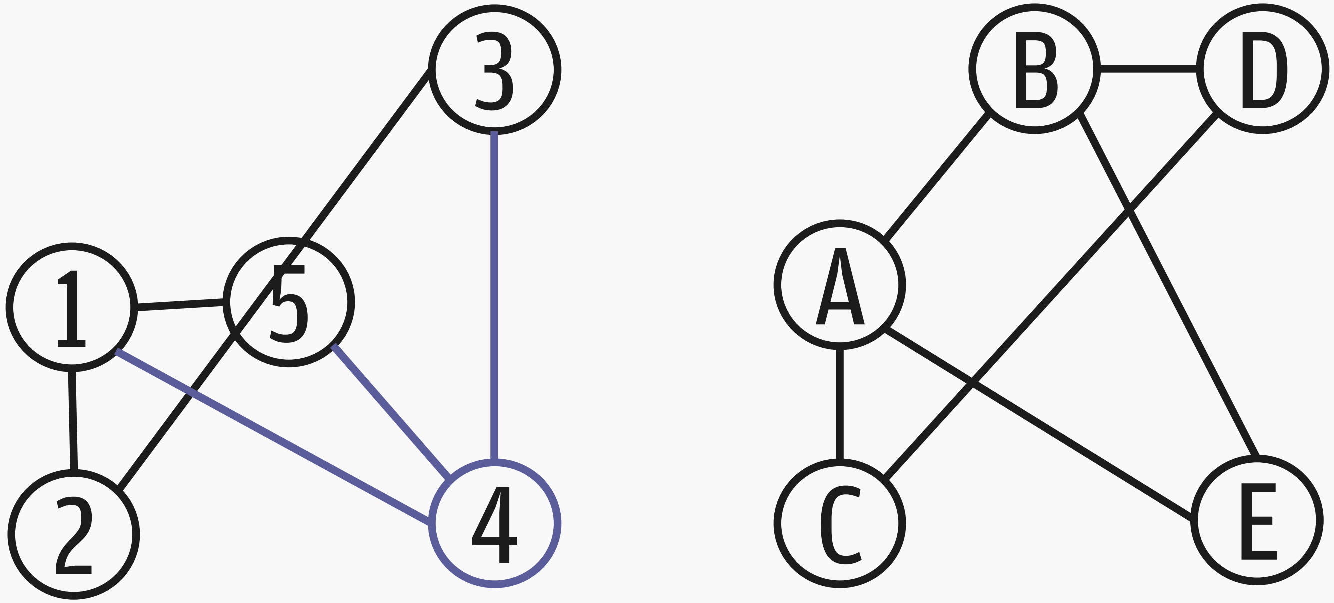

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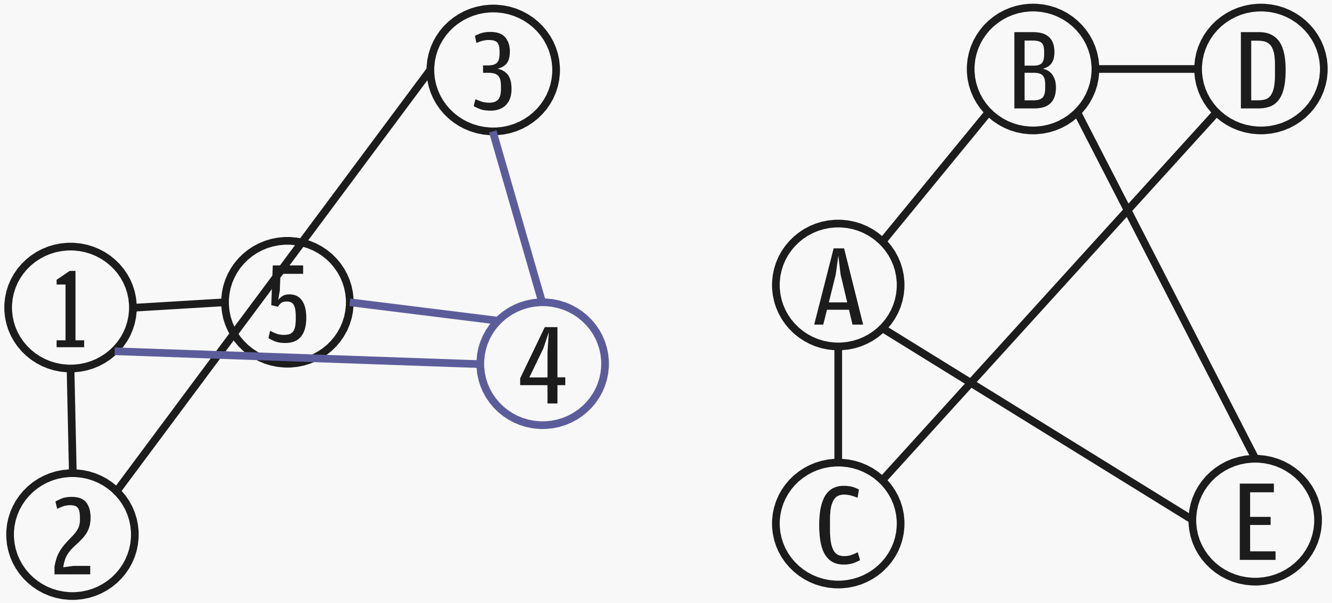

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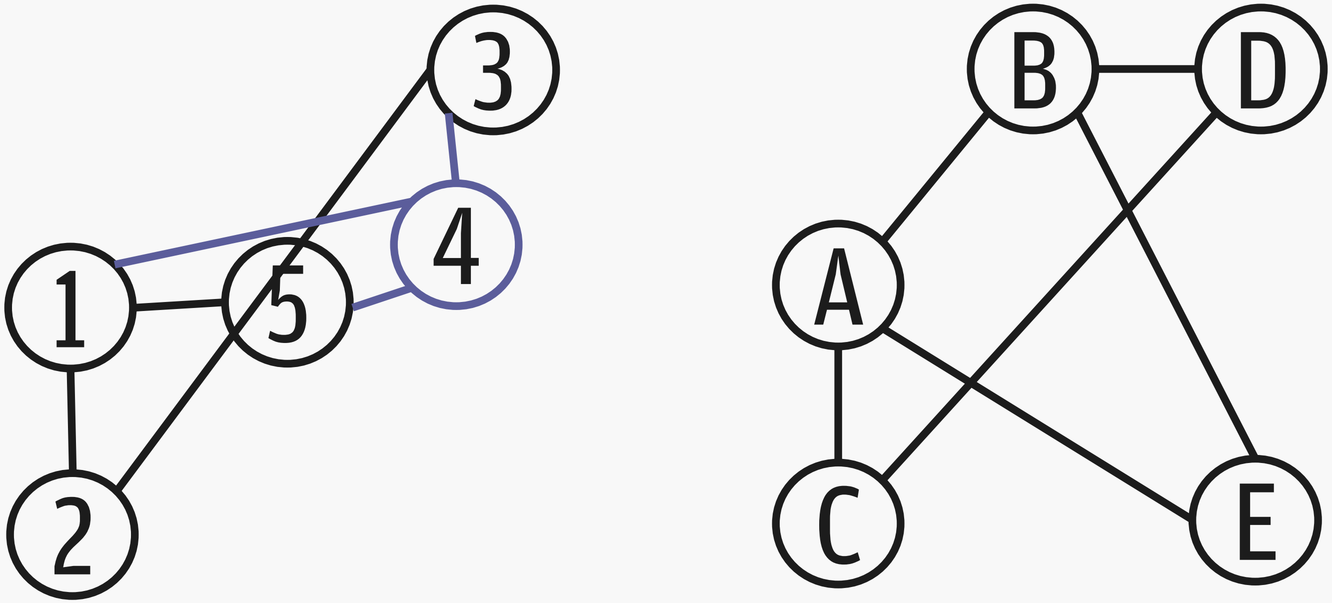

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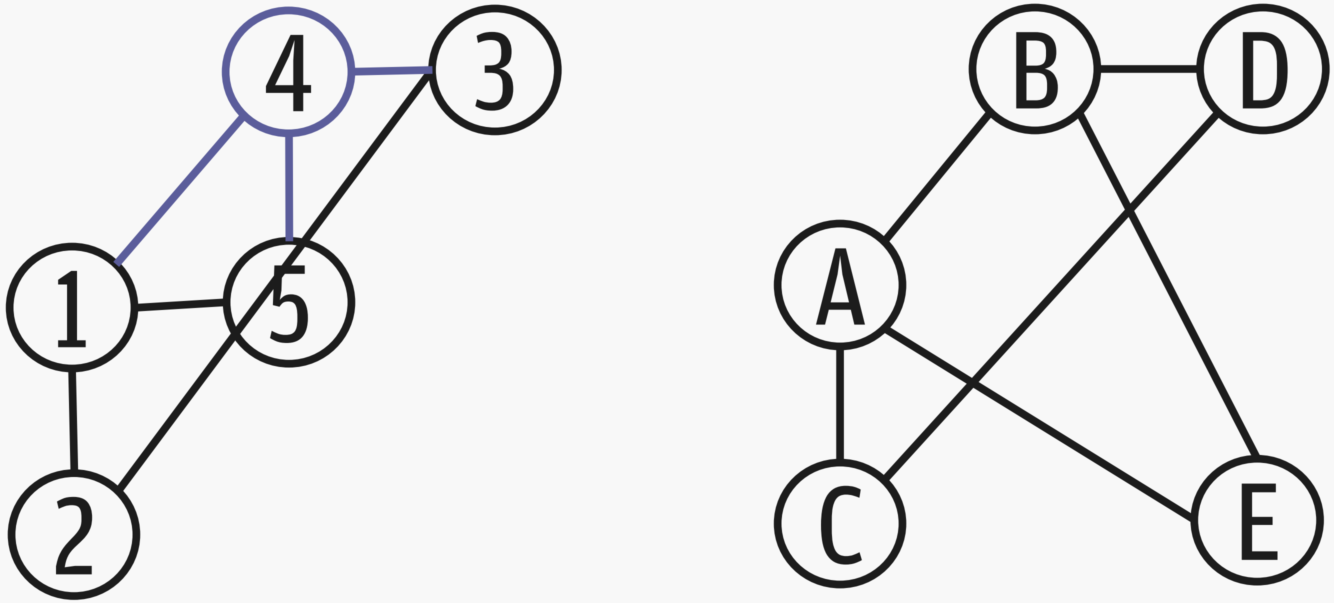

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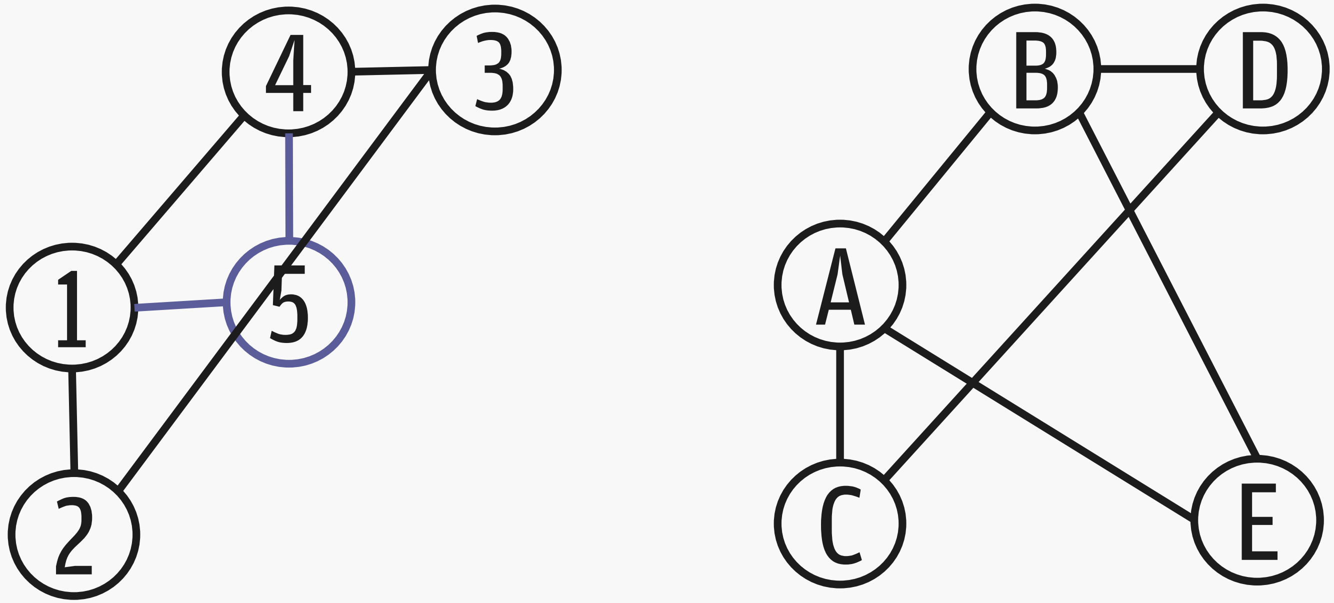

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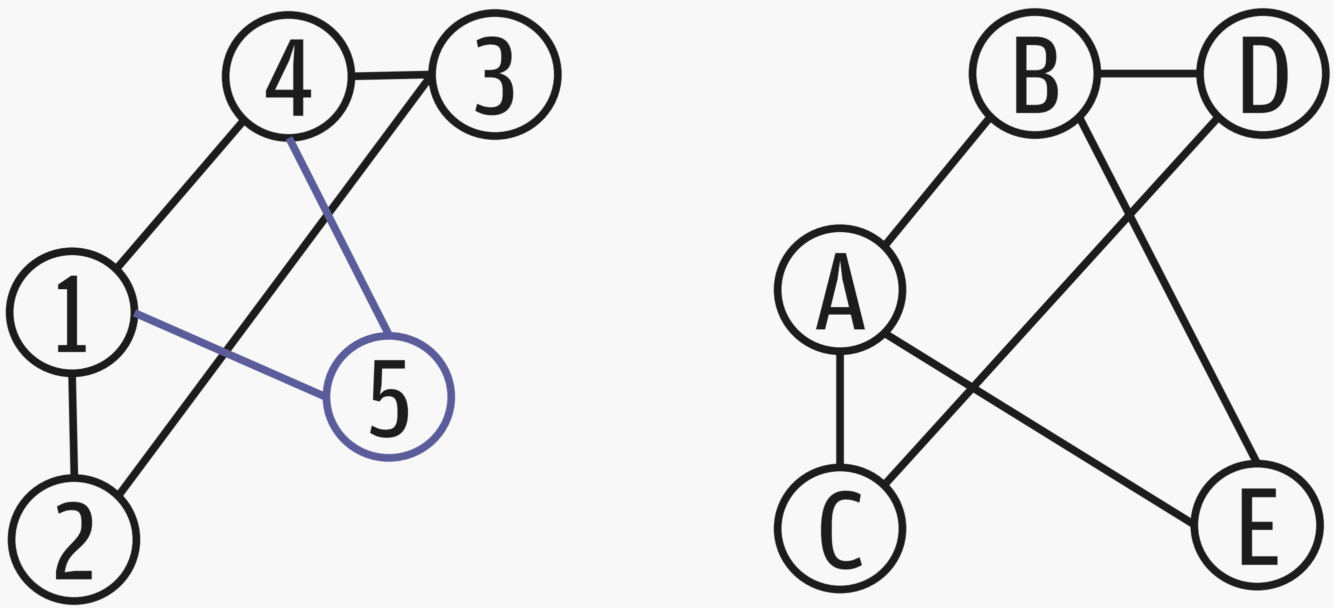

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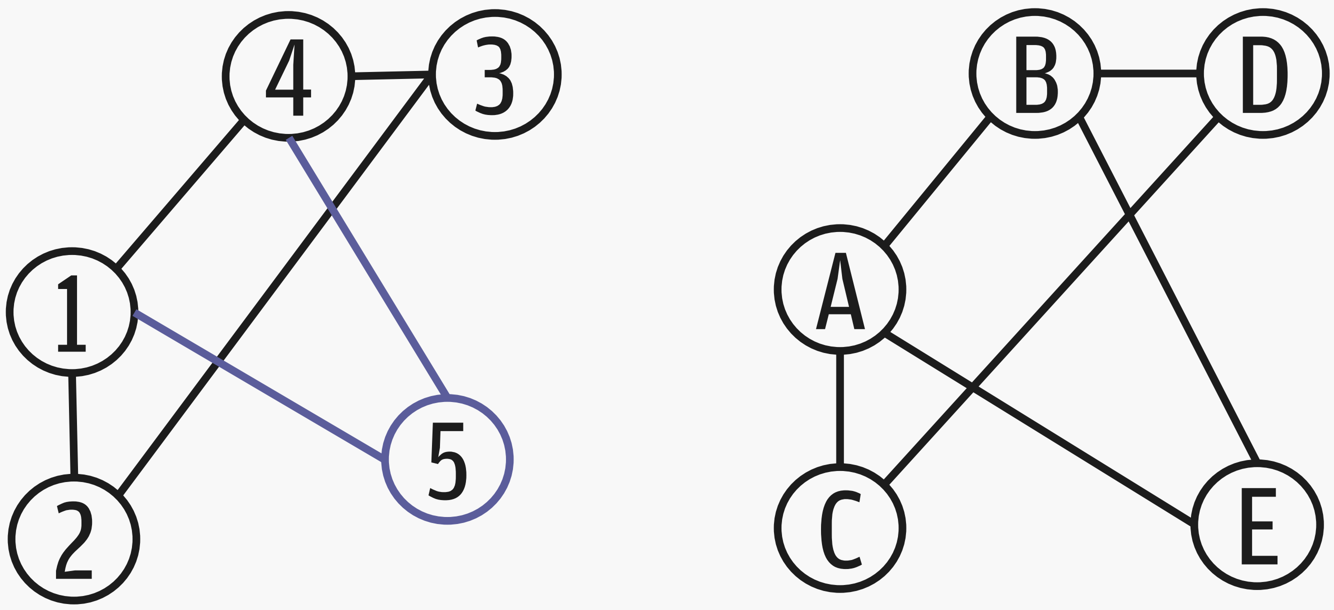

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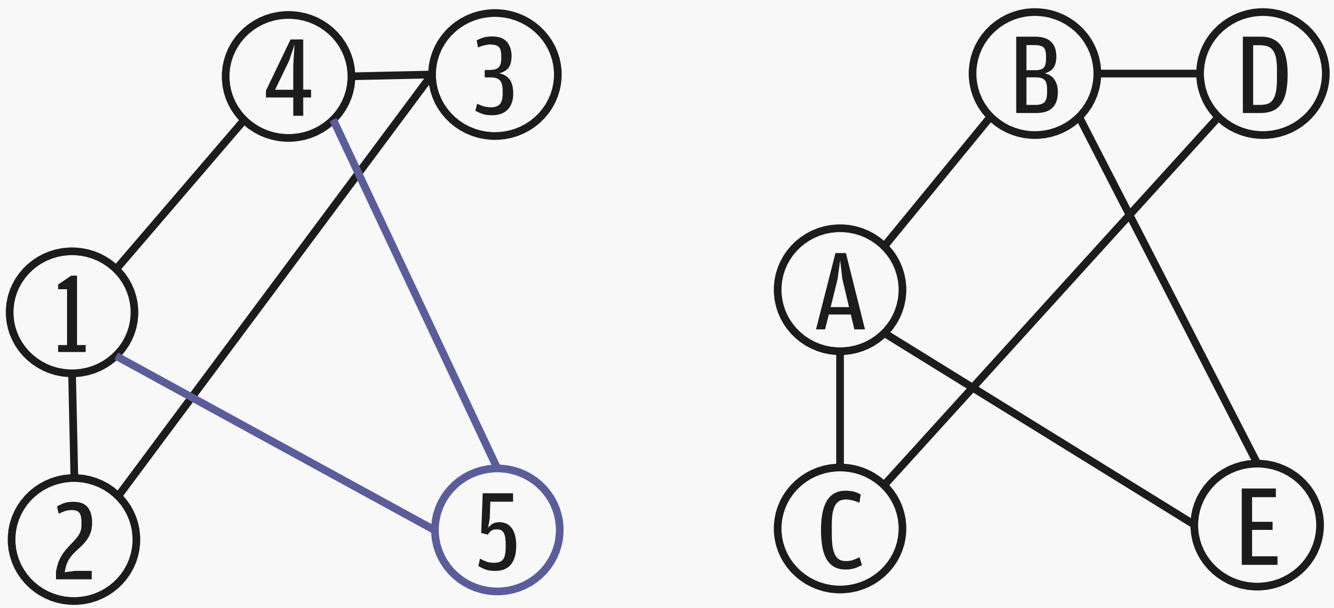

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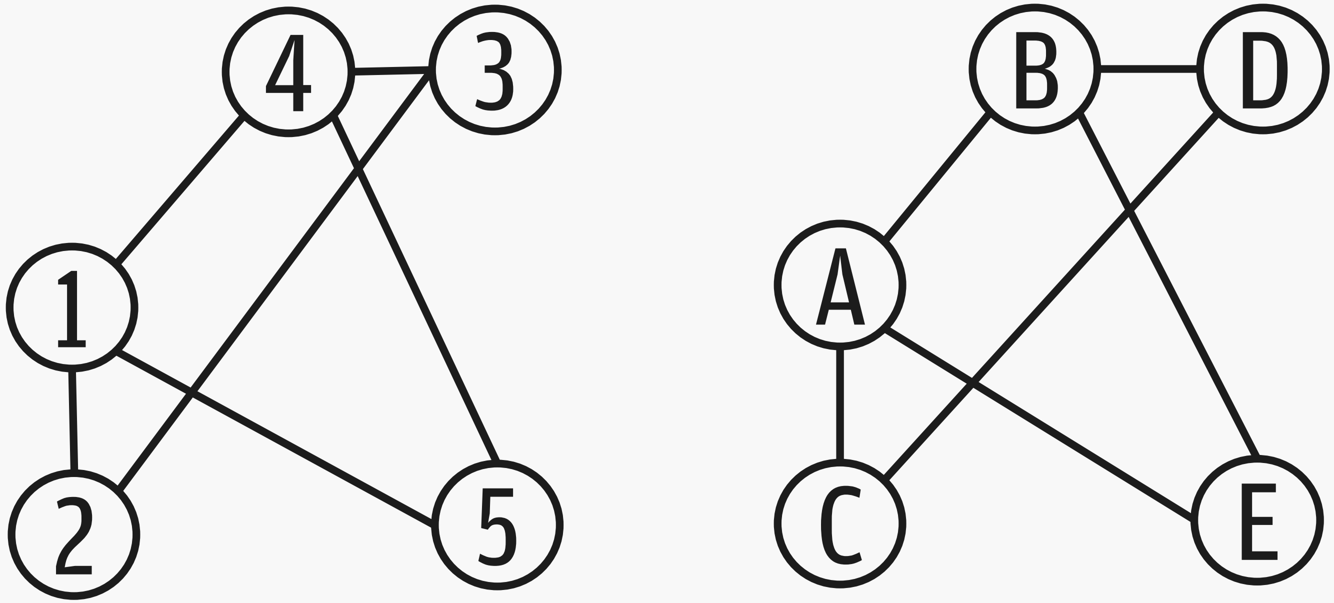

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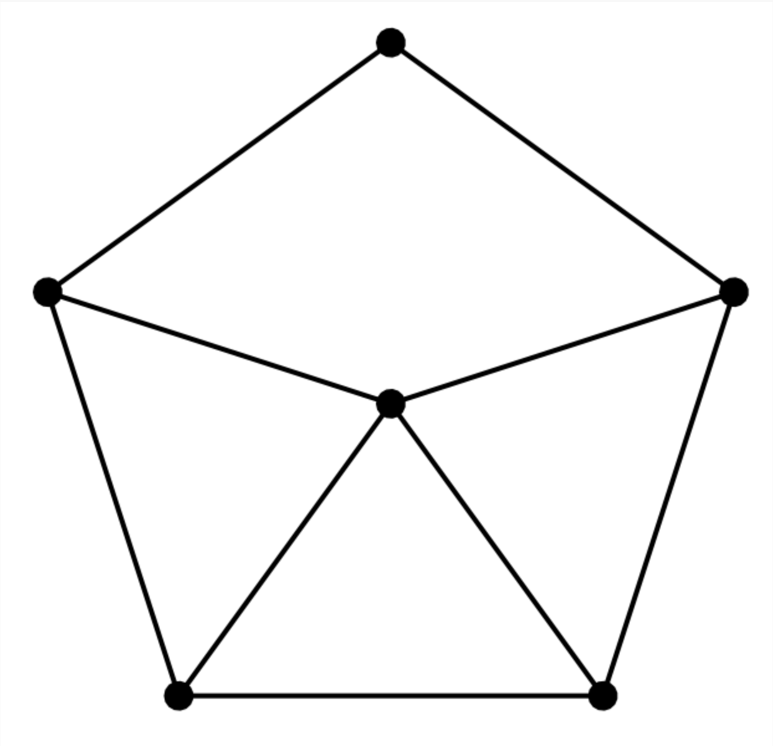
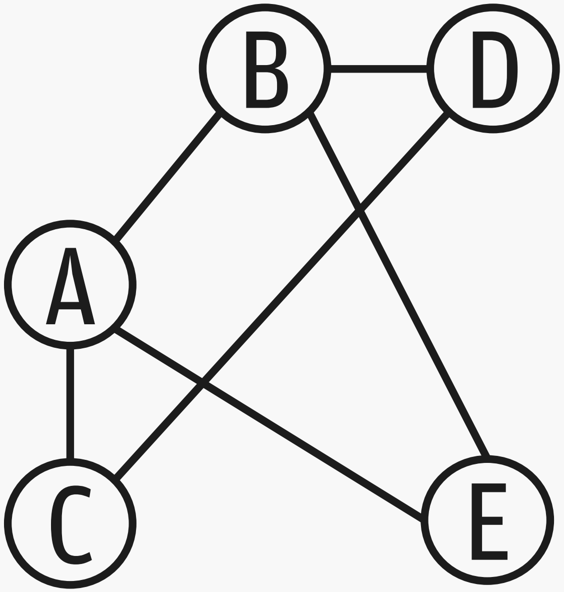
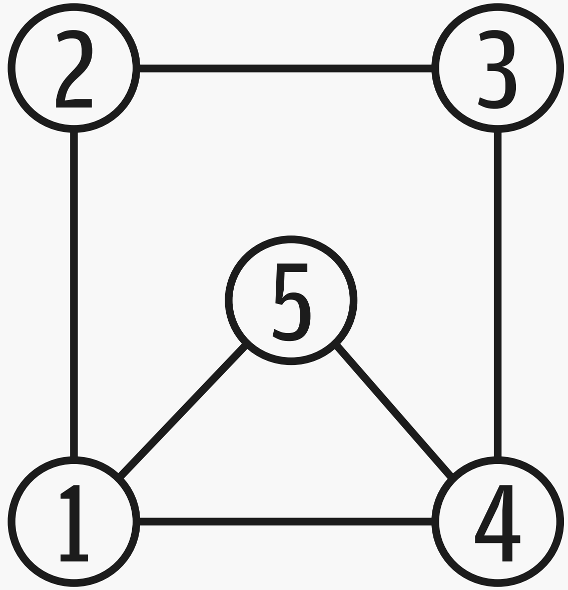


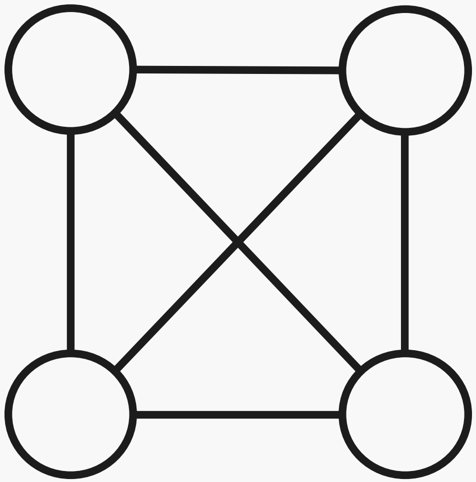
- A graph which can be drawn so that none of its edges cross is called **planar**.



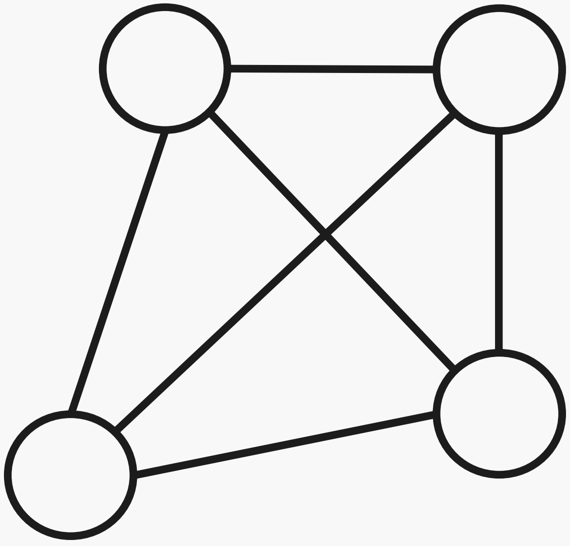
This is a planar graph…

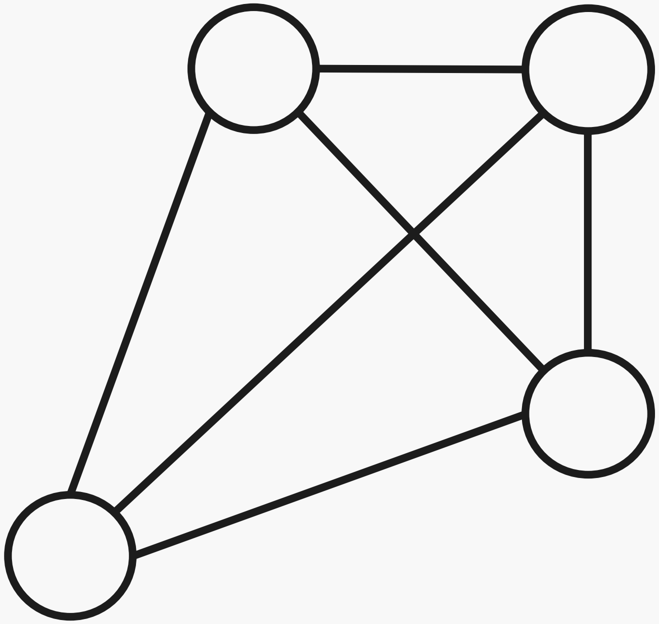


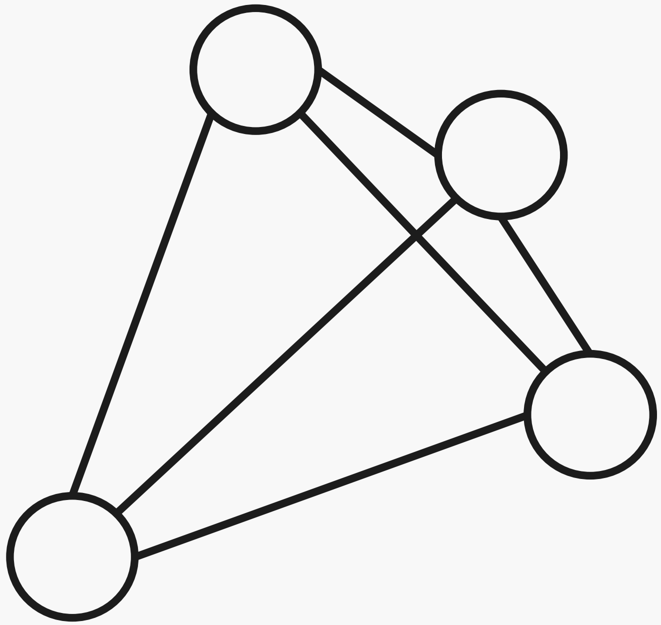
Because it is the same as this graph.

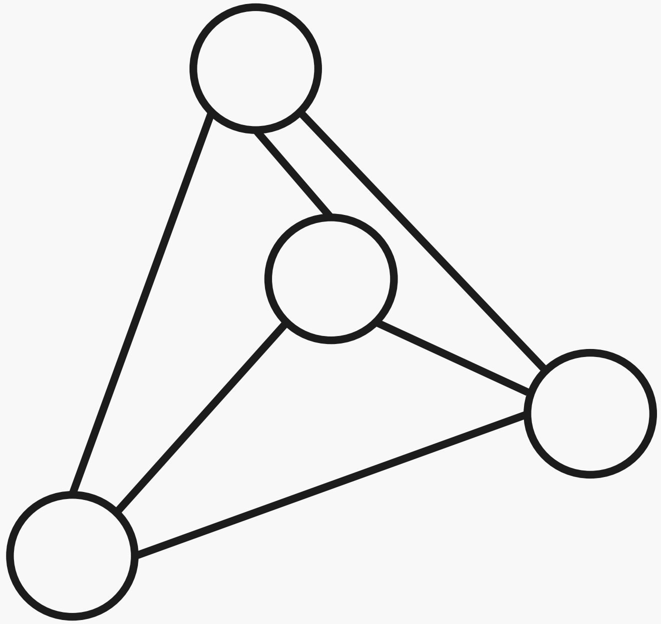


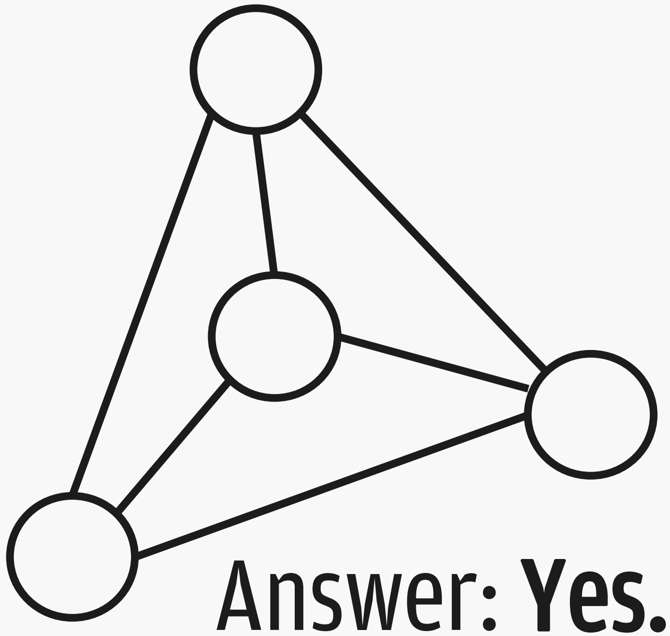
Is this graph planar?



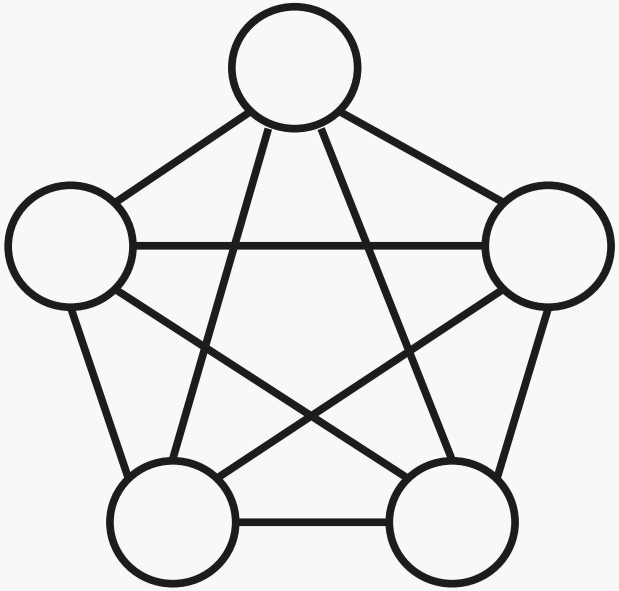




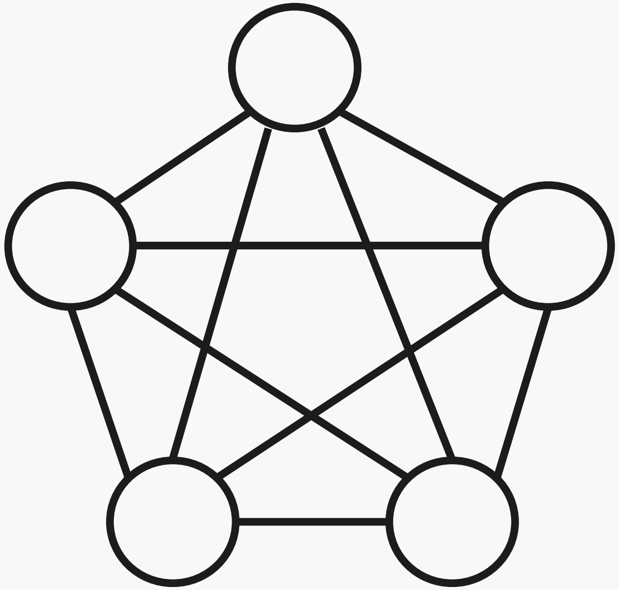




Answer: **Yes.**



Five houses problem: is this graph planar?



Answer: **No**. But why?

November 2016

- We wanted to study **planar graphs.**

November 2016

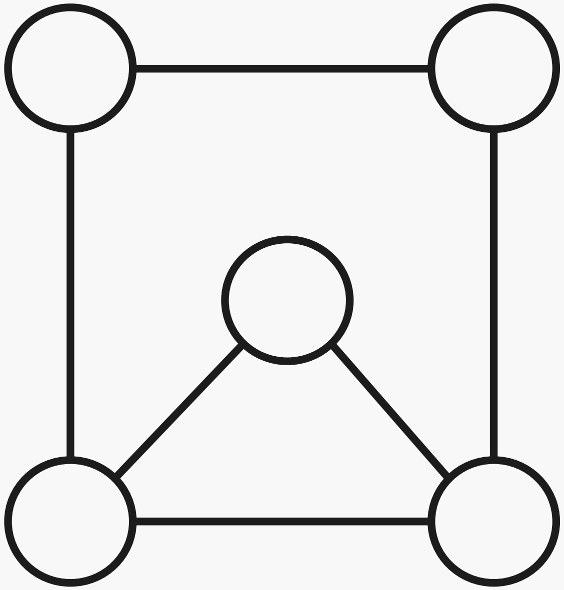
- We wanted to study **planar graphs**.
- But the problem is, we already know a lot about planar graphs.

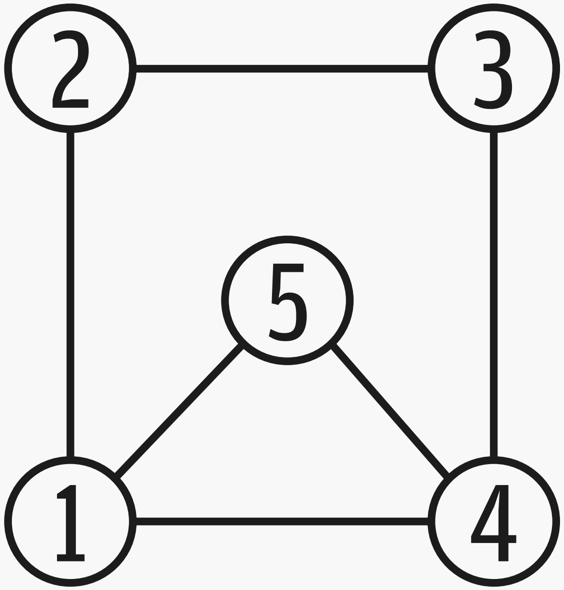
November 2016

- We wanted to study **planar graphs**.
- But the problem is, we already know a lot about planar graphs.
- For example. . .

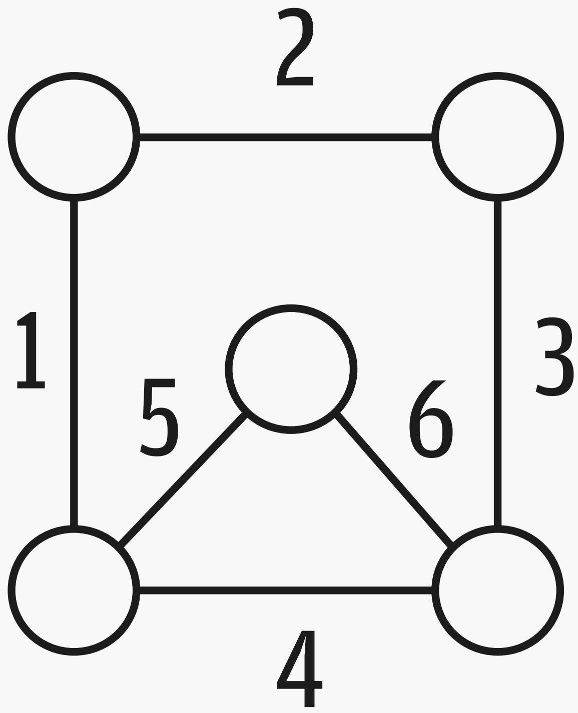
Euler's formula

In a planar graph, let V be the number of vertices, E be the number of edges, and F be the number of faces. Then $V - E + F = 2$.

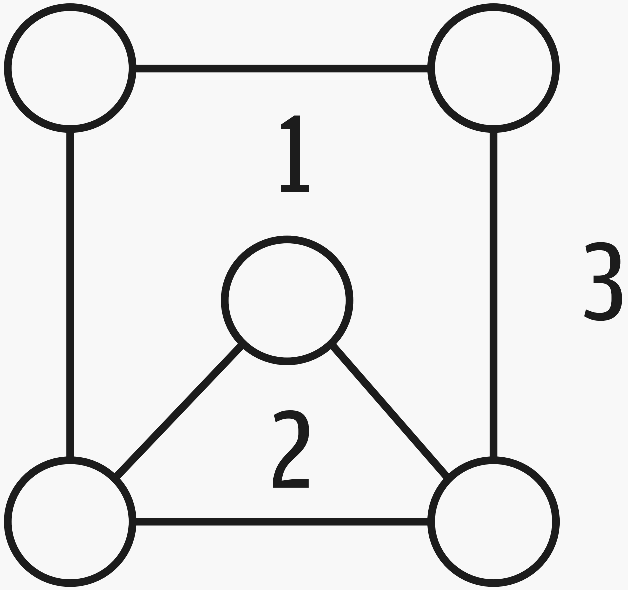




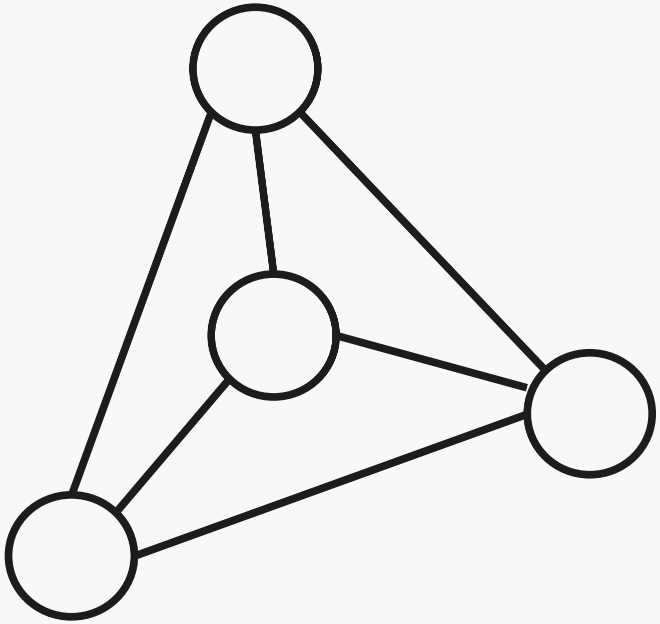
Vertices: 5

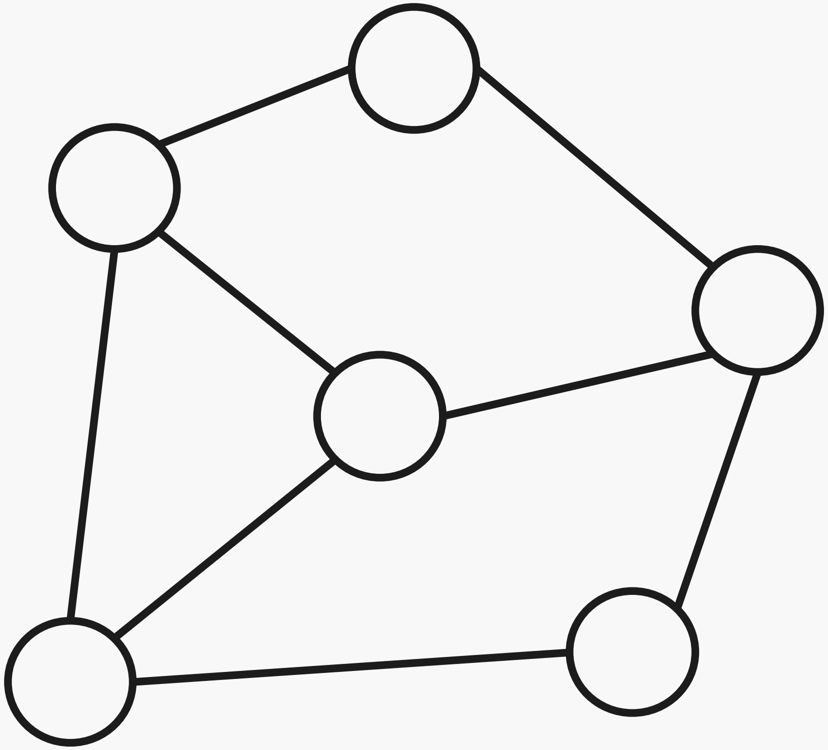


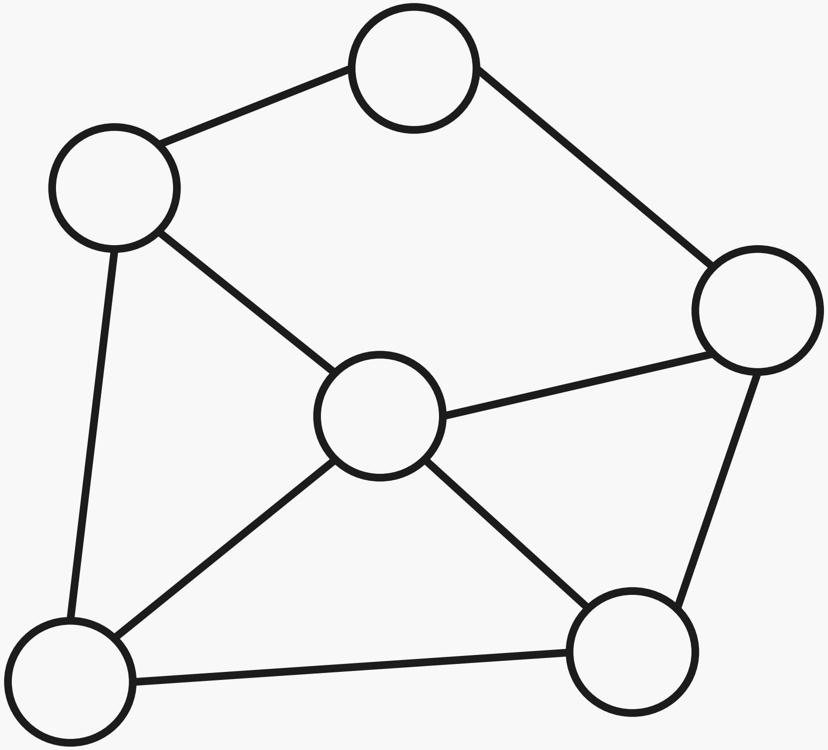
Vertices: 5
Edges: 6



Vertices: 5
Edges: 6
Faces: 3







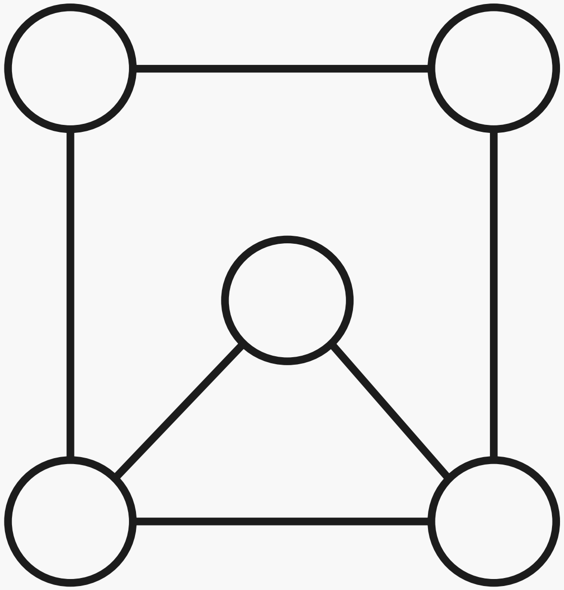
Maximal planar graphs

A **maximal planar graph** is a planar graph where we can't add any more edges to keep it planar.

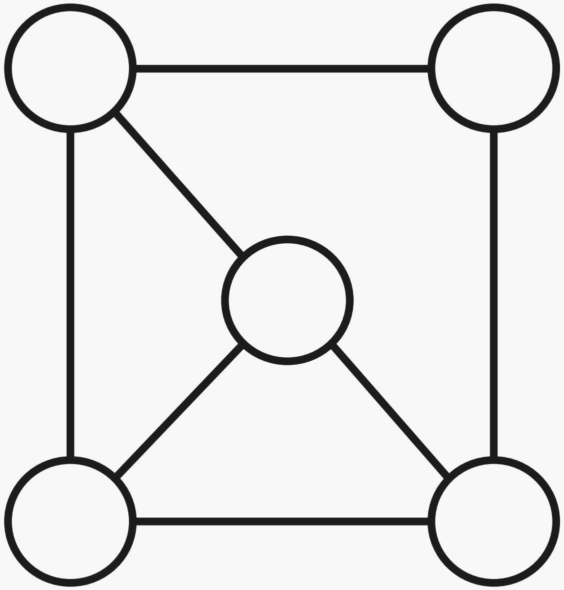
Maximal planar graphs

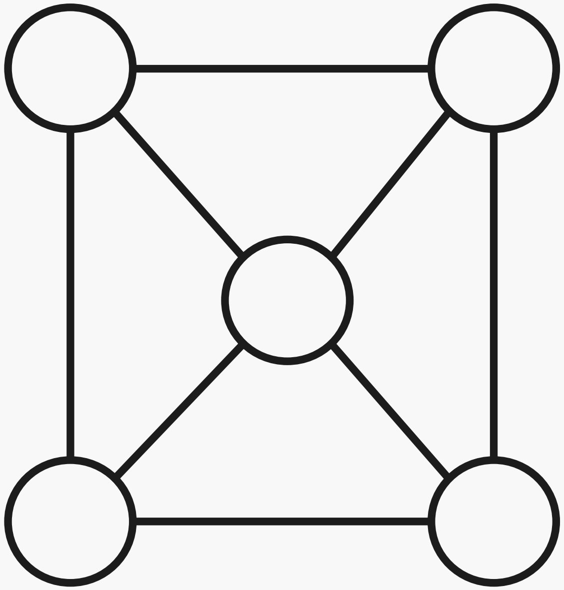
A **maximal planar graph** is a planar graph where we can't add any more edges to keep it planar.

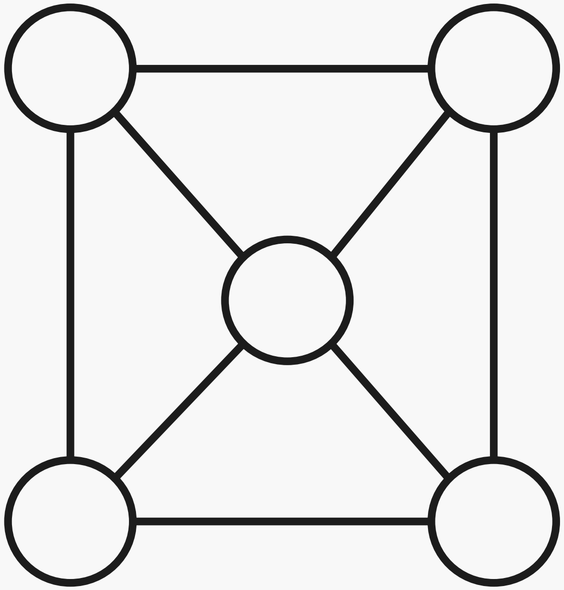
In a maximal planar graph, all the faces are enclosed by three edges.



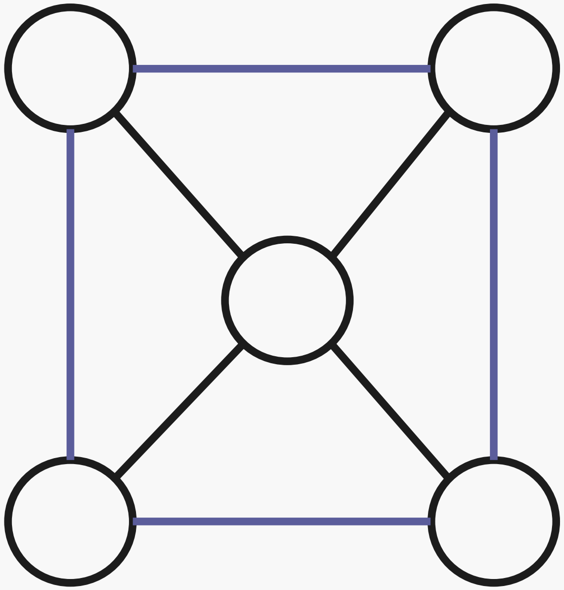
Not maximally planar: we can add more edges.



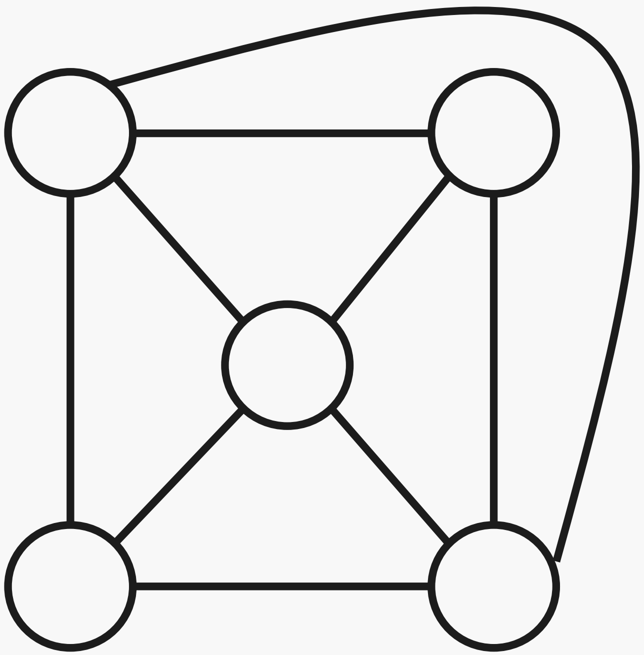




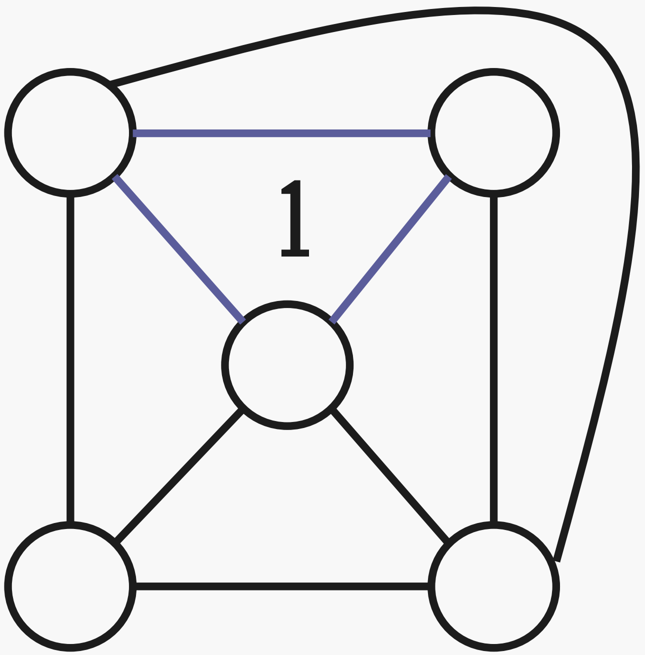
Still not maximally planar!

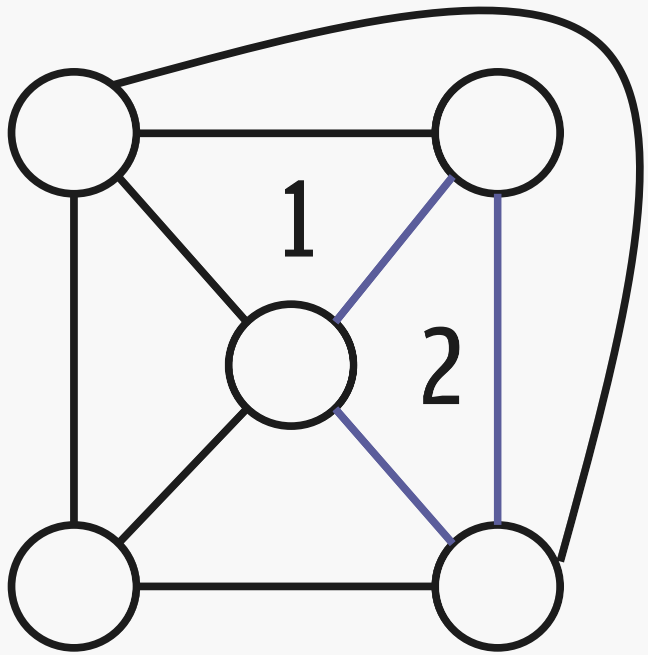


It has a face with four edges: the outside face.



Now it is maximally planar.


1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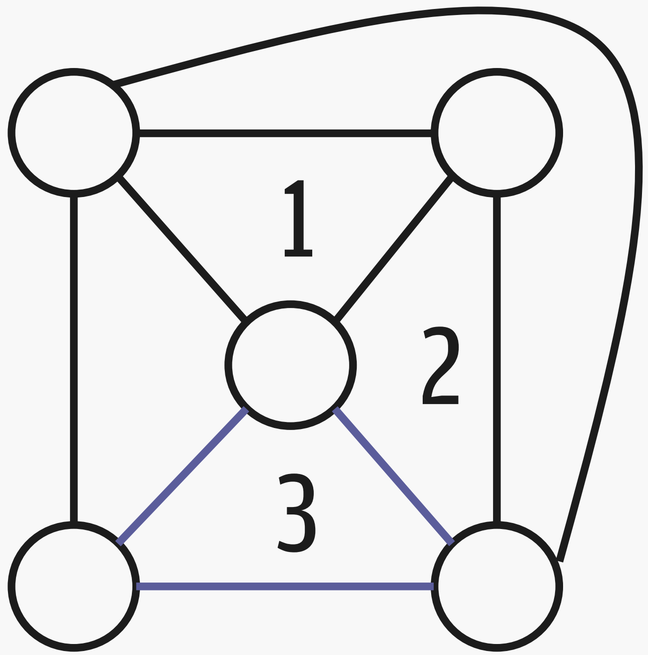
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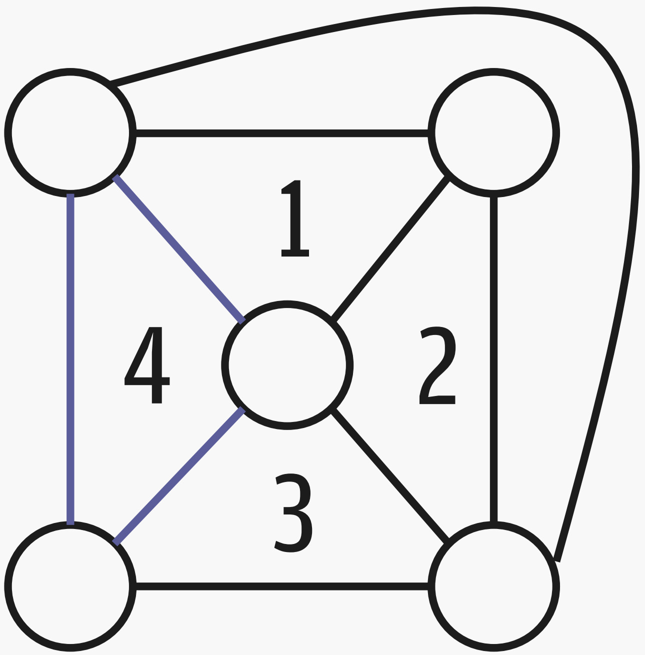
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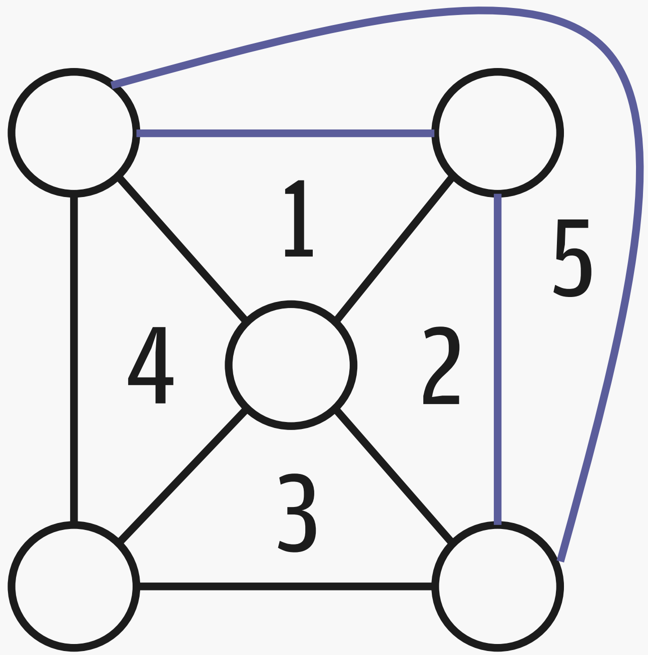
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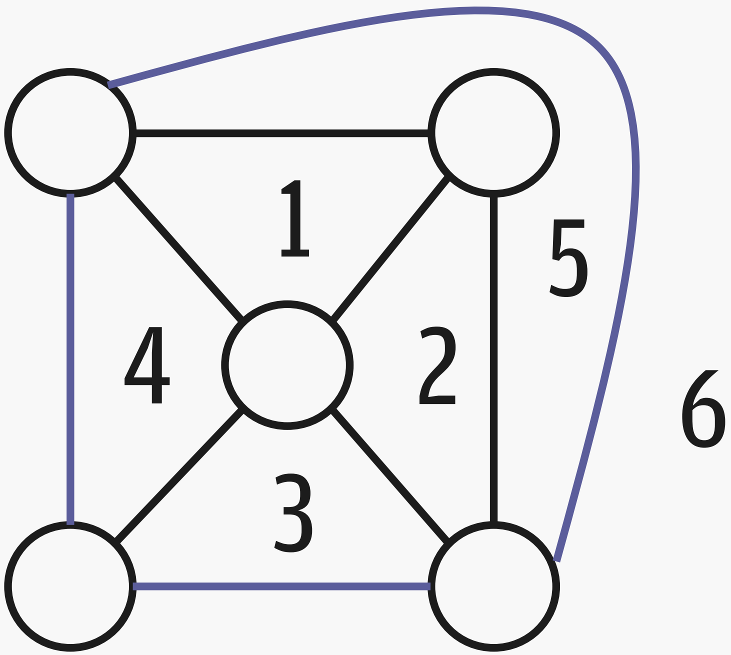
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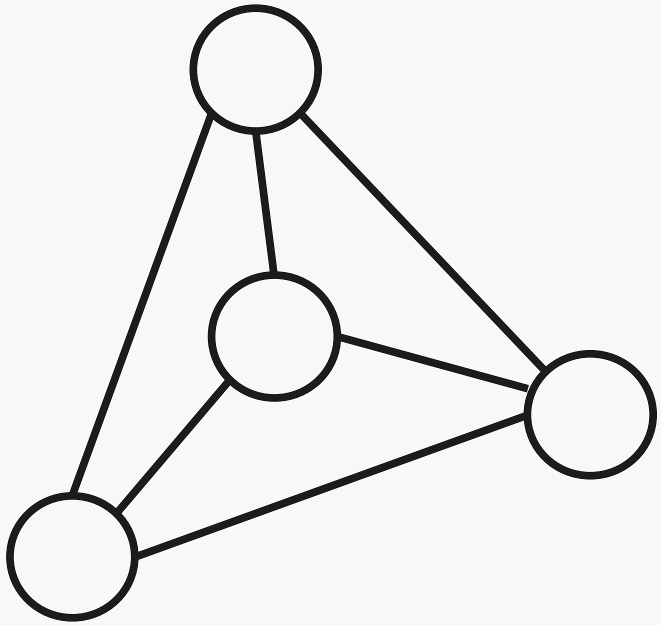
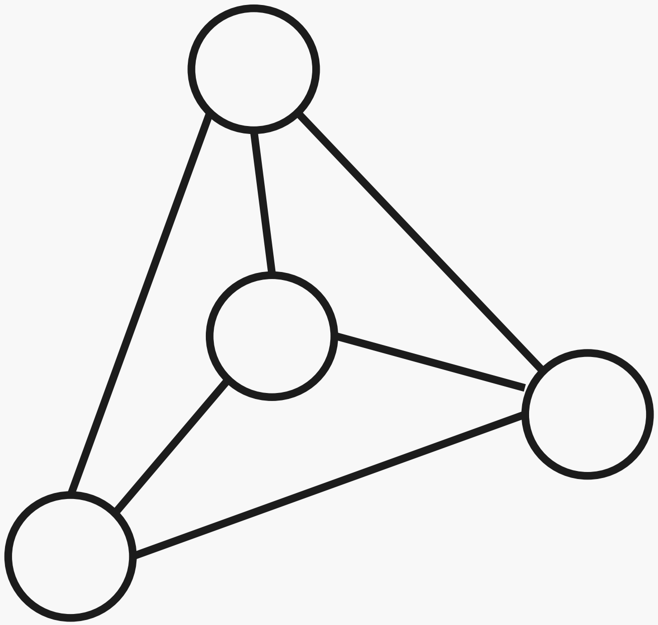
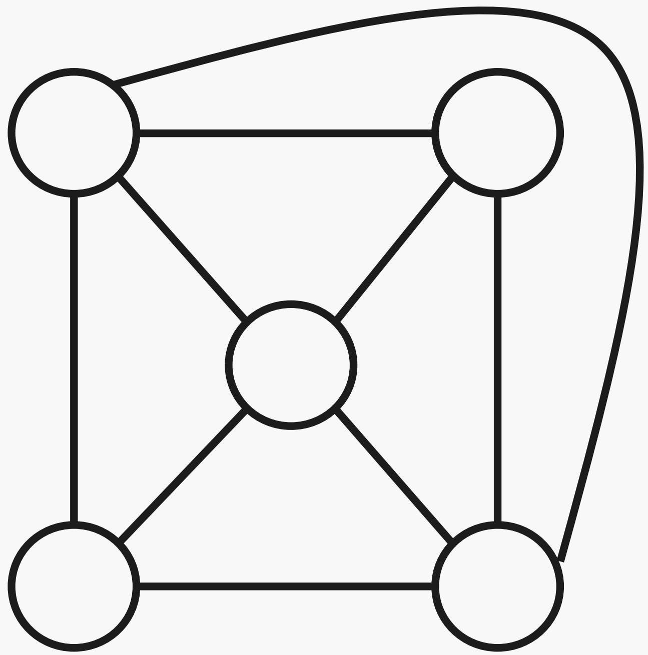


Maximal planar graphs

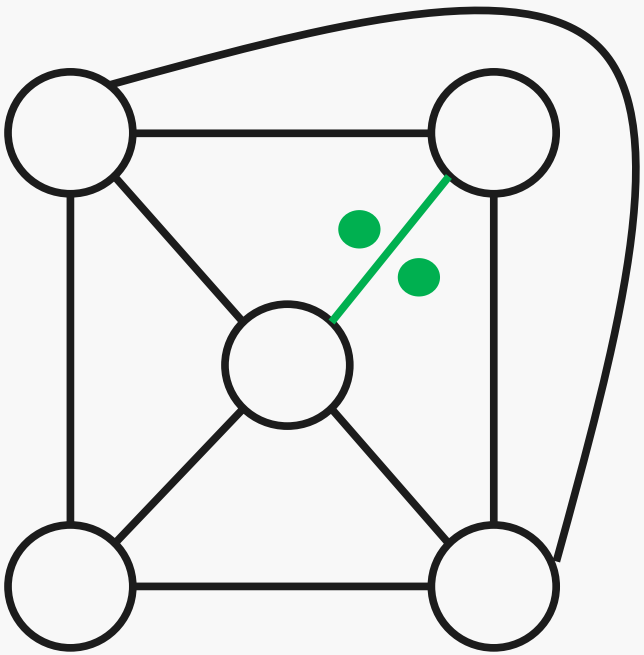
In a maximal planar graph, $2E = 3F$.



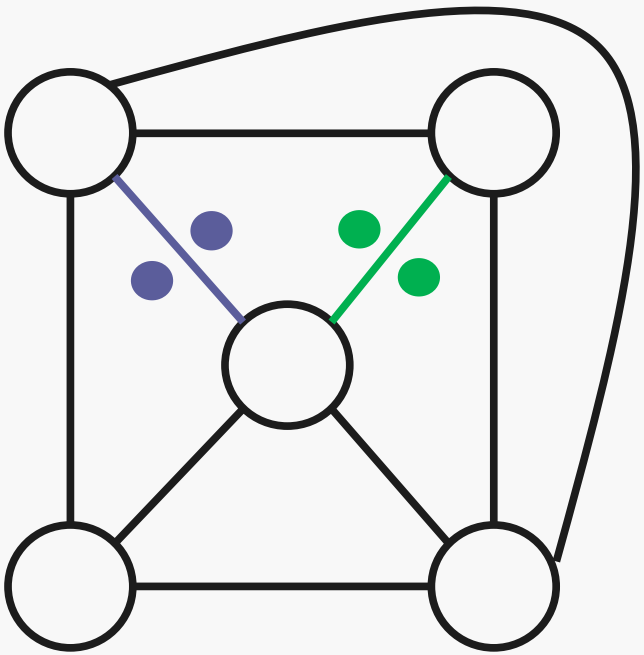
Here E = 6 and F = 4.

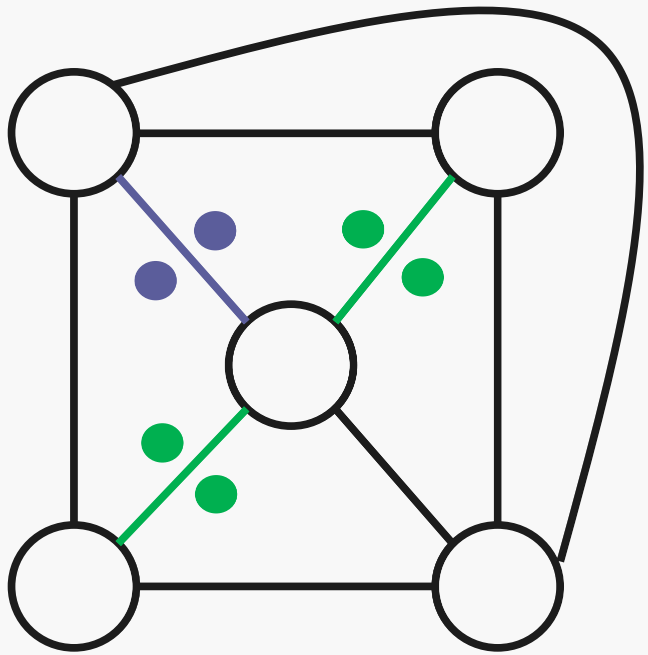


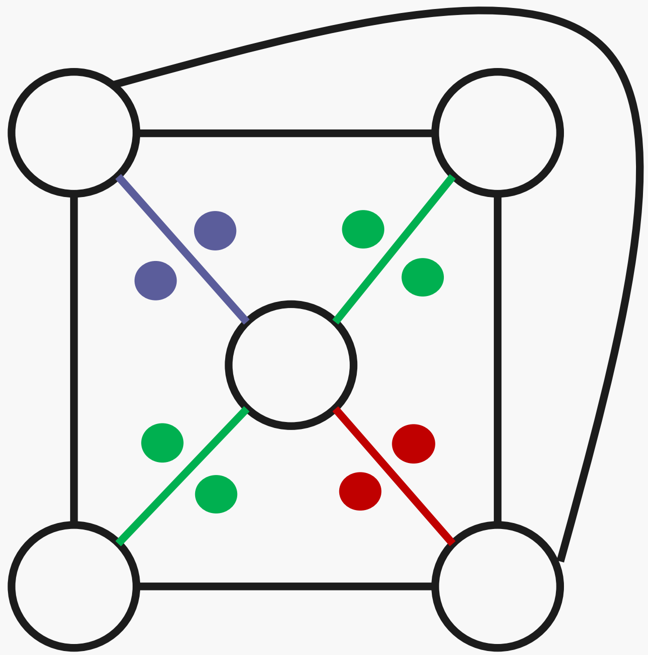
Here E = 9 and F = 6.

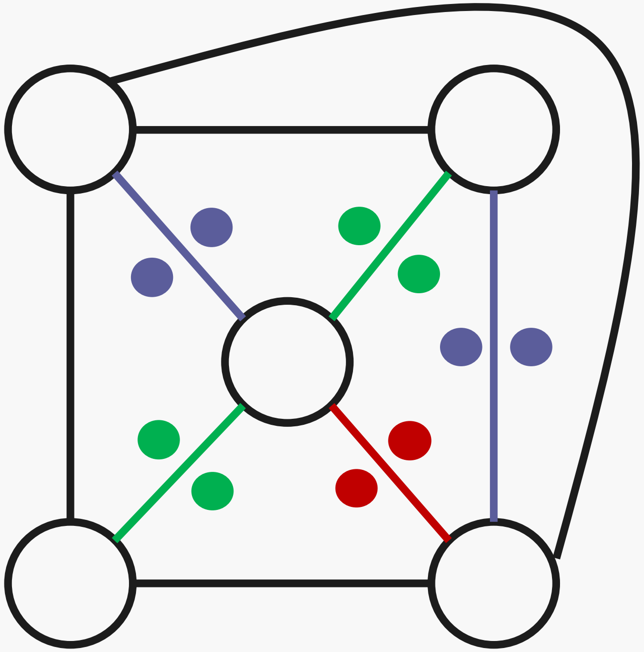


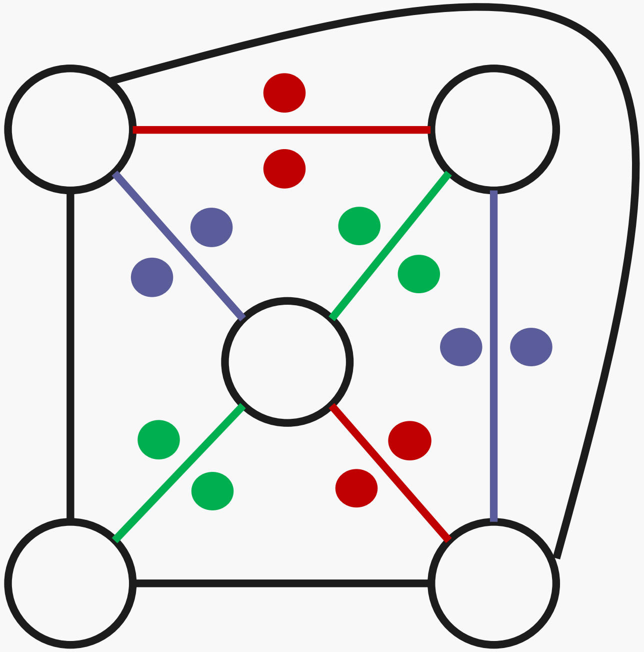
We can place a dot on both sides of each edge.

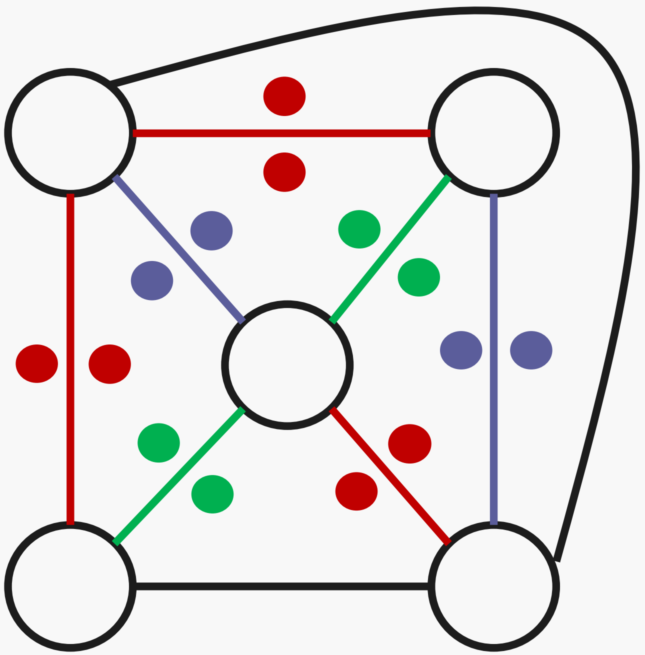


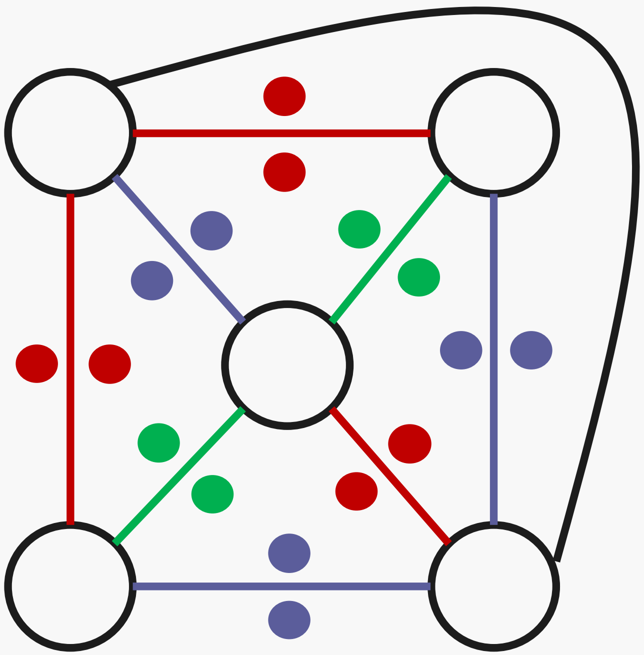


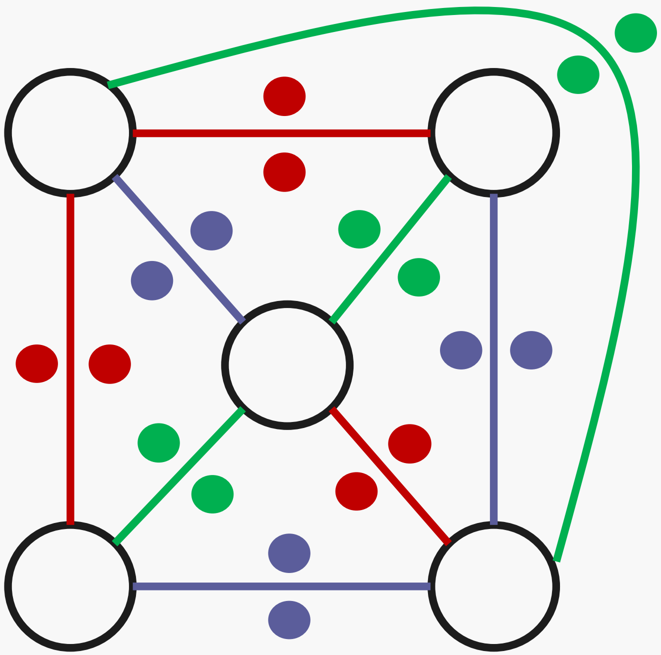


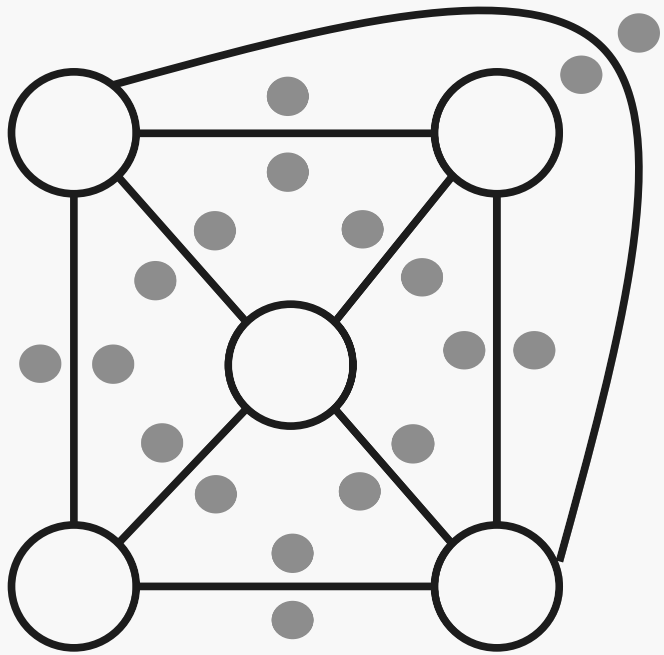


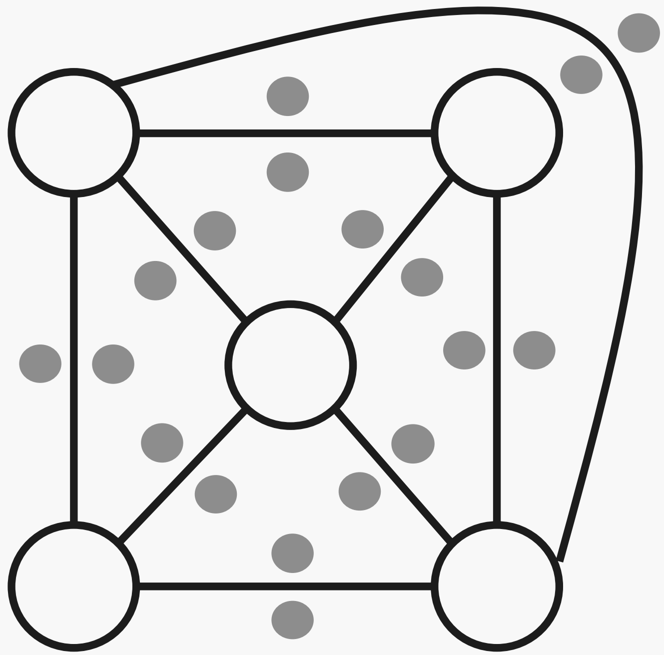




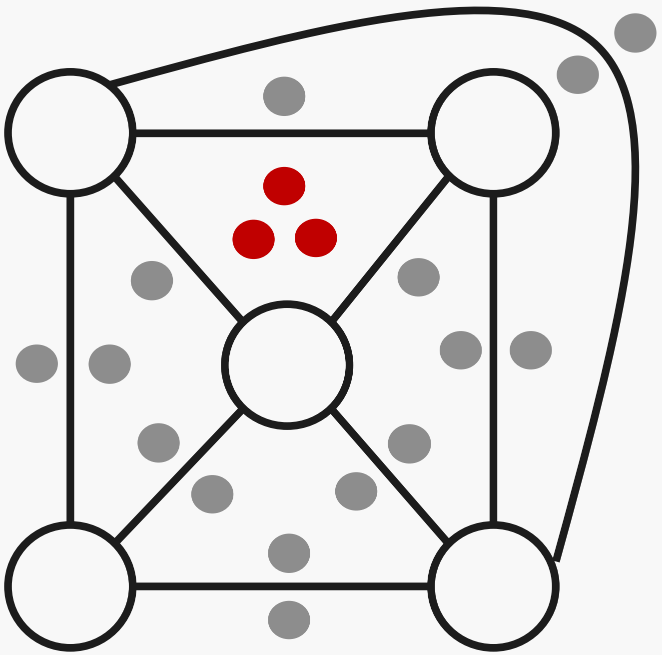


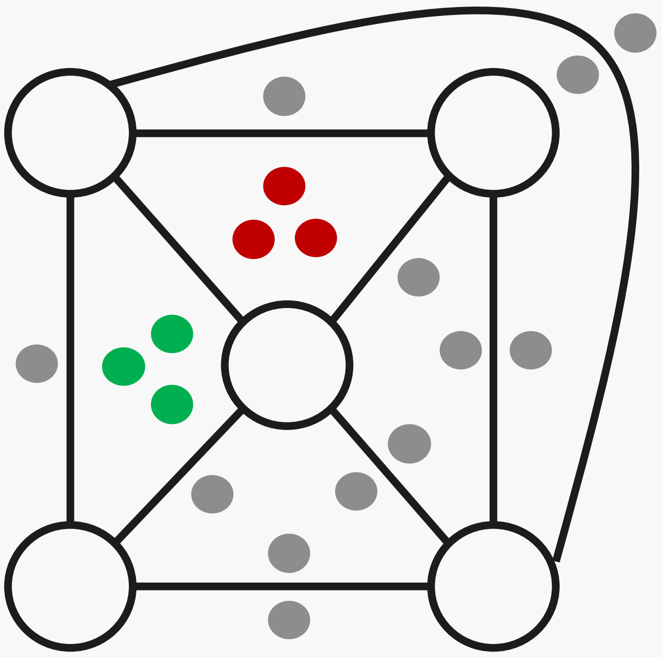


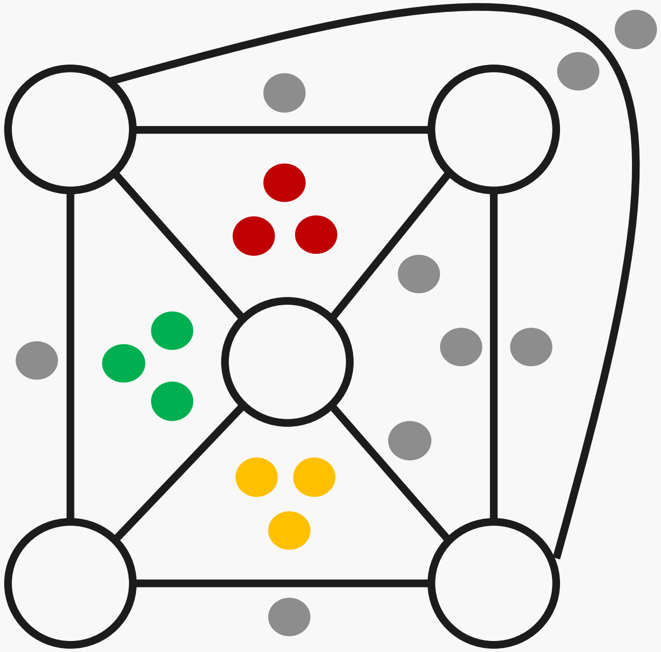


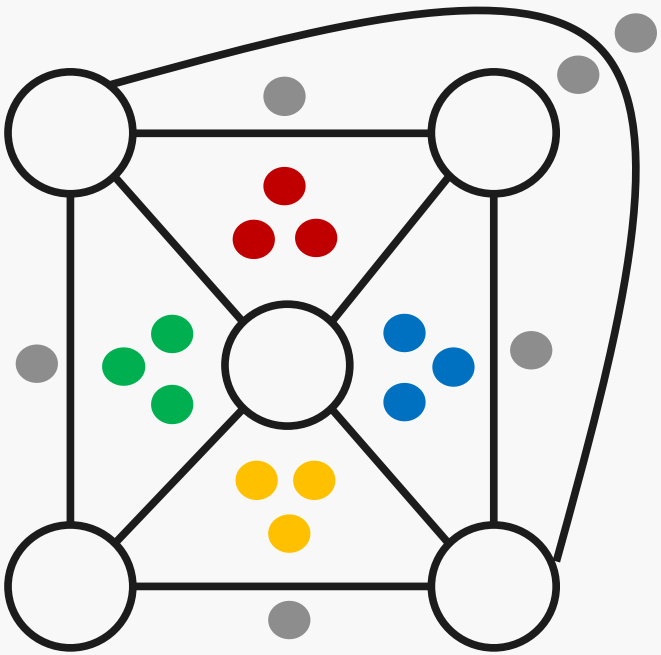


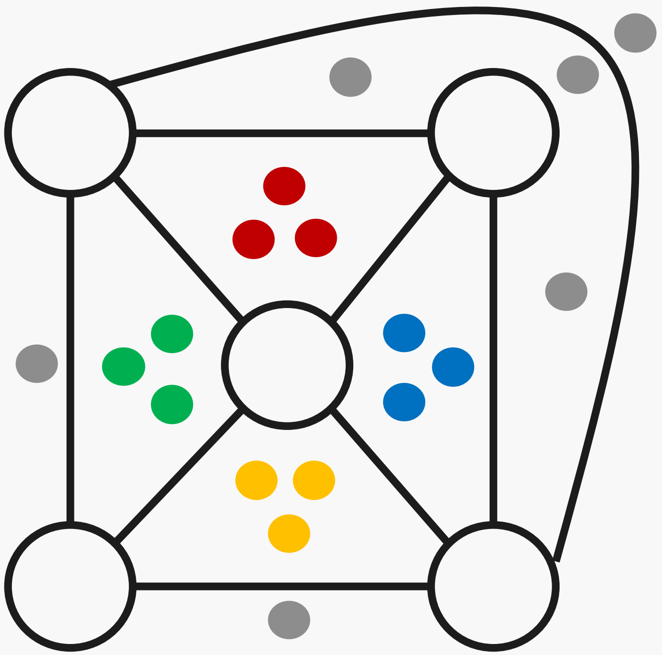
But notice that each face now has three dots.

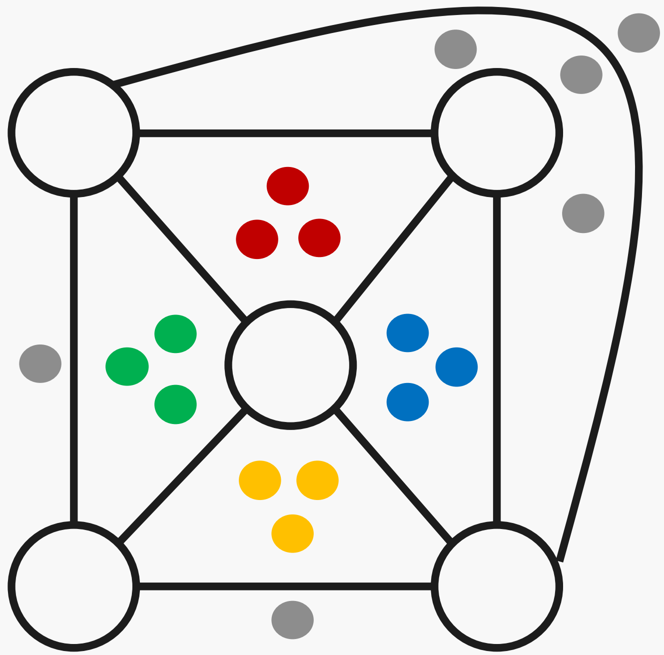


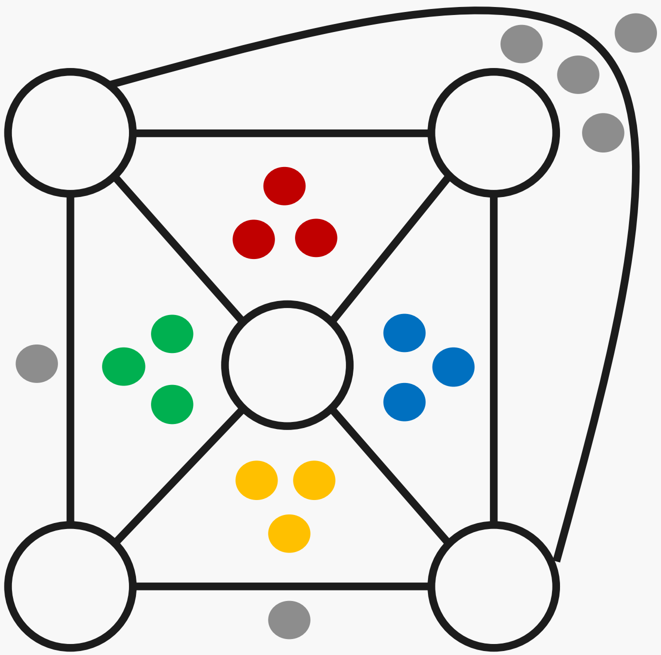


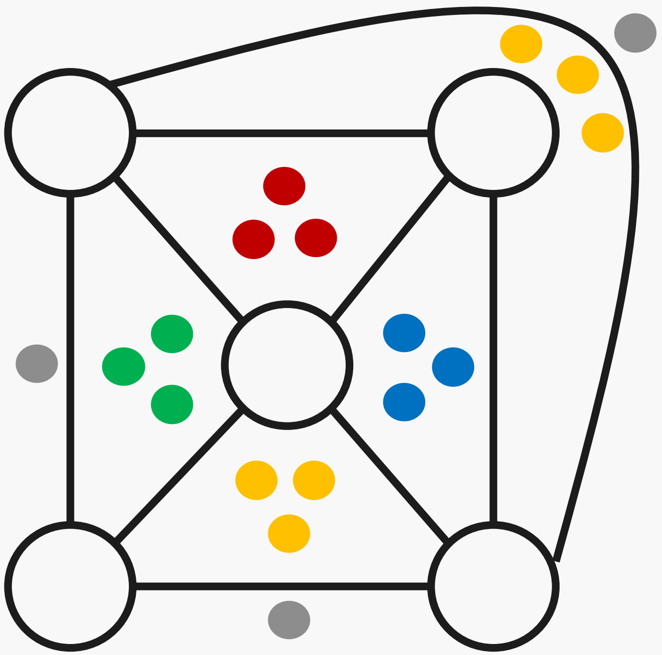


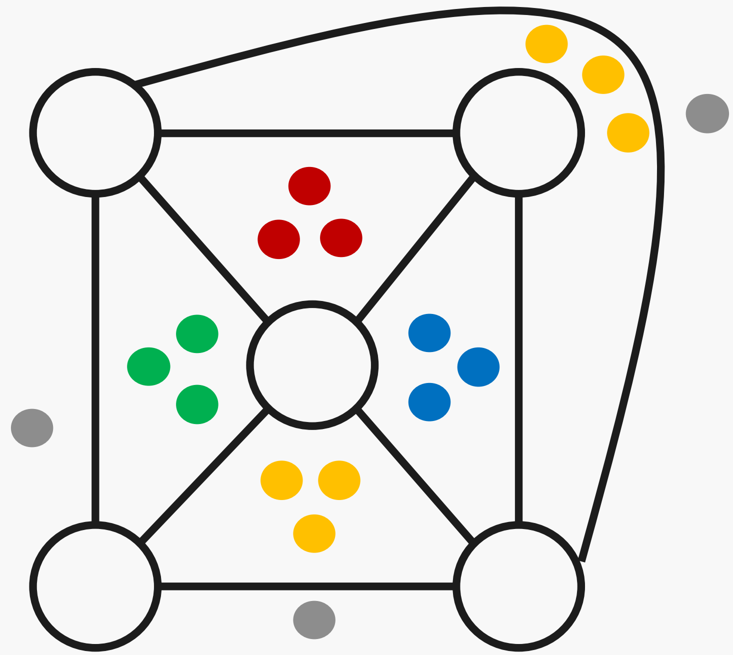


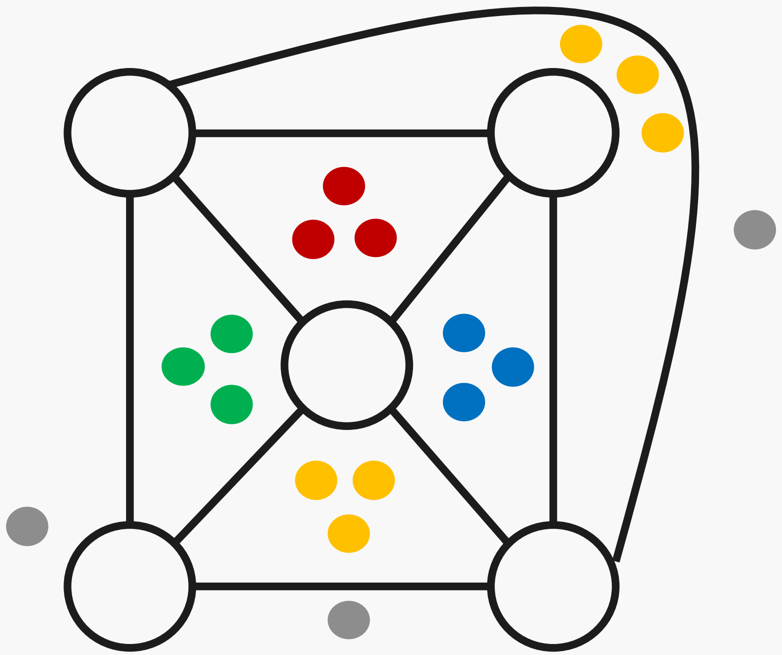


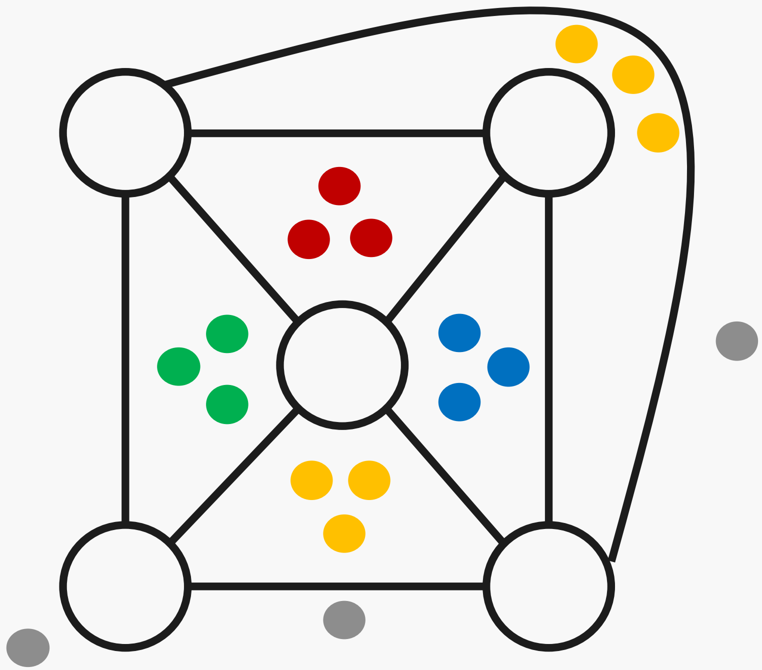


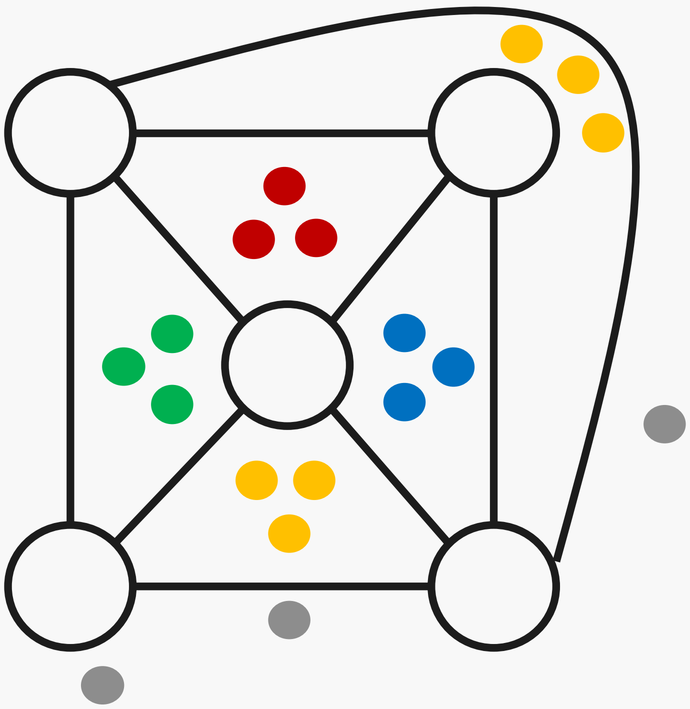


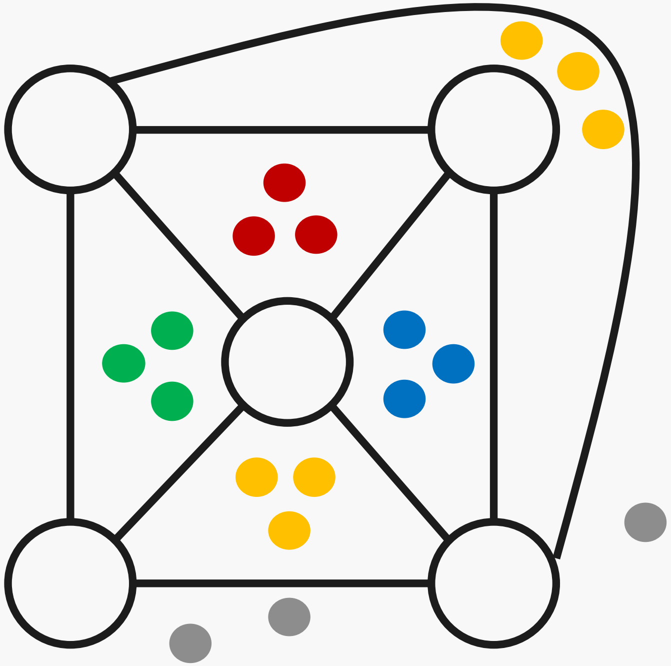


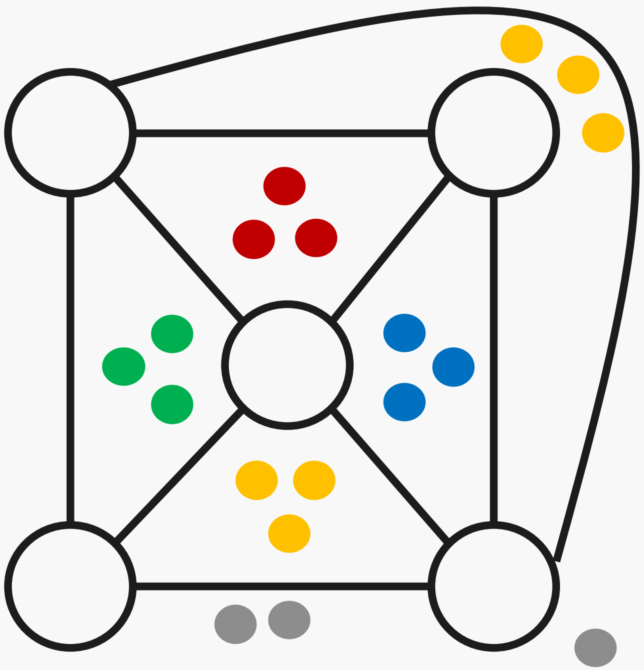


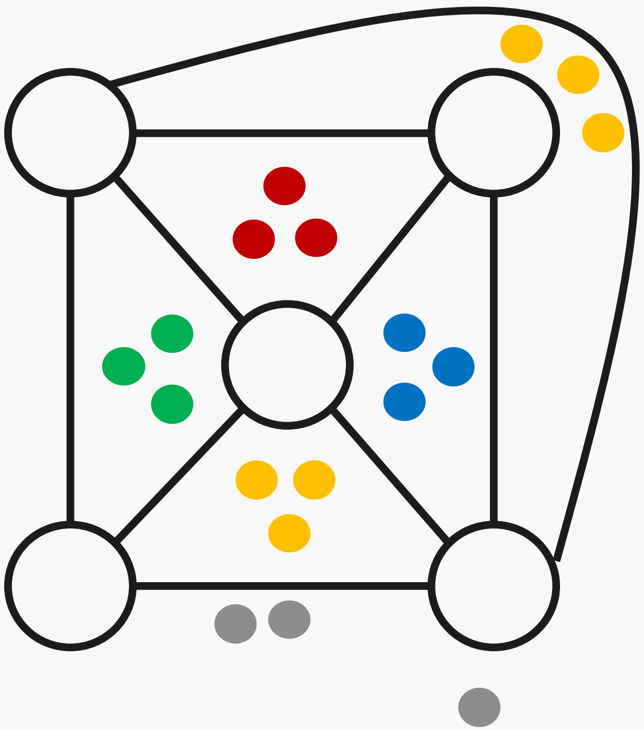


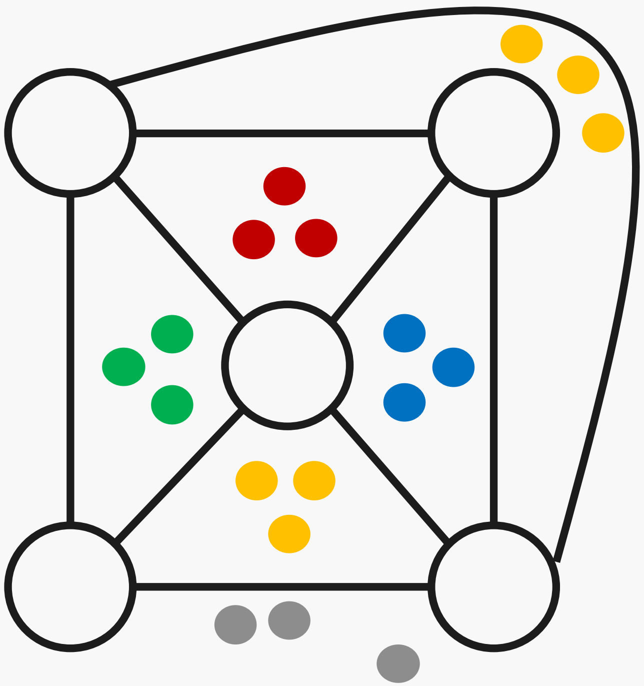


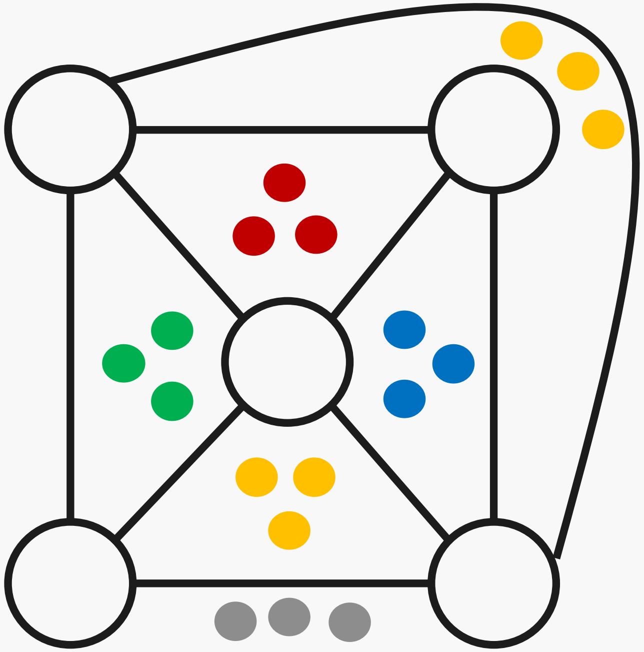


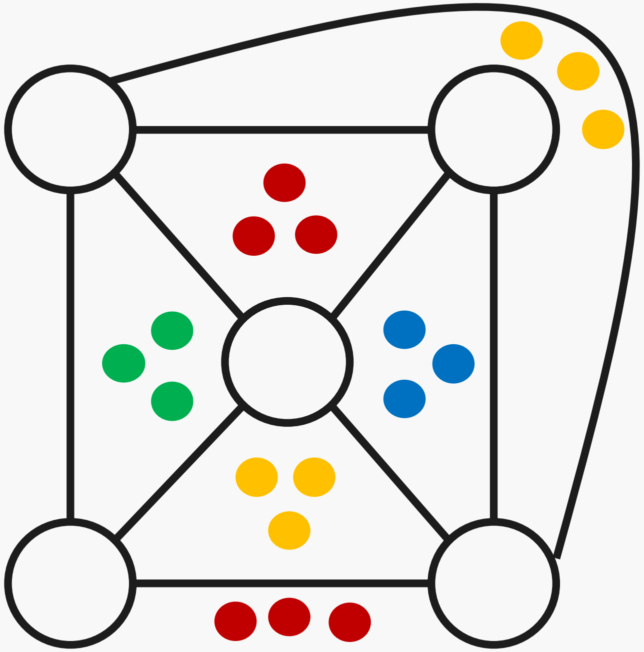


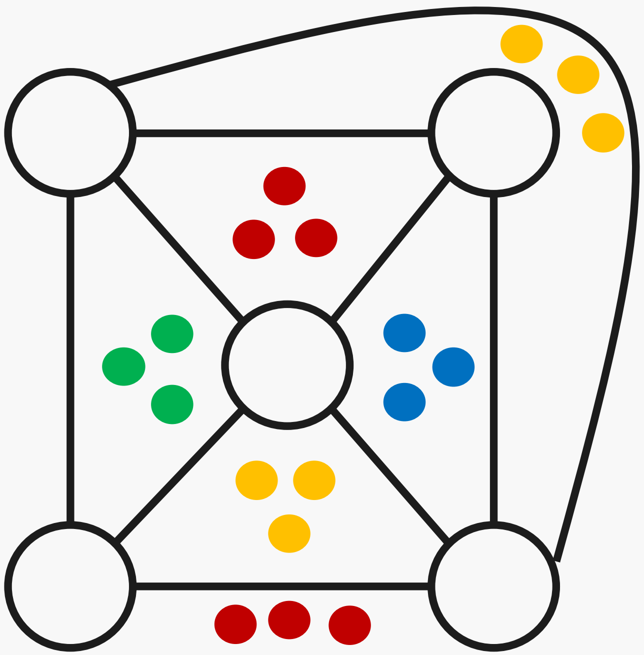




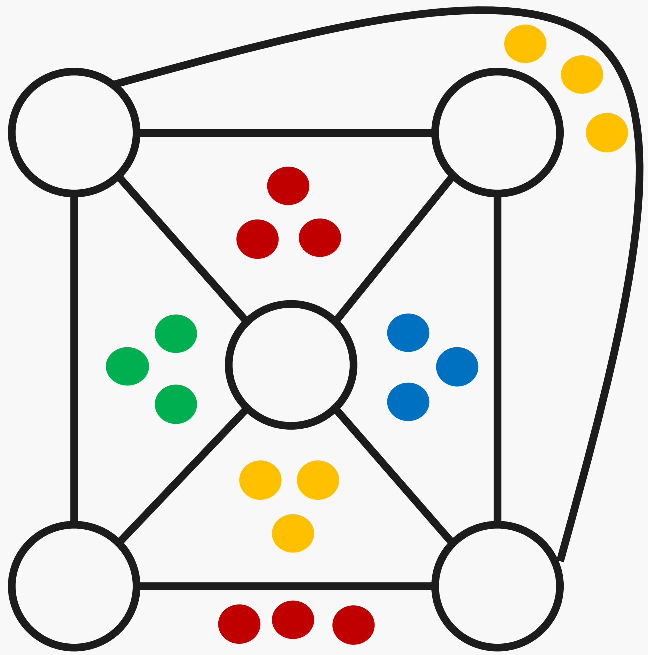




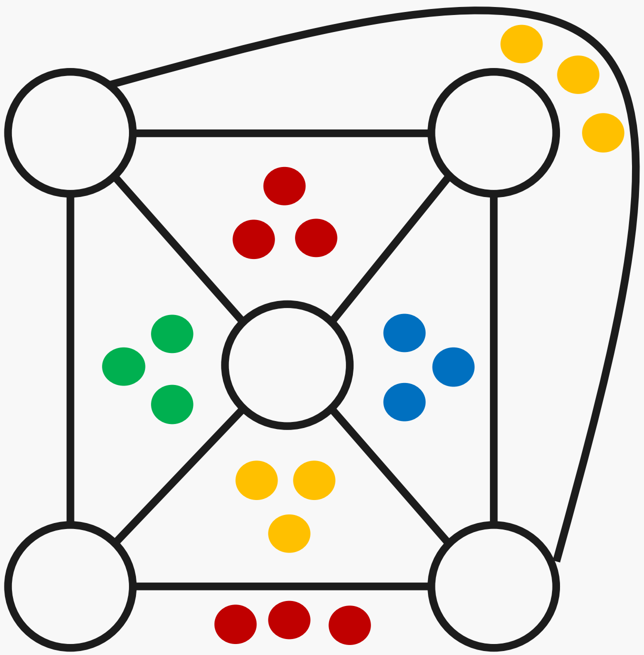




We put 2E dots, since we put 2 for each edge.



We put 2E dots, since we put 2 for each edge.
But this is also 3F, since there are 3 for each face.



We put 2E dots, since we put 2 for each edge.
But this is also 3F, since there are 3 for each face.
So 2E = 3F.

Maximal planar graphs

In a maximal planar graph, $2E = 3F$.

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$$3V - E = 6$$
$$\mathbf{E = 3V - 6.}$$

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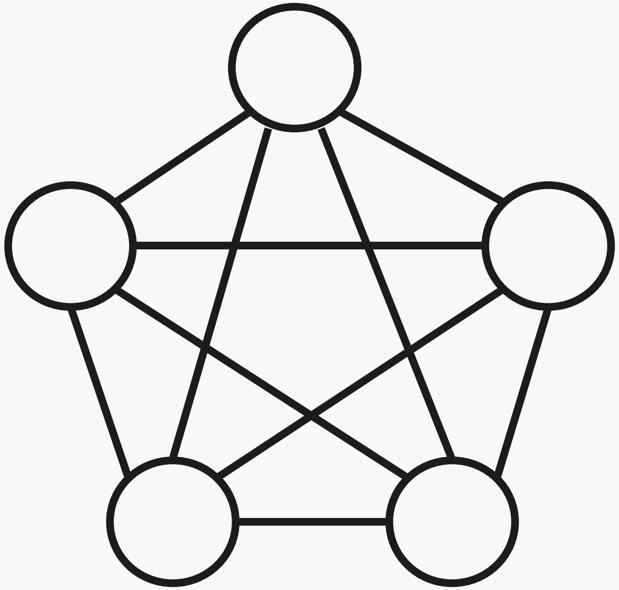
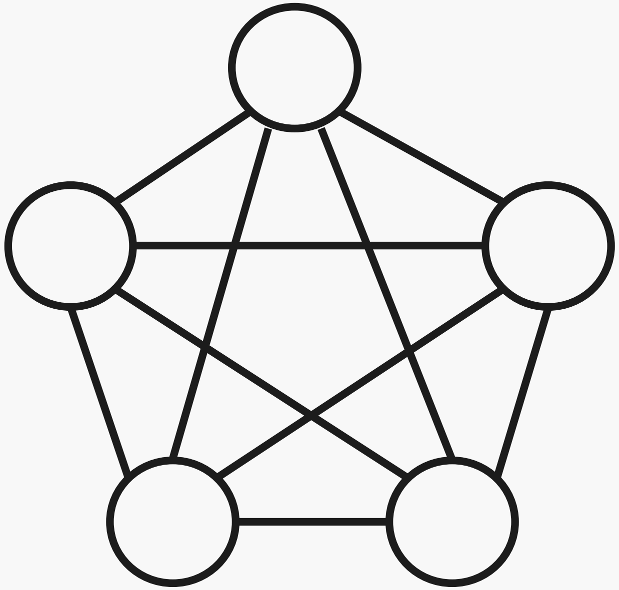
A maximal planar graph has the most number of edges.

So for **any planar graph,**

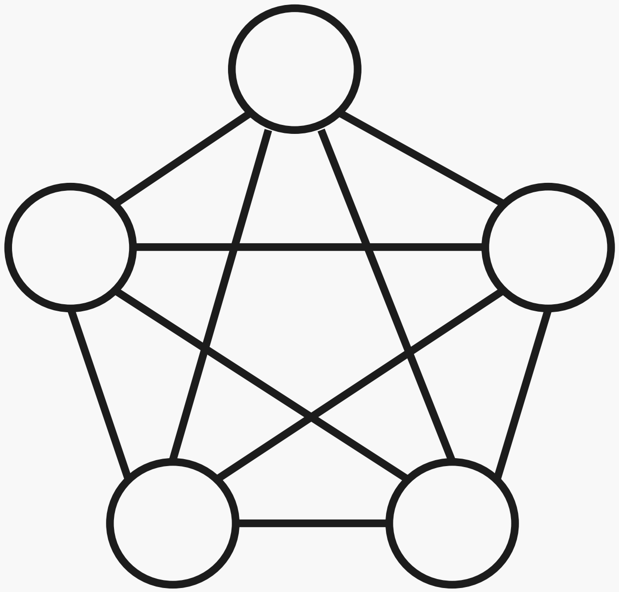
$$\mathbf{E \leq 3V - 6.}$$



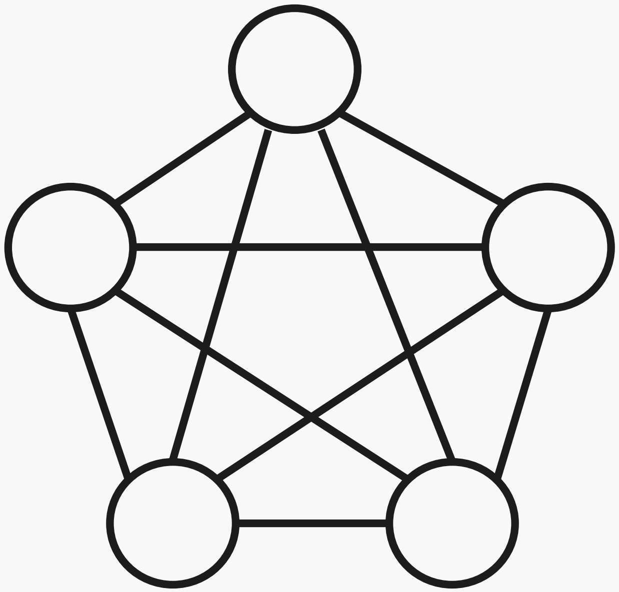
Five houses problem: is this graph planar?



It has 5 vertices and 10 edges.



It has 5 vertices and 10 edges.
If it were planar, $E \leq 3V - 6$.



It has 5 vertices and 10 edges.
If it were planar, $E \leq 3V - 6$.
But 10 is greater than $3(5) - 6 = 9$.

November 2016

- We wanted to study **planar graphs**.
- But the problem is, we already know a lot about planar graphs.
- For example:

November 2016

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- For example:
    - Euler's formula: $V - E + F = 2$.

November 2016

- We wanted to study **planar graphs**.
- But the problem is, we already know a lot about planar graphs.
- For example:
  - Euler's formula: $V - E + F = 2$.
  - Also, $E \leq 3V - 6$.

November 2016

- How do you find something that no one has found before?

November 2016

- How do you find something that no one has found before?
- Answering open problems is hard.

November 2016

- How do you find something that no one has found before?
- Answering open problems is hard.
- Idea: **Answer a problem no one has asked before.**

November 2016

- At Ateneo, I saw a paper by Ian Garces and Jose Rosario.
- It was called Computing the Metric Dimension of Truncated Wheels.

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Computing the Metric Dimension of Truncated Wheels

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Department of Mathematics
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The Philippines

Jose B. Rosario

Department of Mathematics

Metric dimension

To explain metric dimension, it’s easiest to start with the idea of GPS.

GPS

The way GPS works is that there are several satellites over the Earth.

GPS

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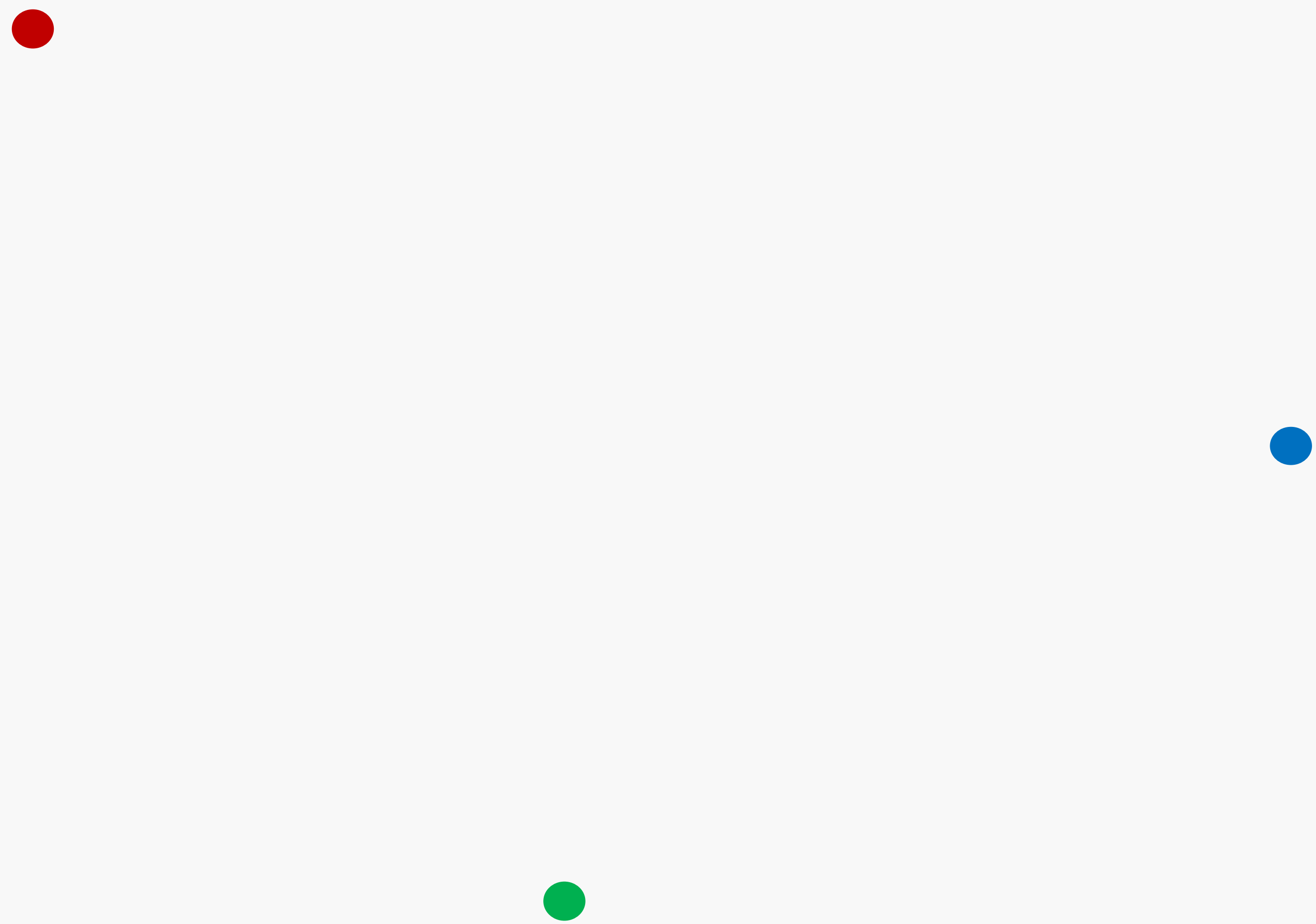
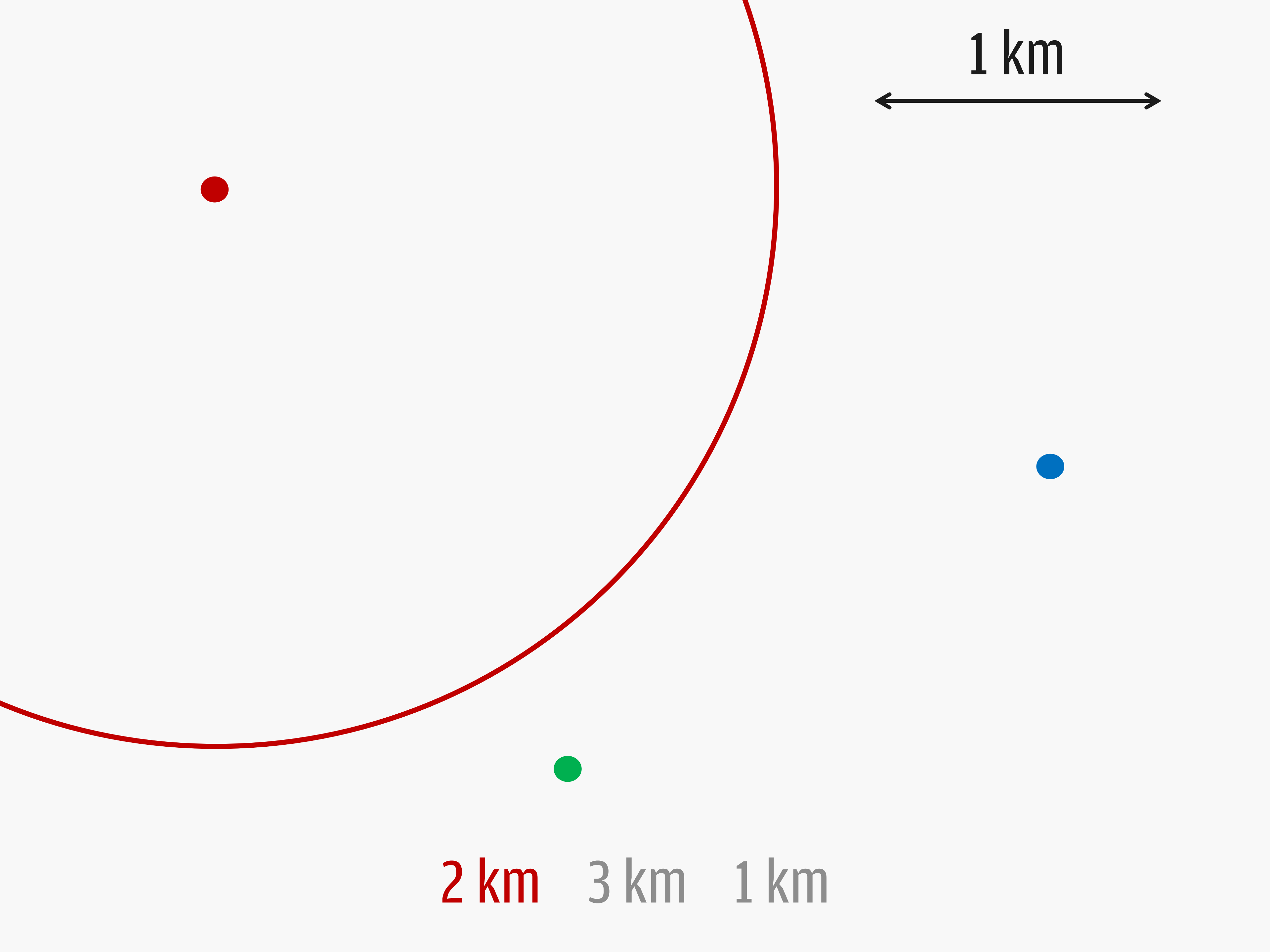
If you know your distance from each satellite, you can determine where you are on the Earth.




1 km
2 km
3 km
1 km


1 km
2 km
3 km
1 km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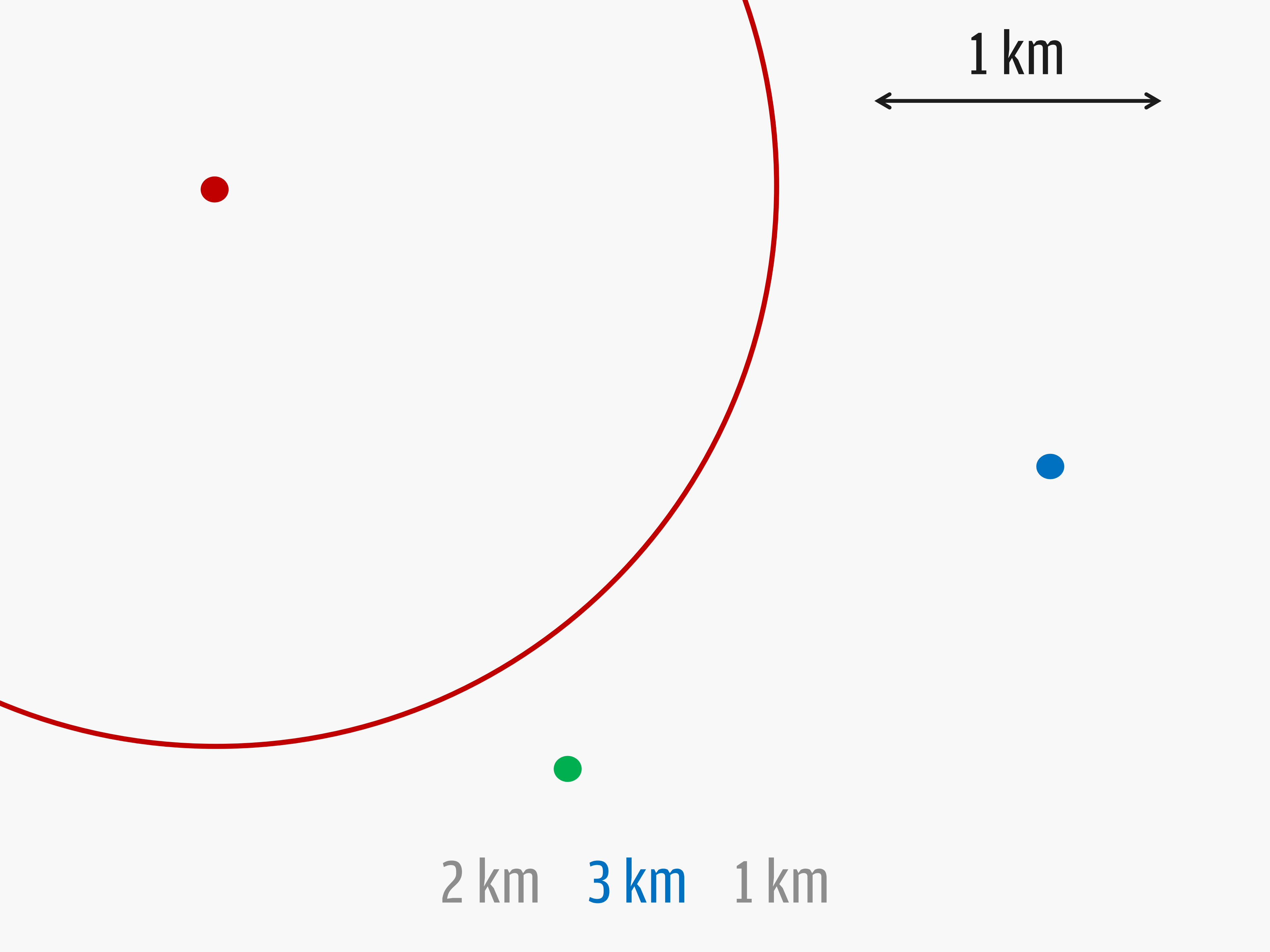
1 km
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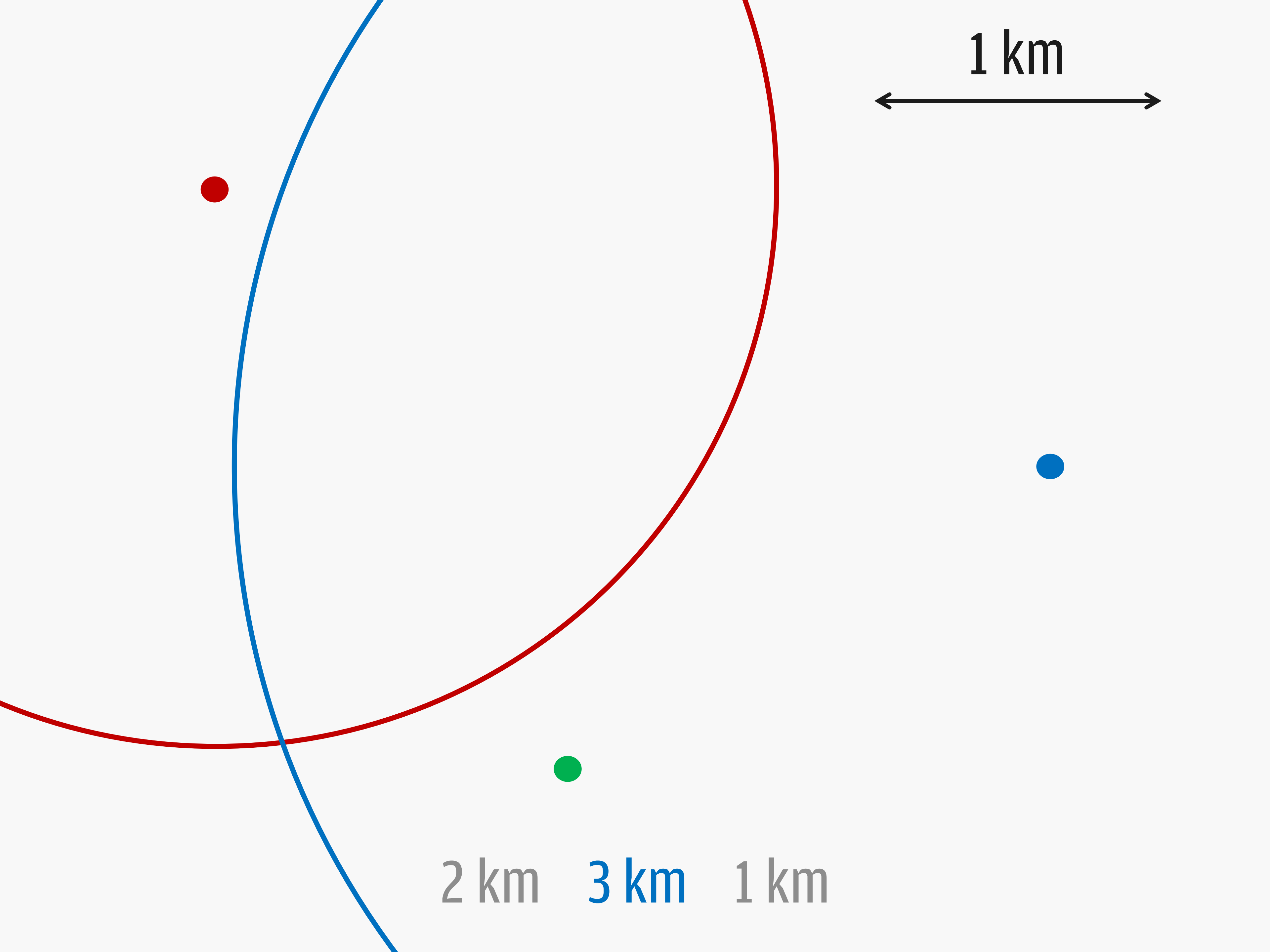
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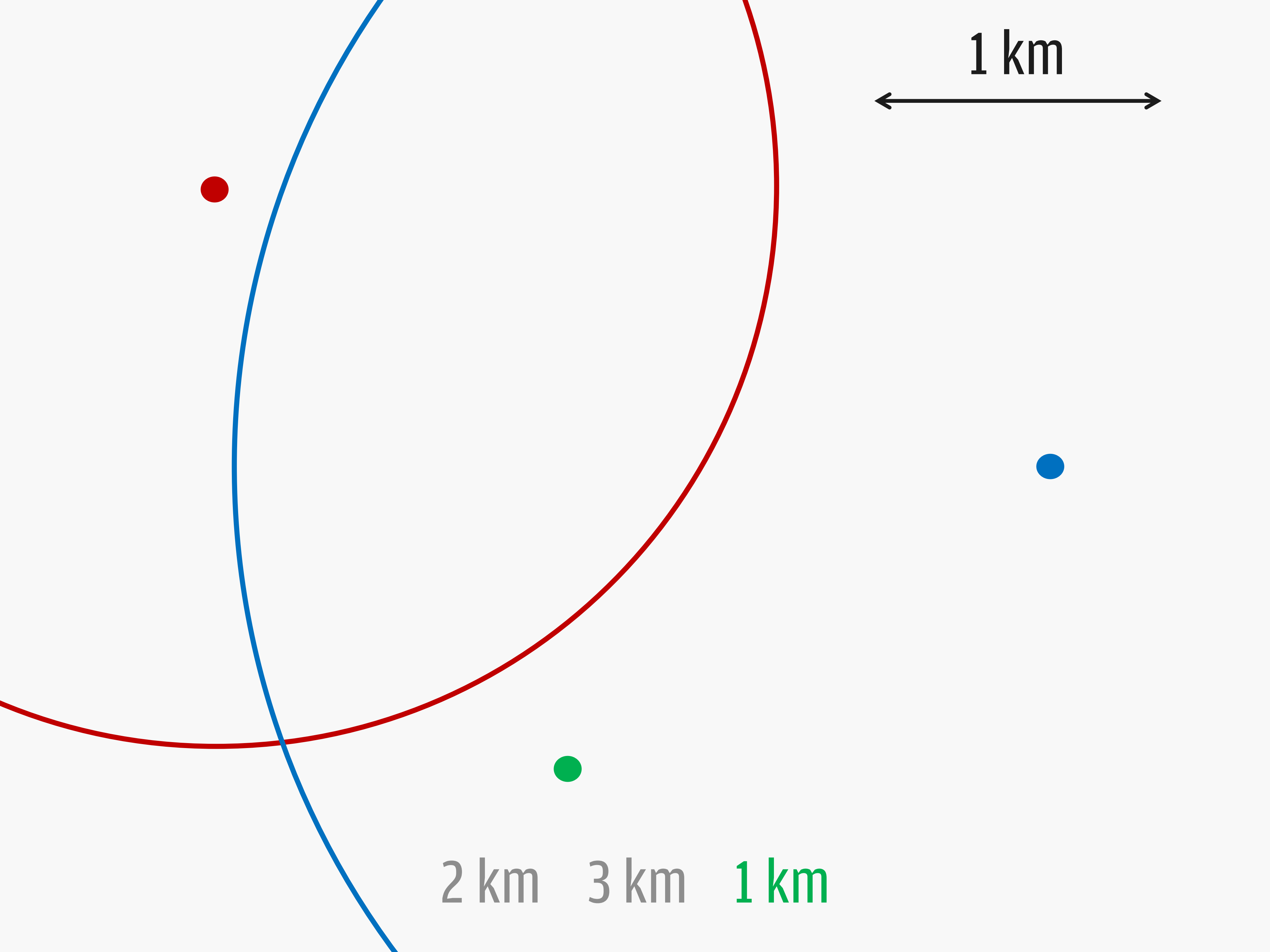
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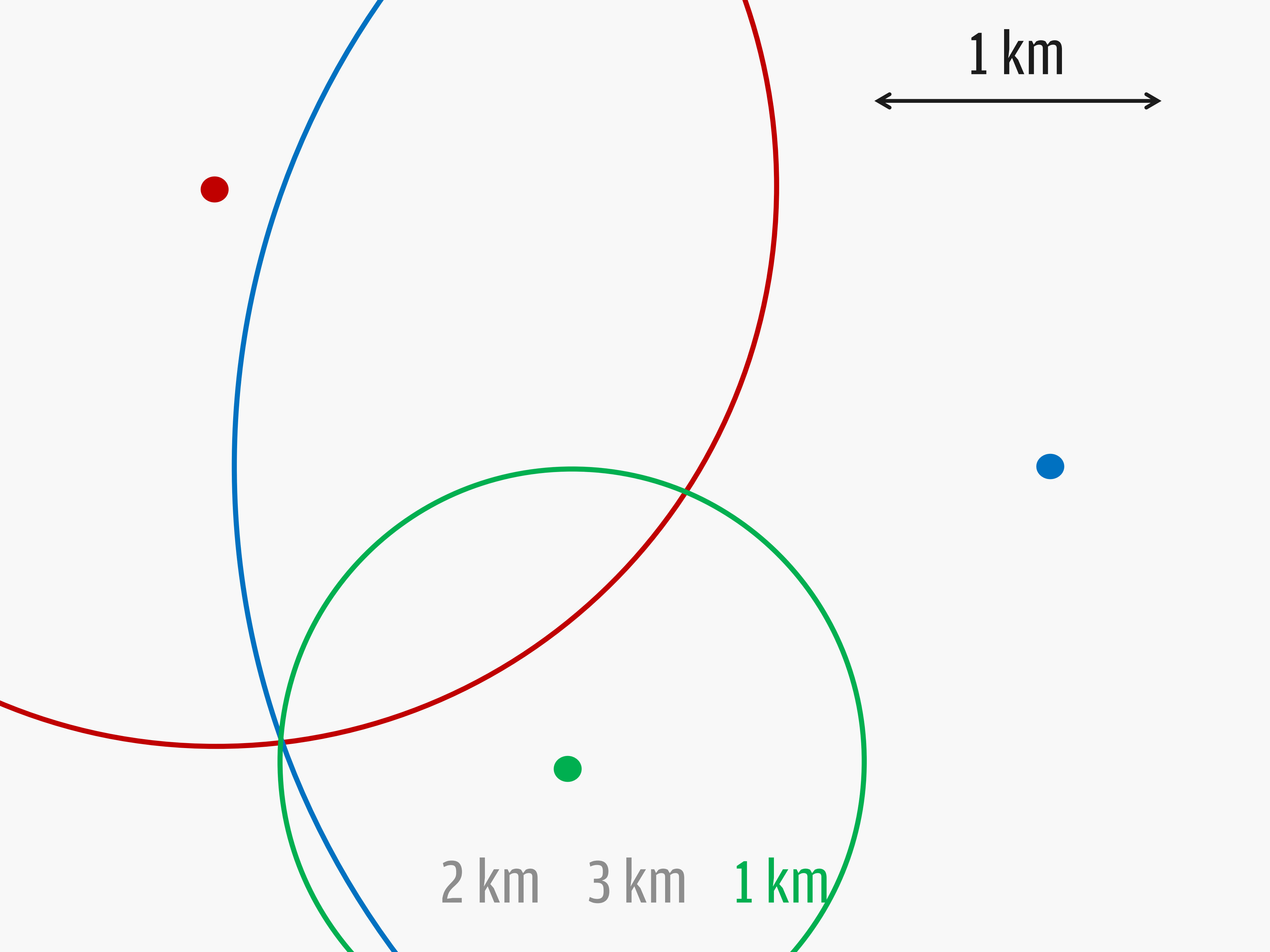
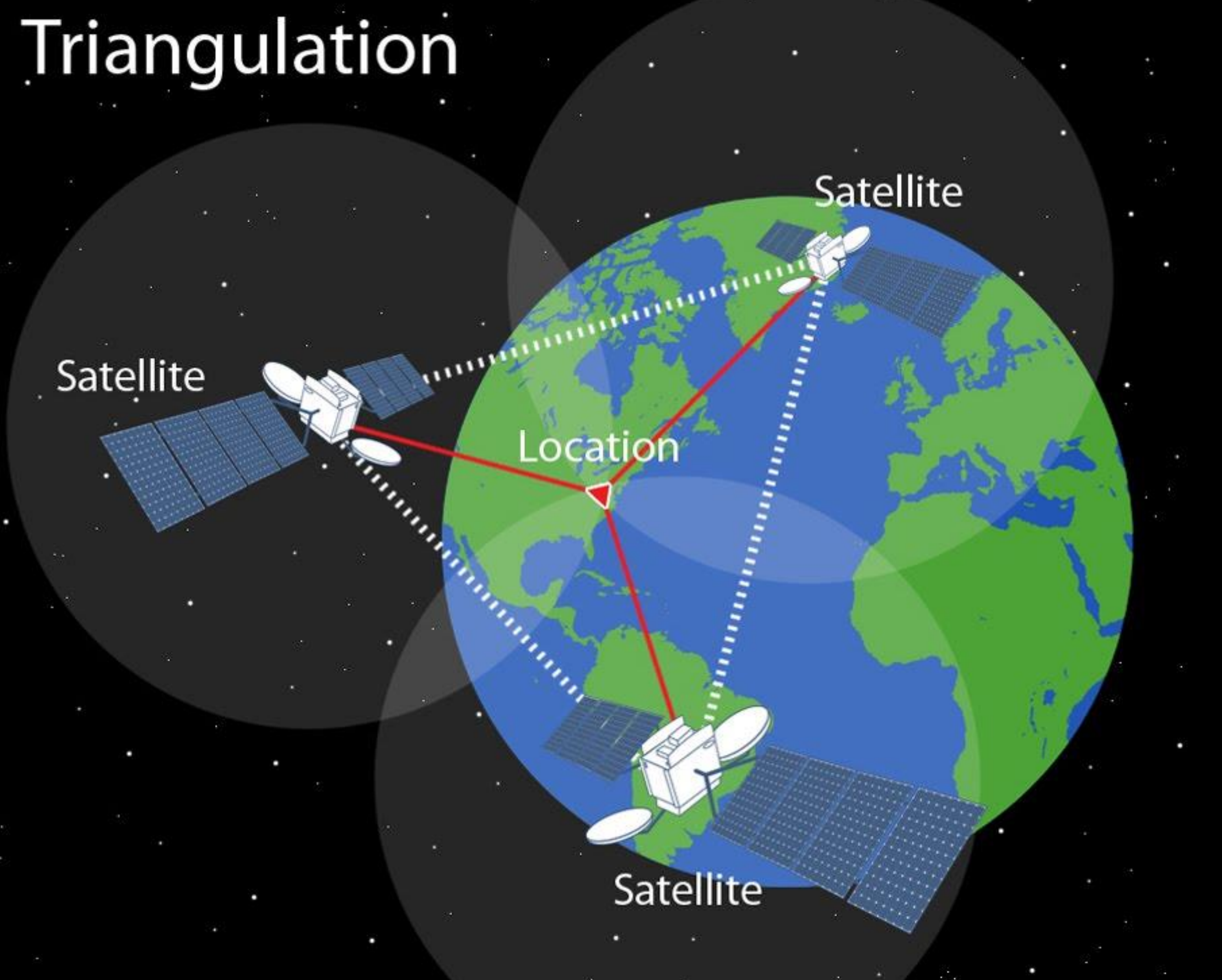


1 km
2 km
3 km
1 km

Triangulation



GPS

The way GPS works is that there are several satellites over the Earth.

If you know your distance from each satellite, you can determine where you are on the Earth.

With GPS, two satellites aren’t enough. You need three.

Metric dimension

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Metric dimension is **GPS on graphs.**

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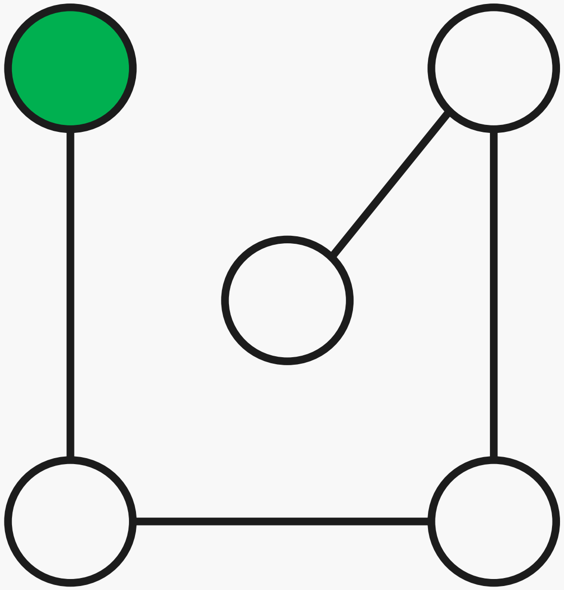
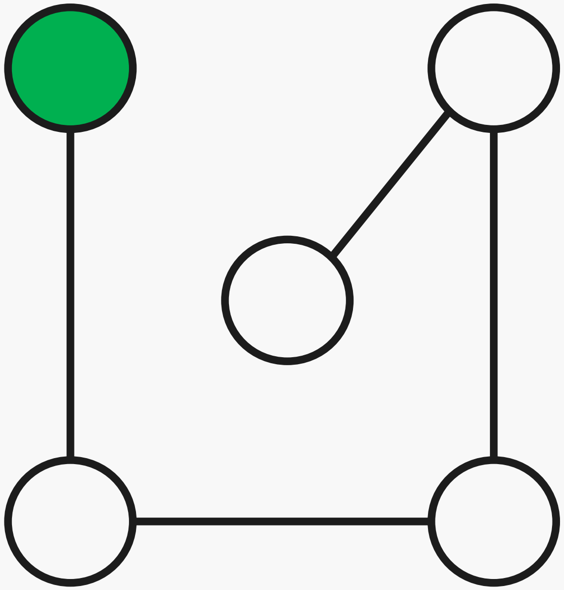
Metric dimension is **GPS on graphs.**

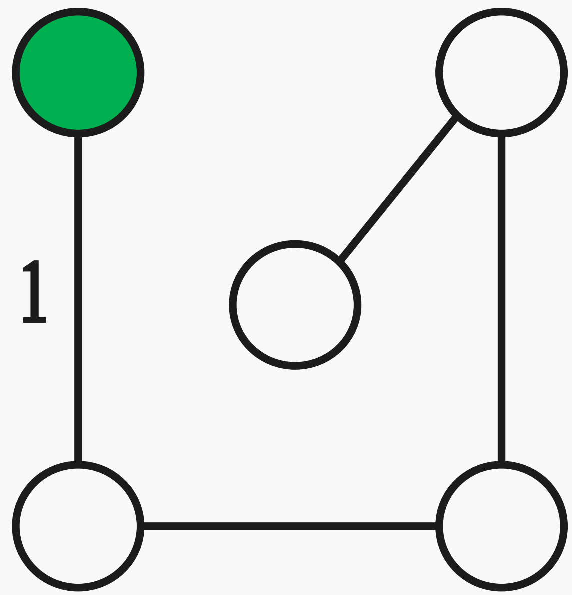
You are on a vertex, and there are satellites. If you know your distance to the satellites, you can determine where you are.



Let’s pick this vertex as a satellite.



Let's pick this vertex as a satellite.
If you are distance 3 to it, where are you?



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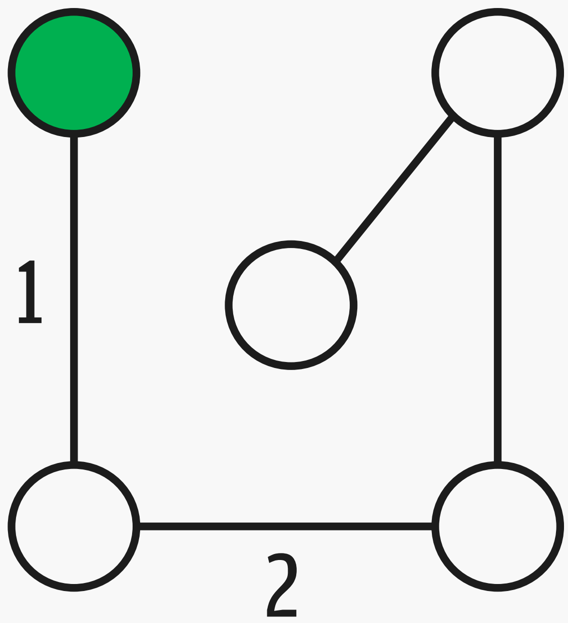

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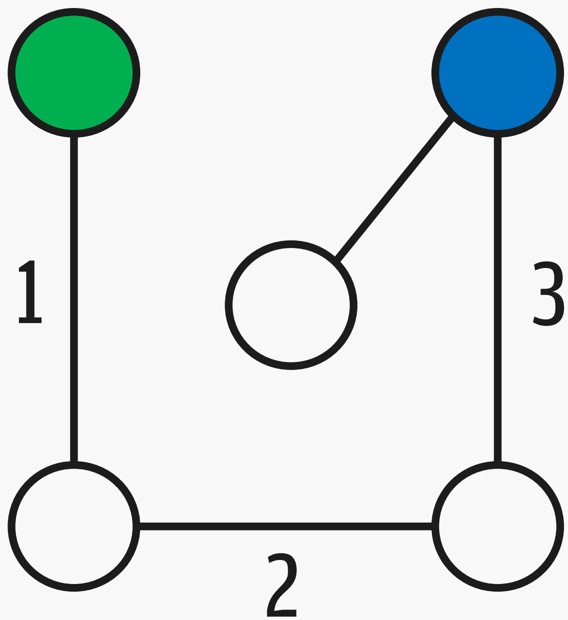

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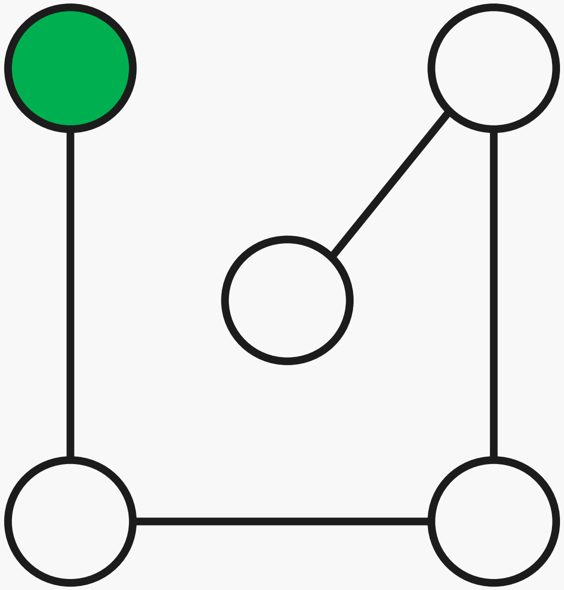
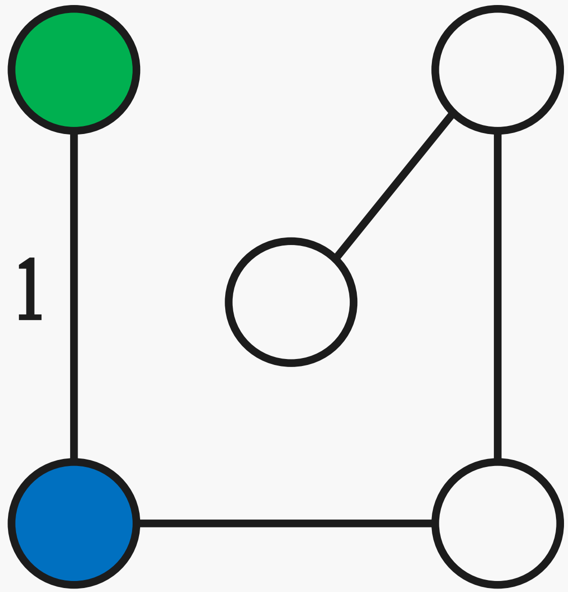


Let’s pick this vertex as a satellite.
If you are distance 1 to it, where are you?



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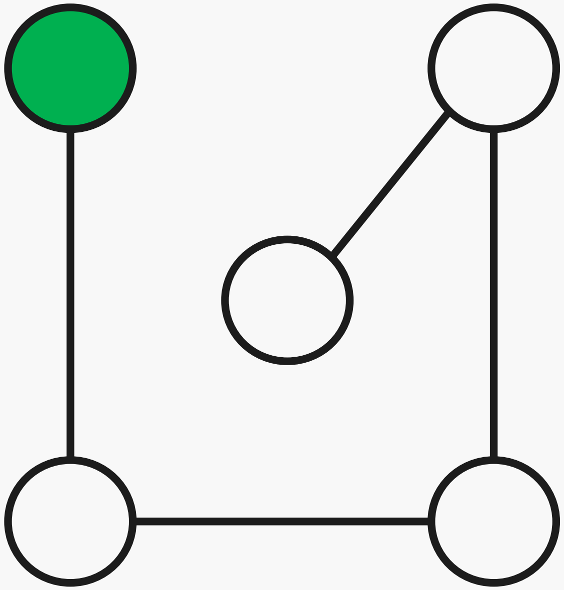
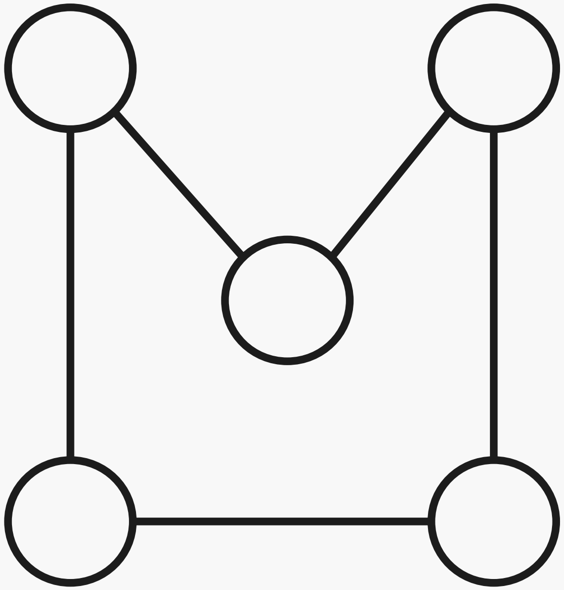
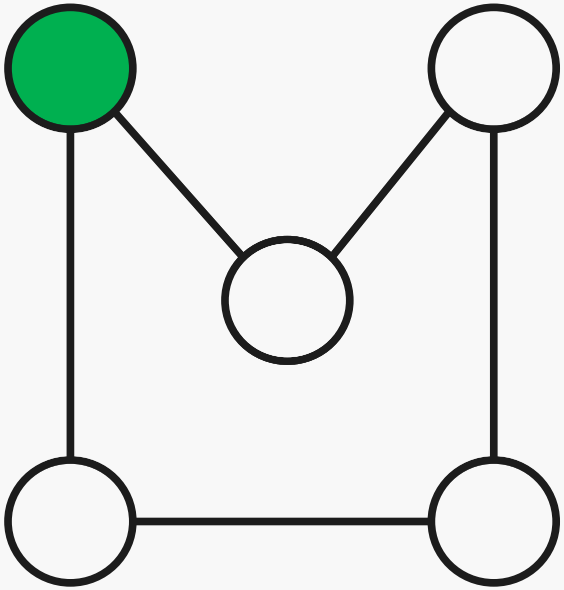


In this graph, one satellite is enough.



How many satellites do we need for this graph?



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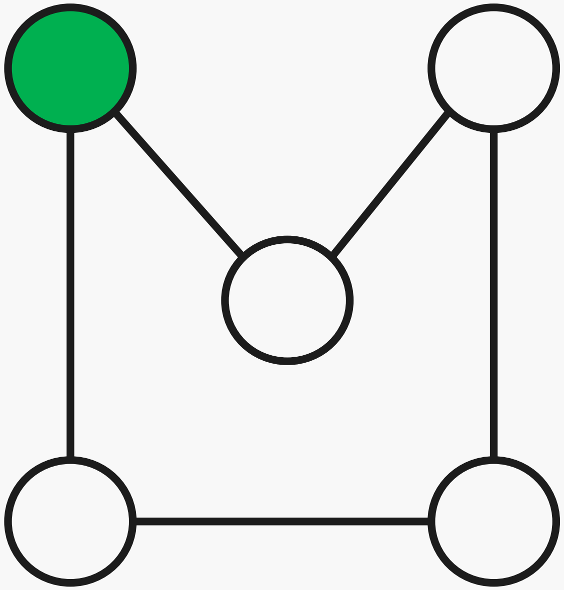
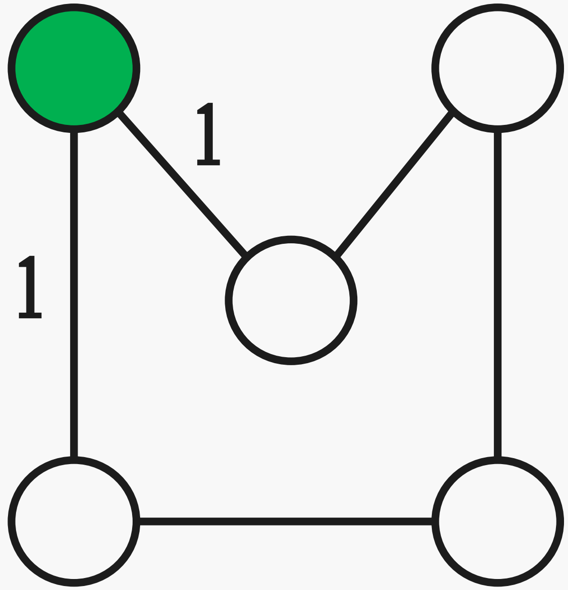


Let’s pick this vertex as a satellite.
If you are distance 2 to it, where are you?



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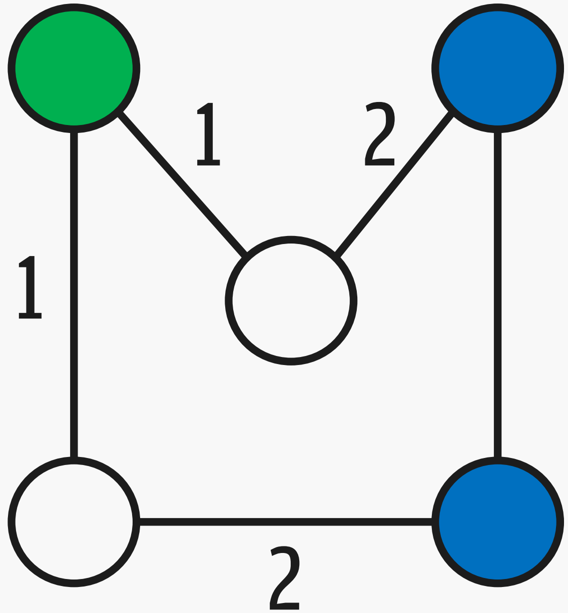

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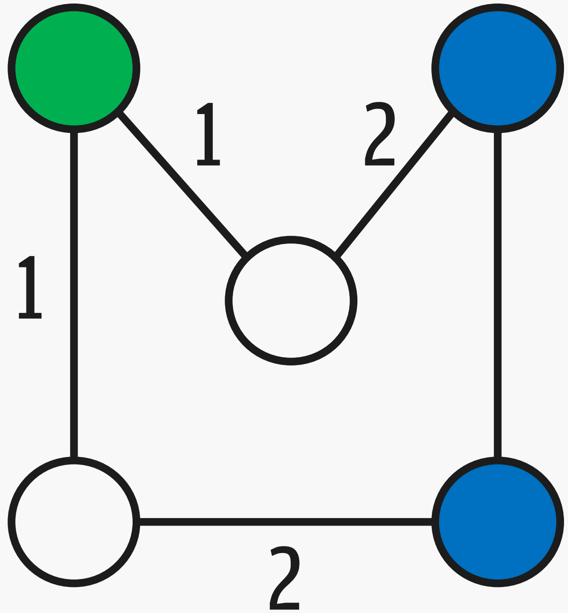
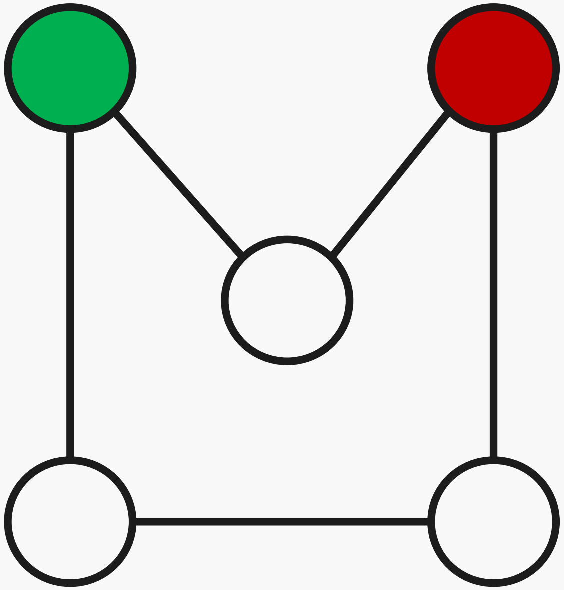
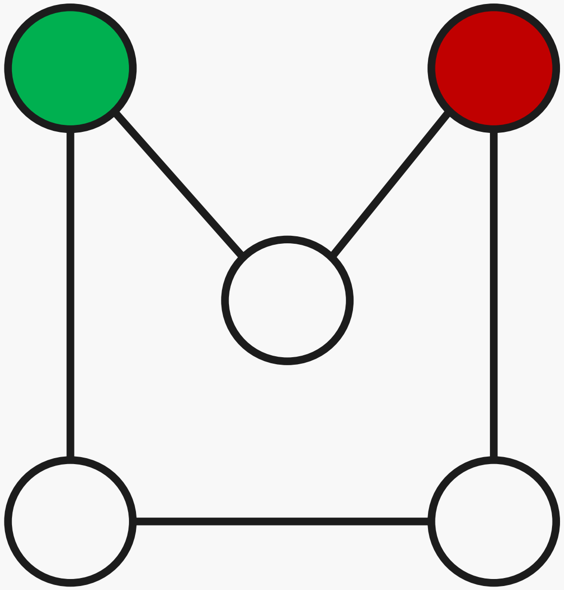


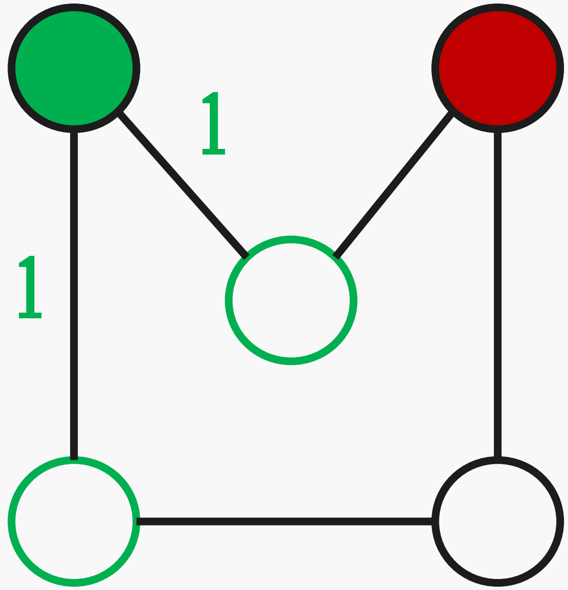
Let's pick this vertex as a satellite.
If you are distance 2 to it, where are you?
We don't know. There's not enough information.



Let’s pick two vertices as satellites.



Let’s pick two vertices as satellites.
You are at distances 1 and 2.



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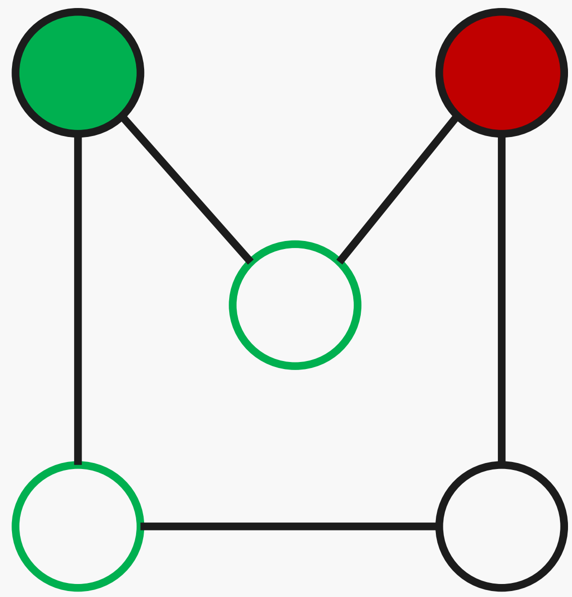
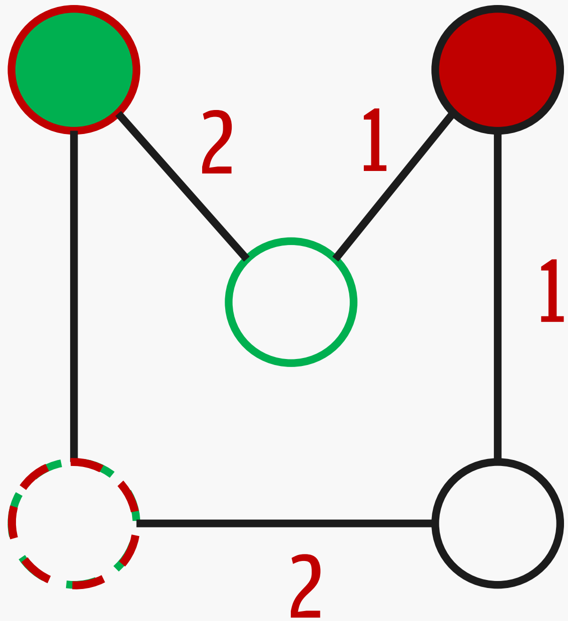


Let’s pick two vertices as satellites.
You are at distances 1 and 2.



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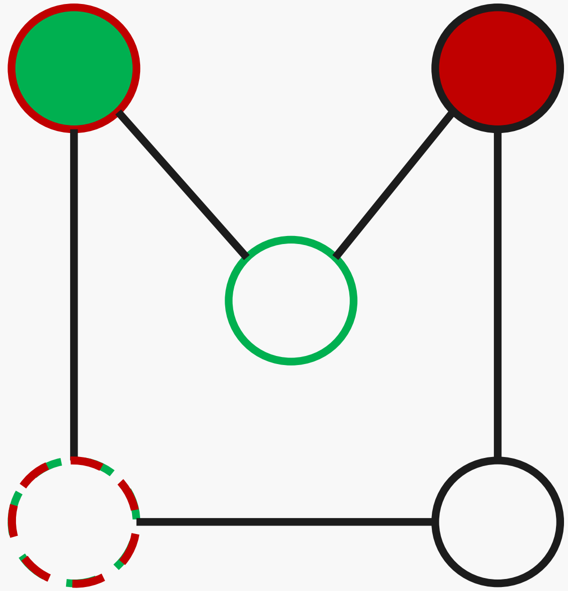

Let’s pick two vertices as satellites.
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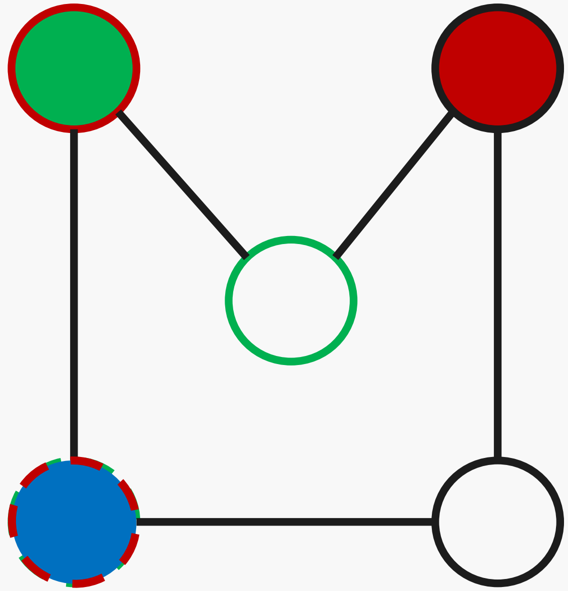
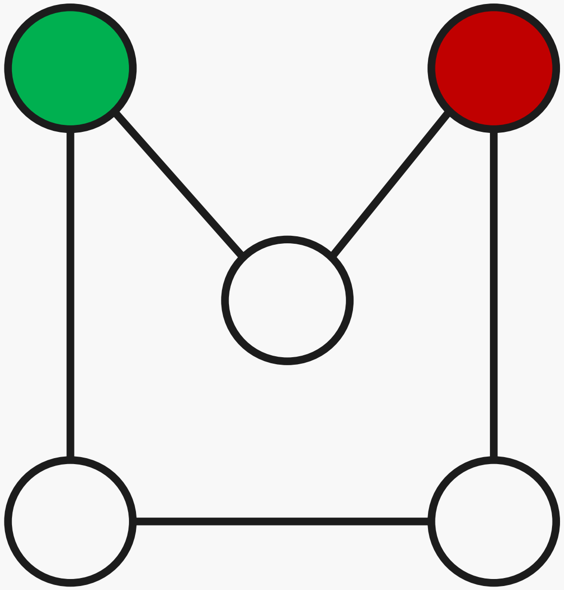


Let's pick two vertices as satellites.
You are at distances 1 and 2.
There's only one vertex with those distances!

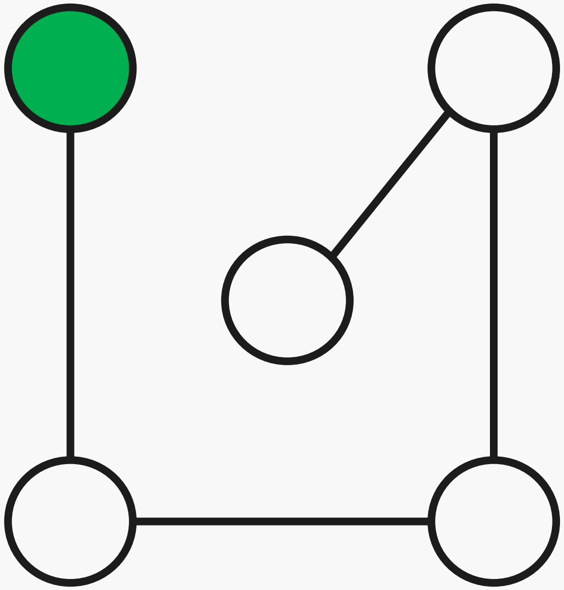


In this graph, you need two satellites.
One satellite isn't enough.

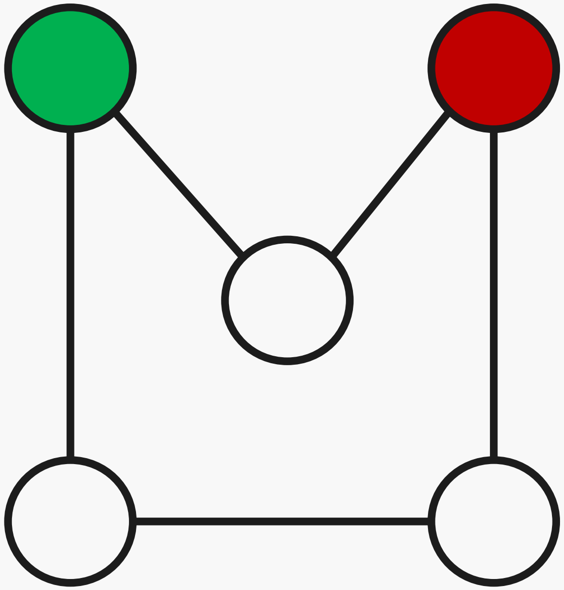
Metric dimension

Satellites are expensive, so you want to use a small number of satellites.

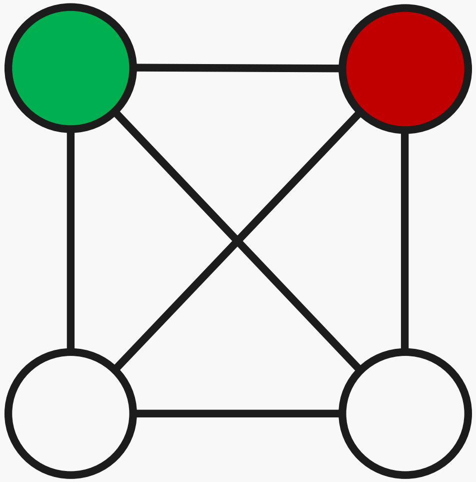
The **metric dimension** is the minimum number of satellites that you need.



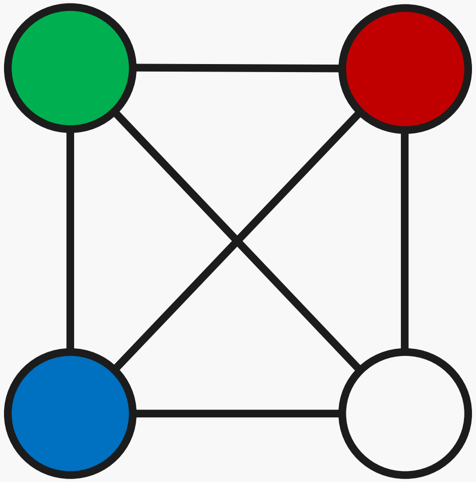
In this graph, one satellite is enough.
This graph has a metric dimension of 1.



In this graph, you need two satellites.
This graph has a metric dimension of 2.



All the vertices are distance 1 from each other.
So two satellites isn't enough.



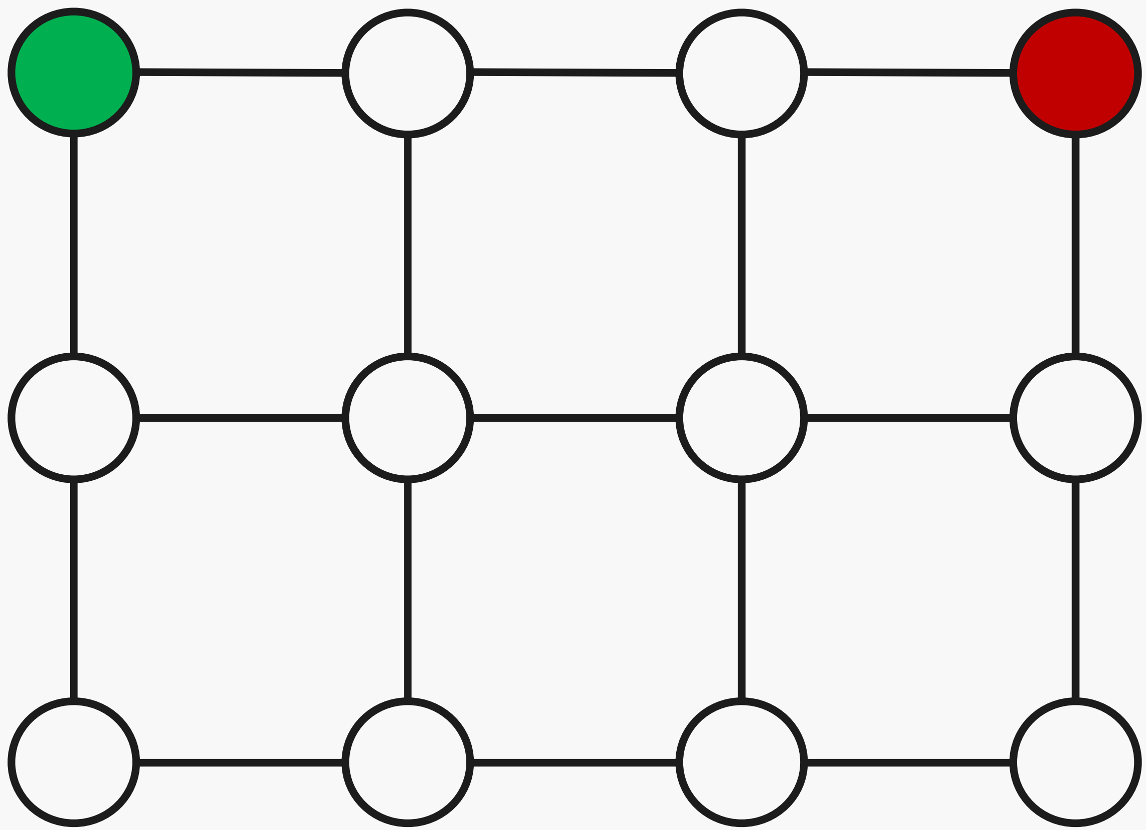
This graph needs three satellites.
It has a metric dimension of 3.

November 2016

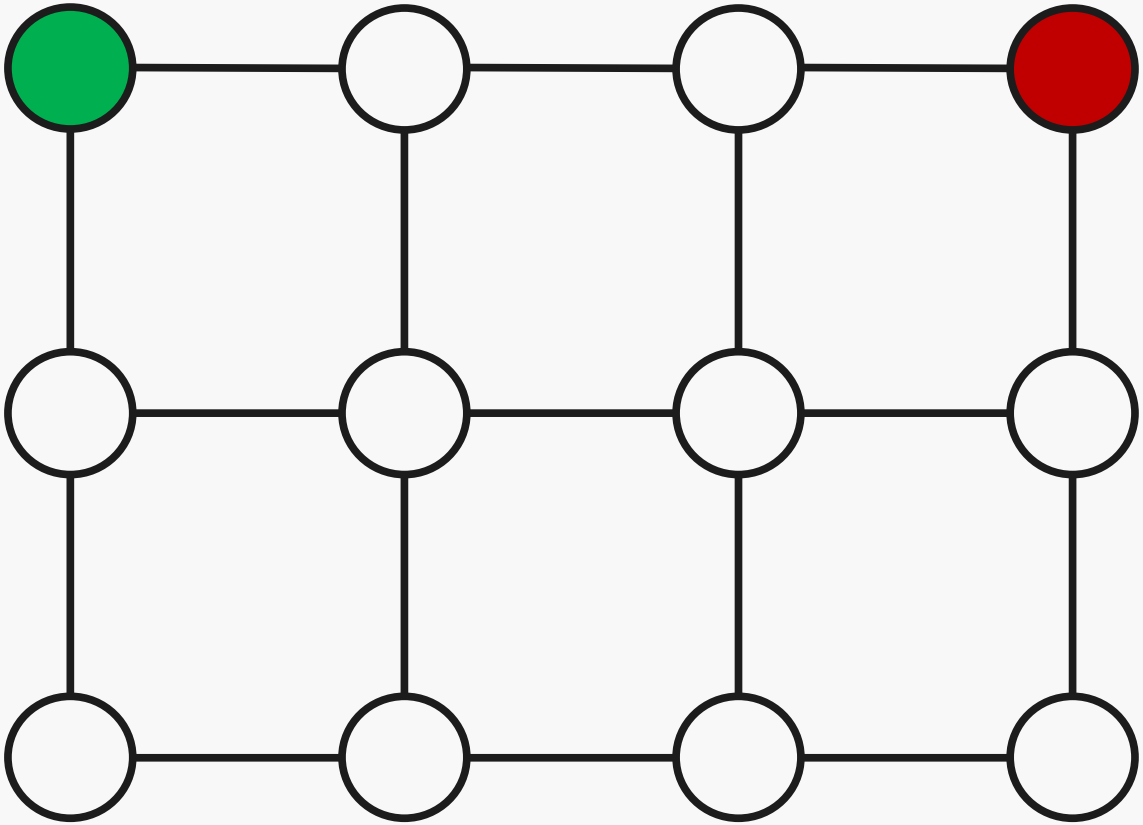
- At Ateneo, I saw a paper by Ian Garces and Jose Rosario.
- It was called Computing the Metric Dimension of Truncated Wheels.

November 2016

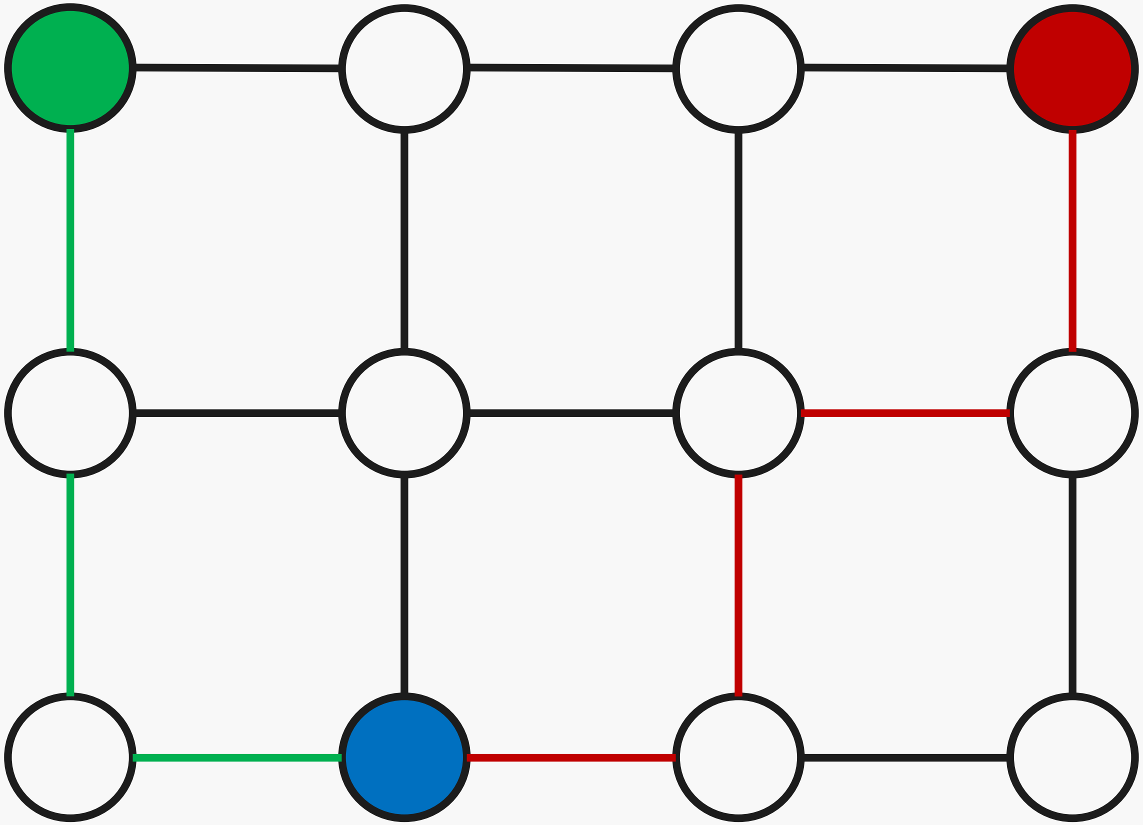
- At Ateneo, I saw a paper by Ian Garces and Jose Rosario.
- It was called Computing the Metric Dimension of Truncated Wheels.
- Most papers about metric dimension calculate it for certain kinds of graphs.



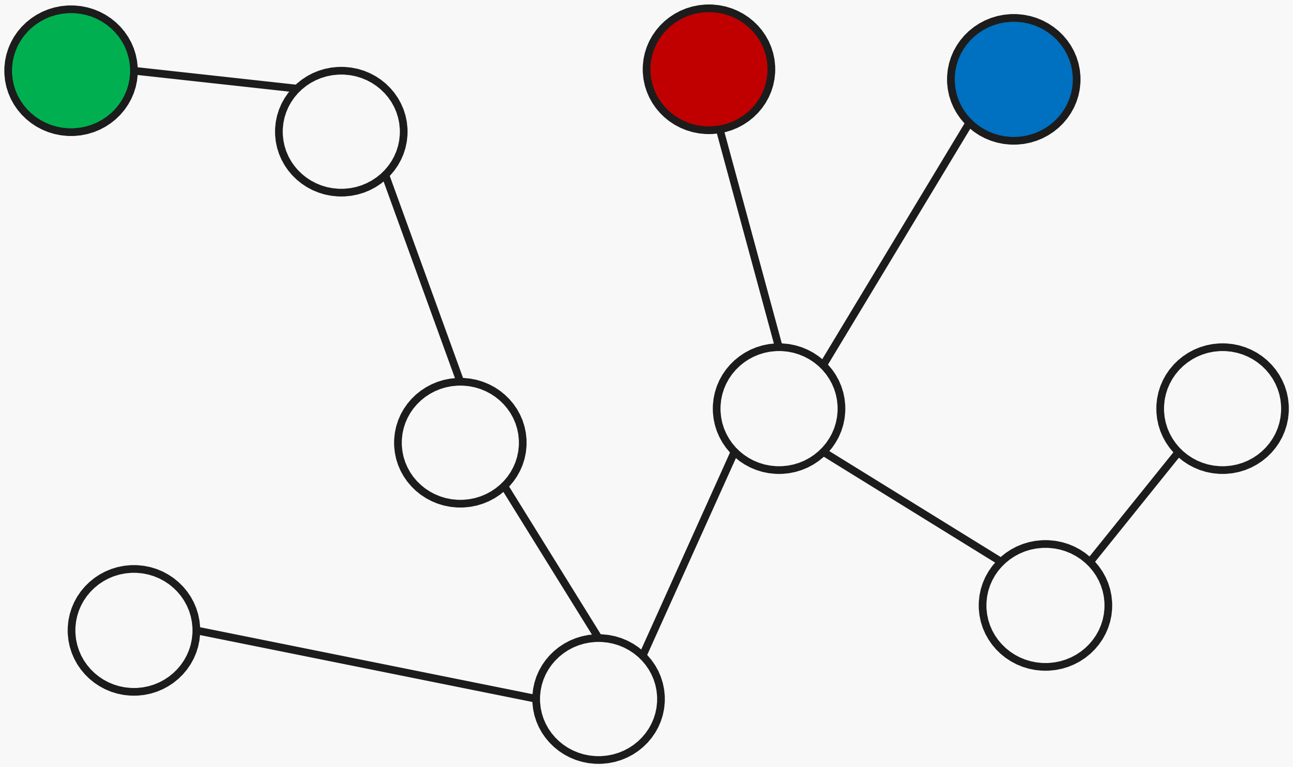
This is a 4×2 grid graph.
In any m×n grid graph, the metric dimension is 2.



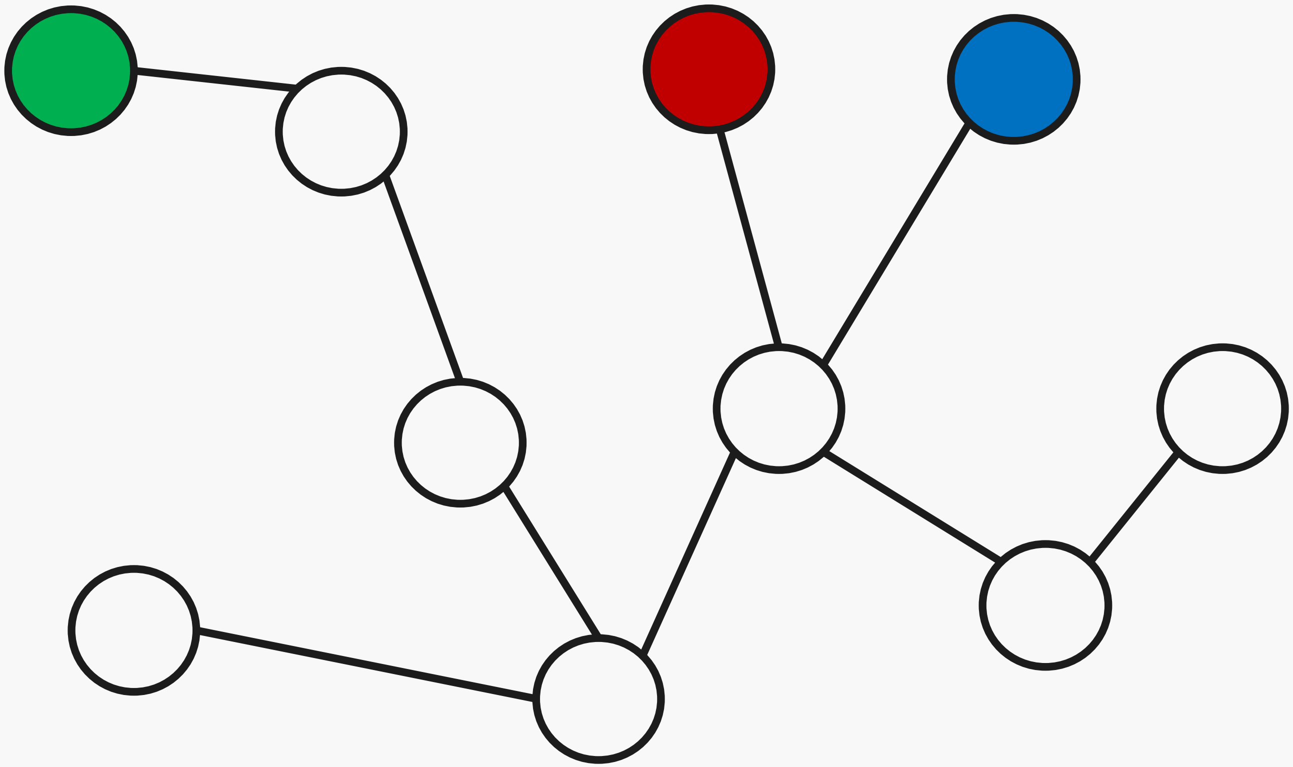
If I'm at distances 3 and 4,



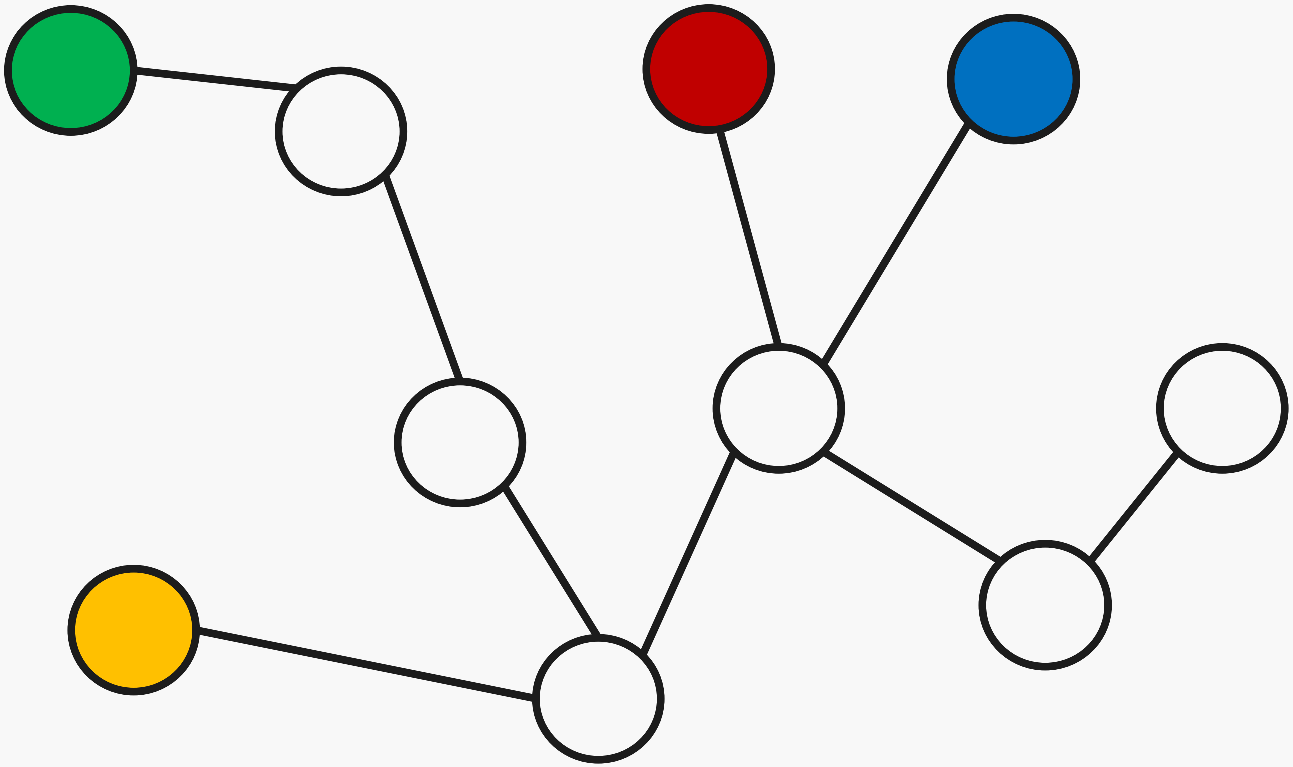
If I'm at distances 3 and 4, I'm here.



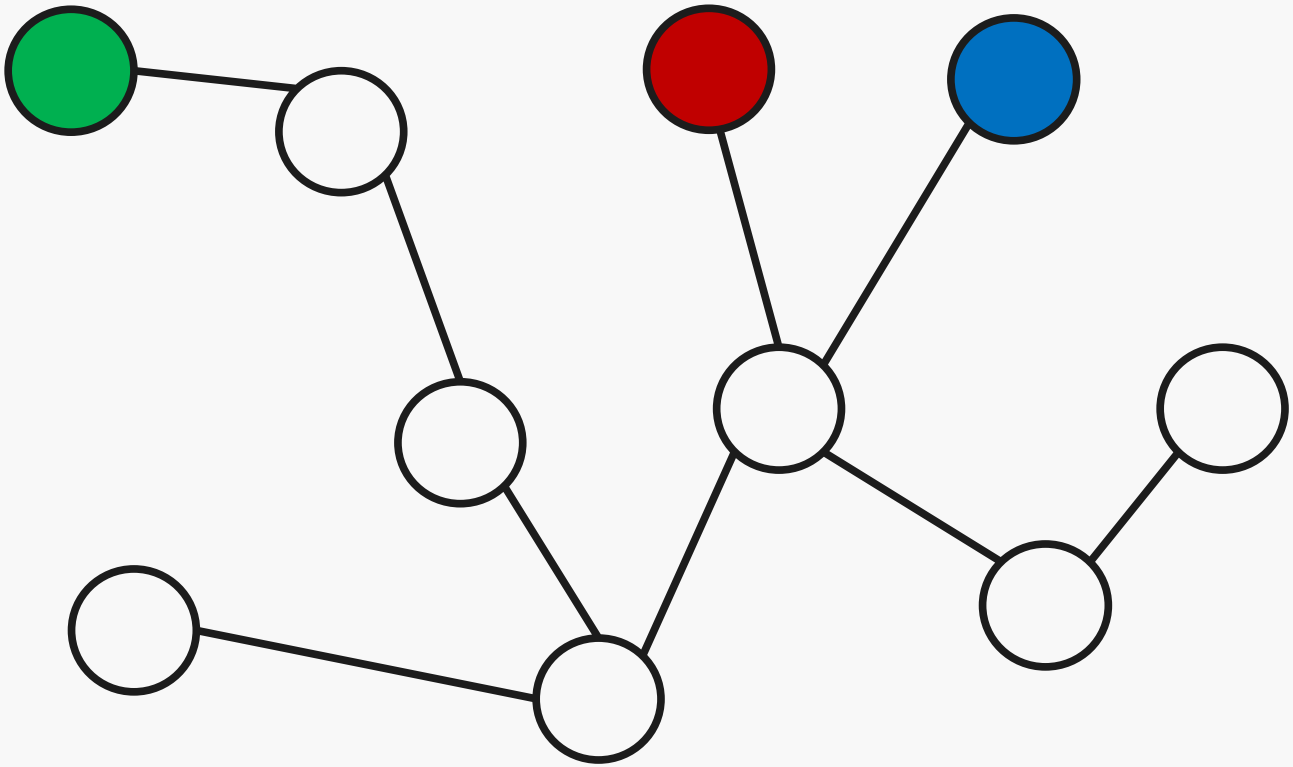
Graphs like these are called trees.
We know how to find metric dimensions of trees.



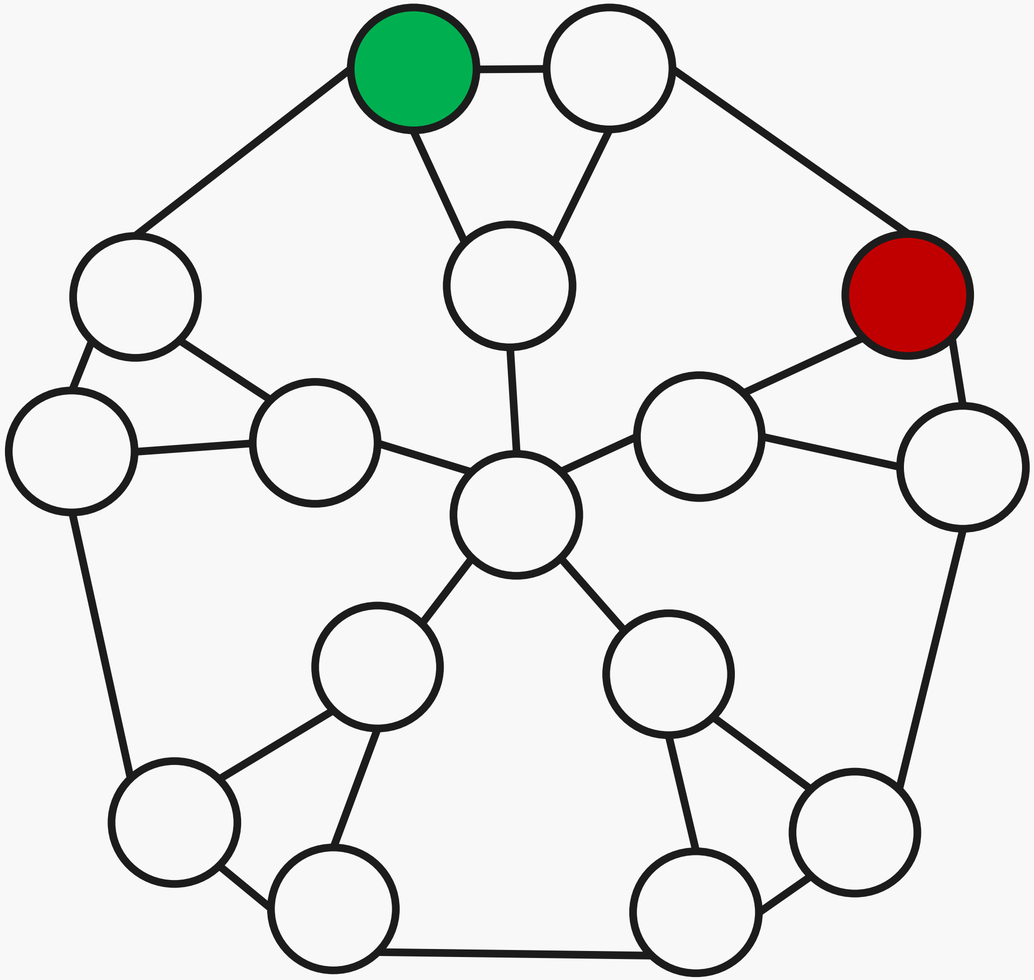
If I'm at distances 4, 3 and 3,



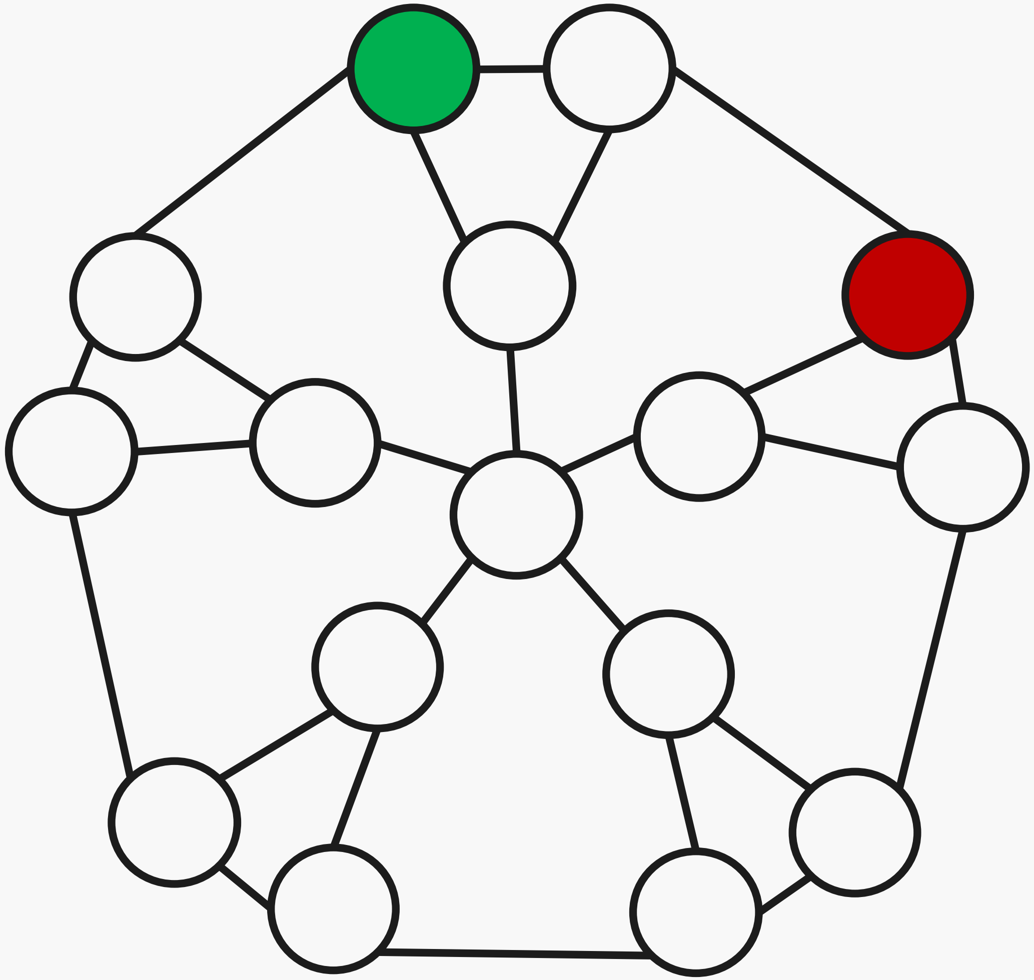
If I'm at distances 4, 3 and 3, I'm here.



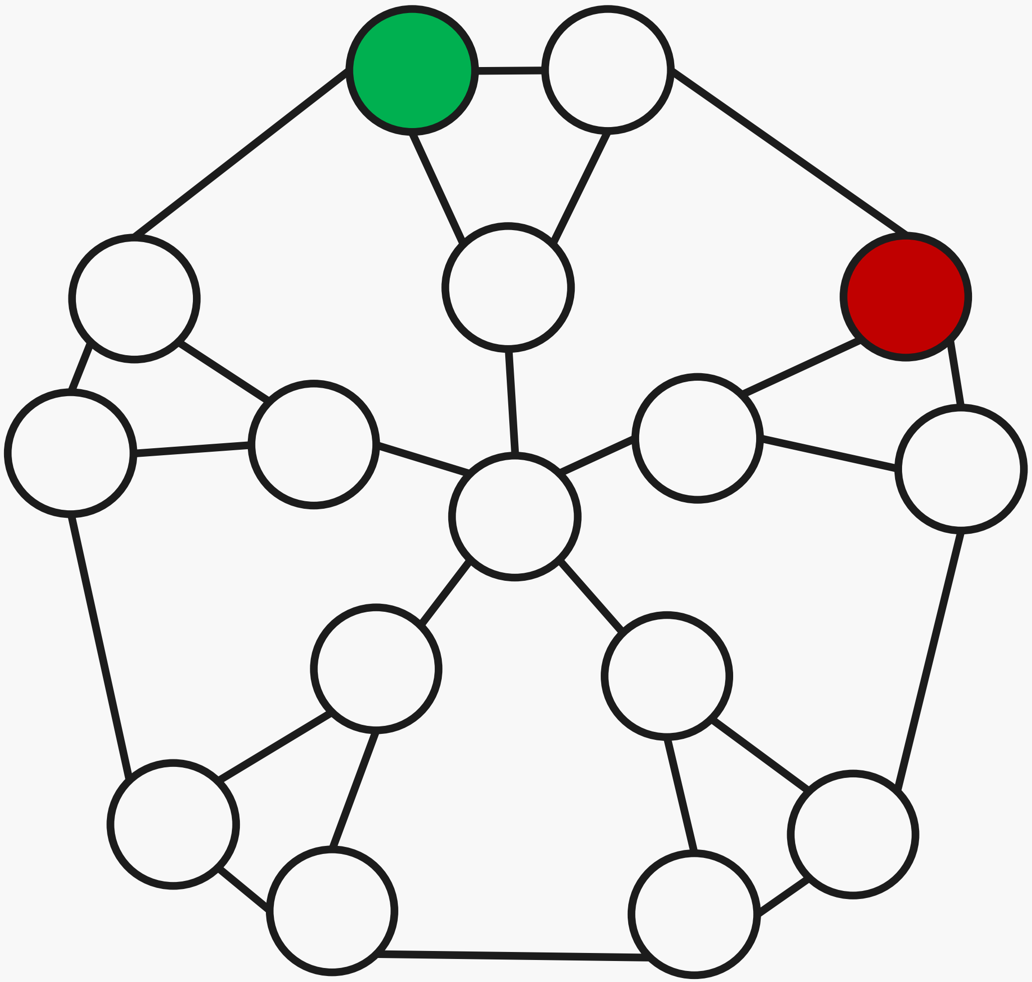
Two satellites aren't enough:
otherwise, we can't tell red and blue apart.



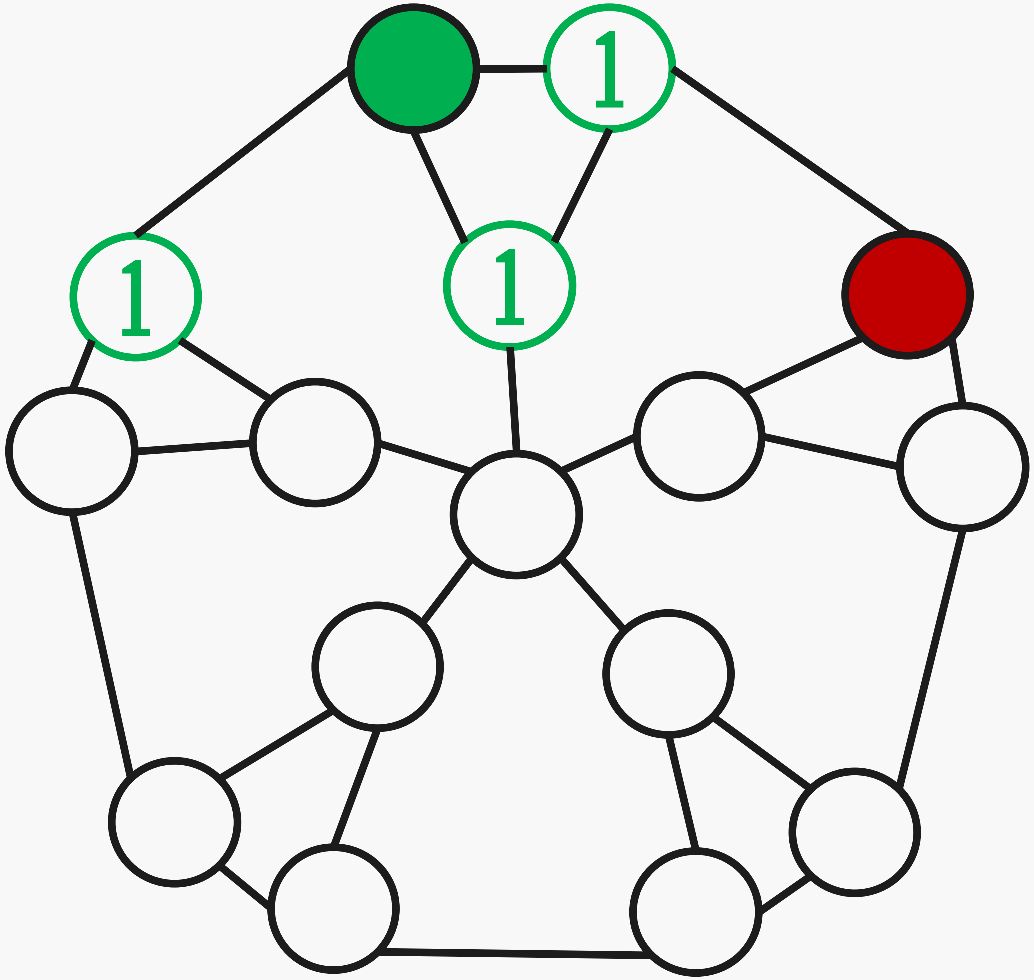
What Garces and Rosario did is find the metric dimension of truncated wheels. This is TW5.



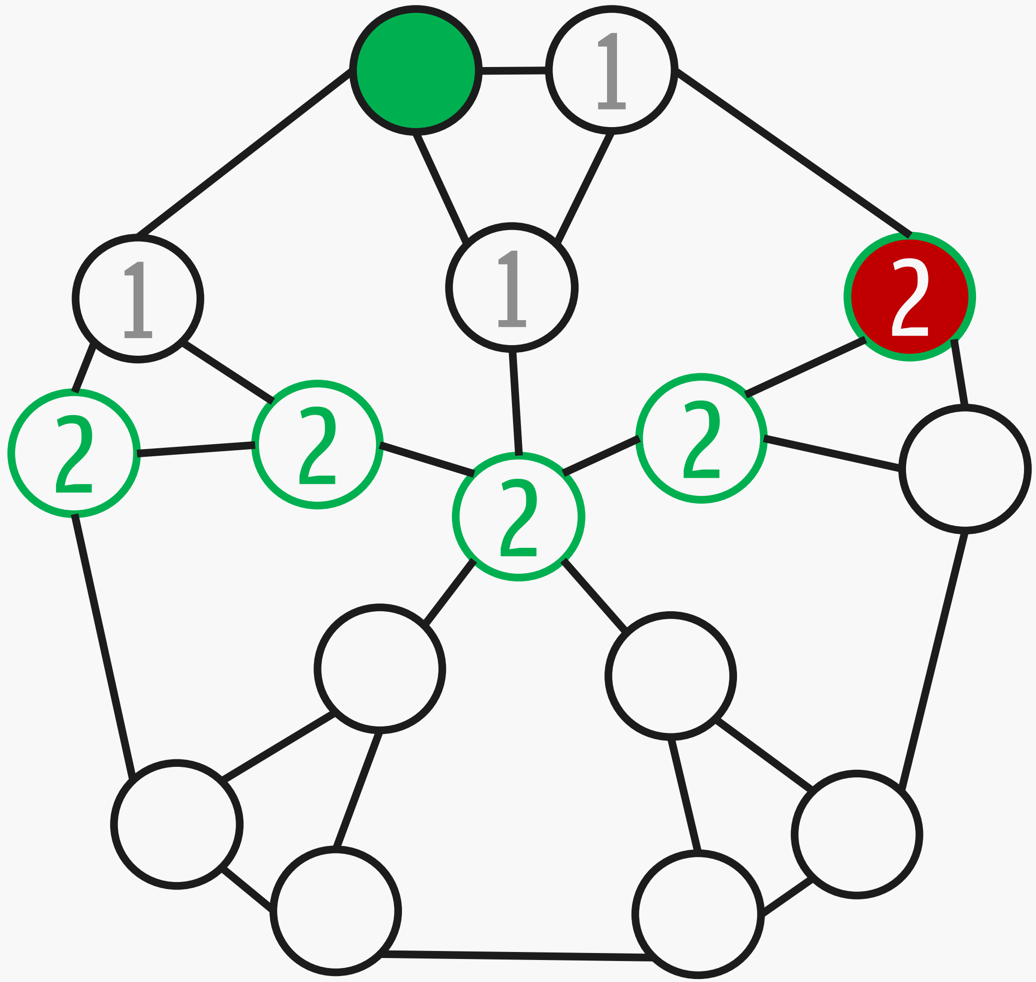
How do you tell when a set of satellites work?
There's a nice way to do it for large graphs like this.



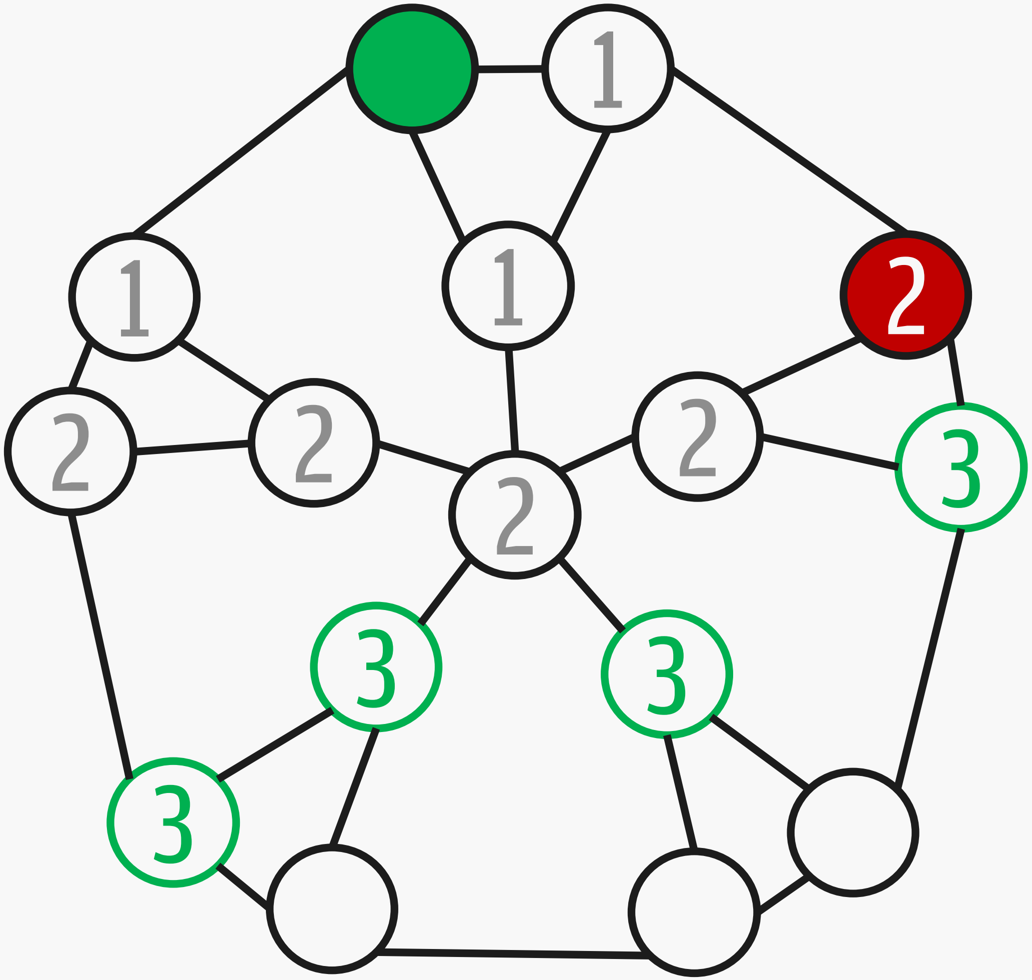
Let's label all the distances from the first satellite.



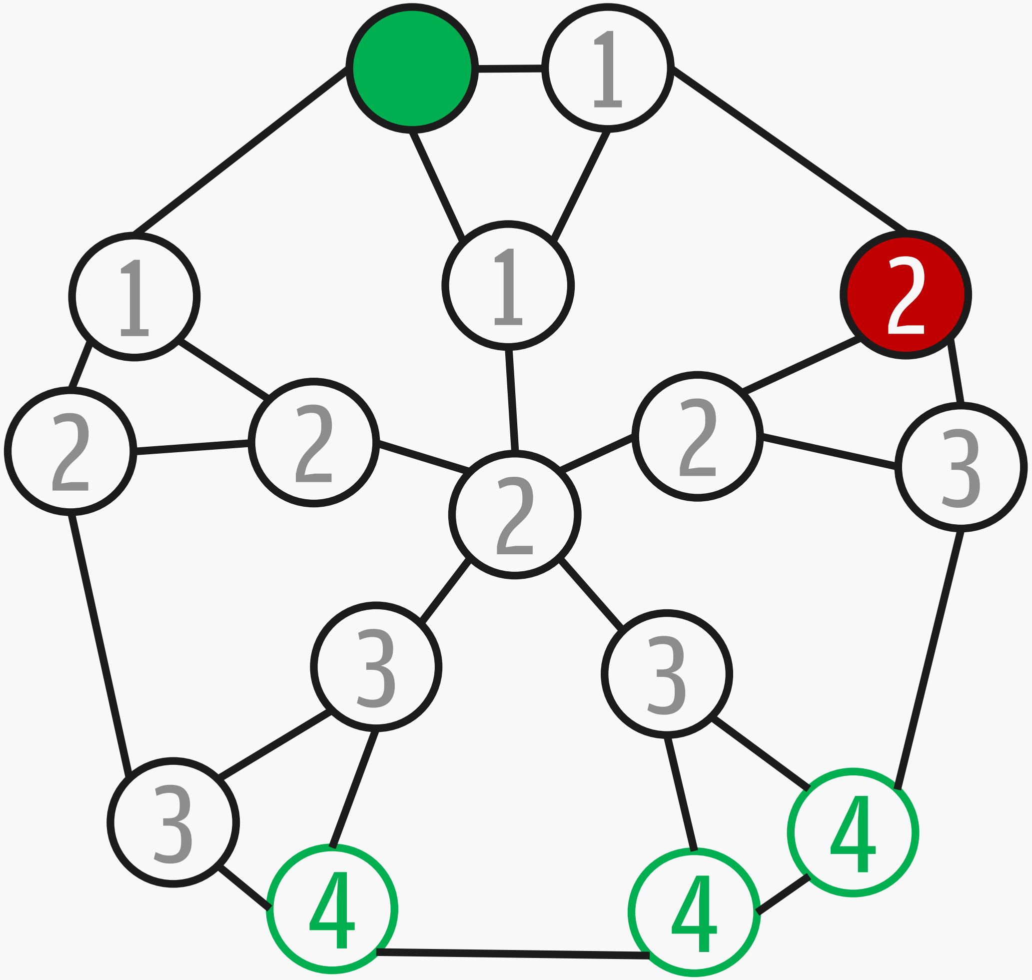
Let's label all the distances from the first satellite.
Here are the vertices with distance 1.



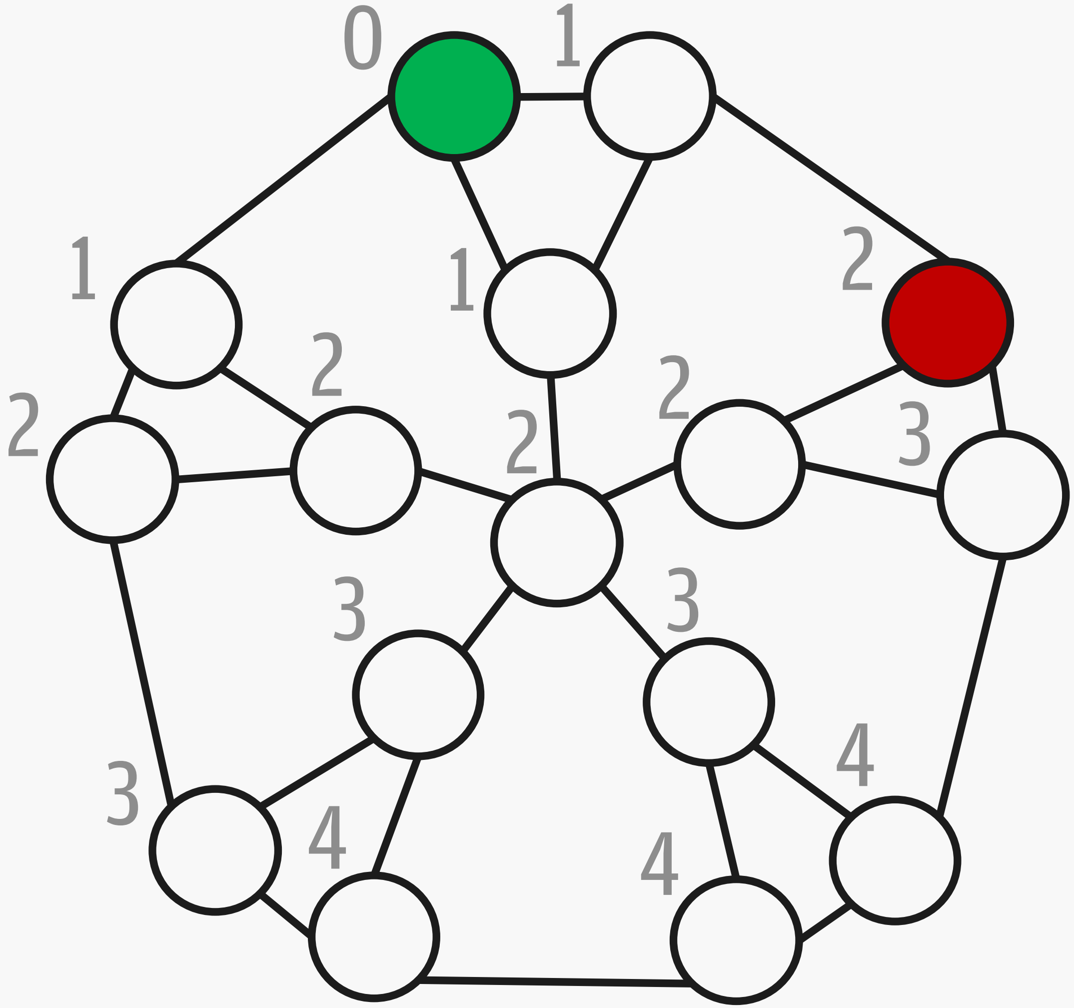
Let's label all the distances from the first satellite.
The vertices with distance 2 are next to distance 1.



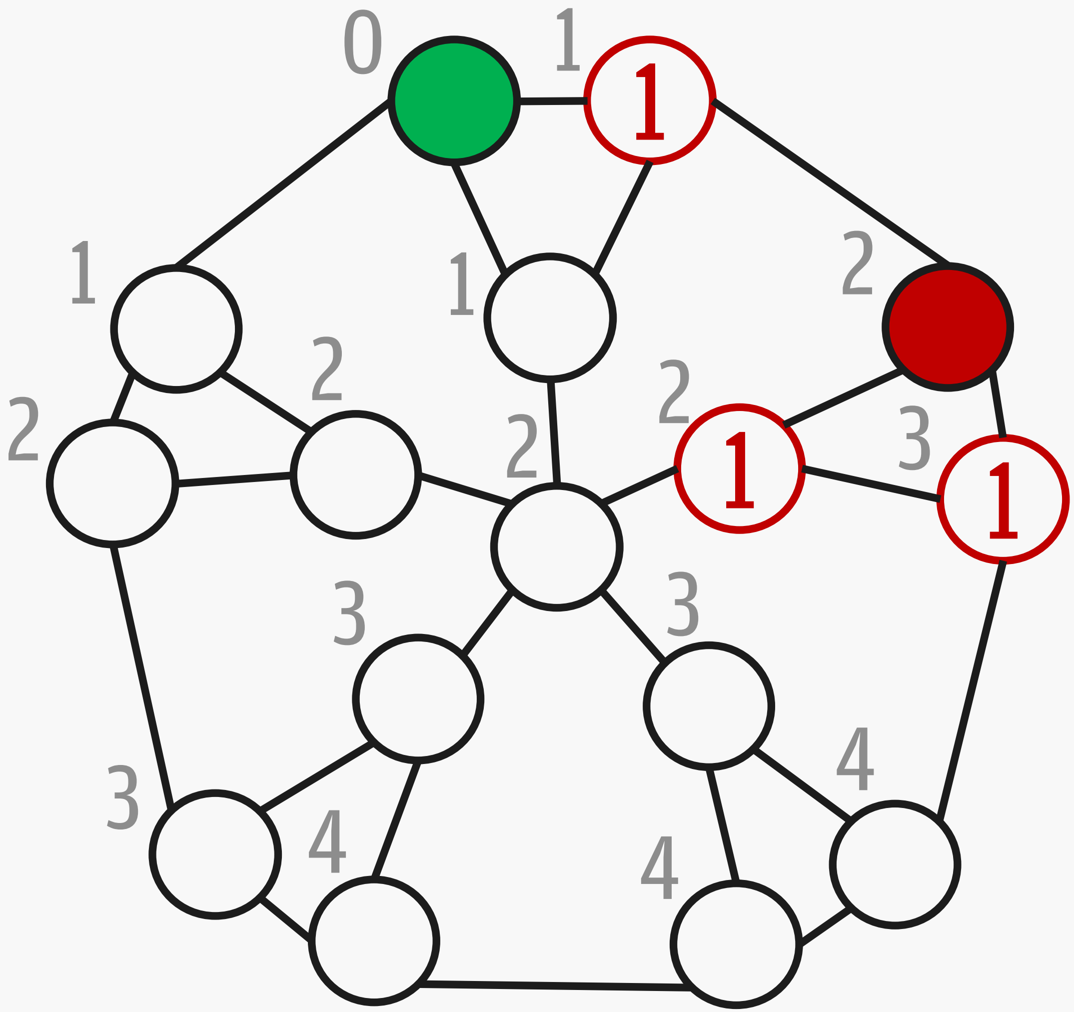
Let's label all the distances from the first satellite.
The vertices with distance 3 are next to distance 2.



Let’s label all the distances from the first satellite.
Finally here are the vertices of distance 4.



Put these distances aside from now.
Let’s label distances from the second satellite.



Put these distances aside from now.
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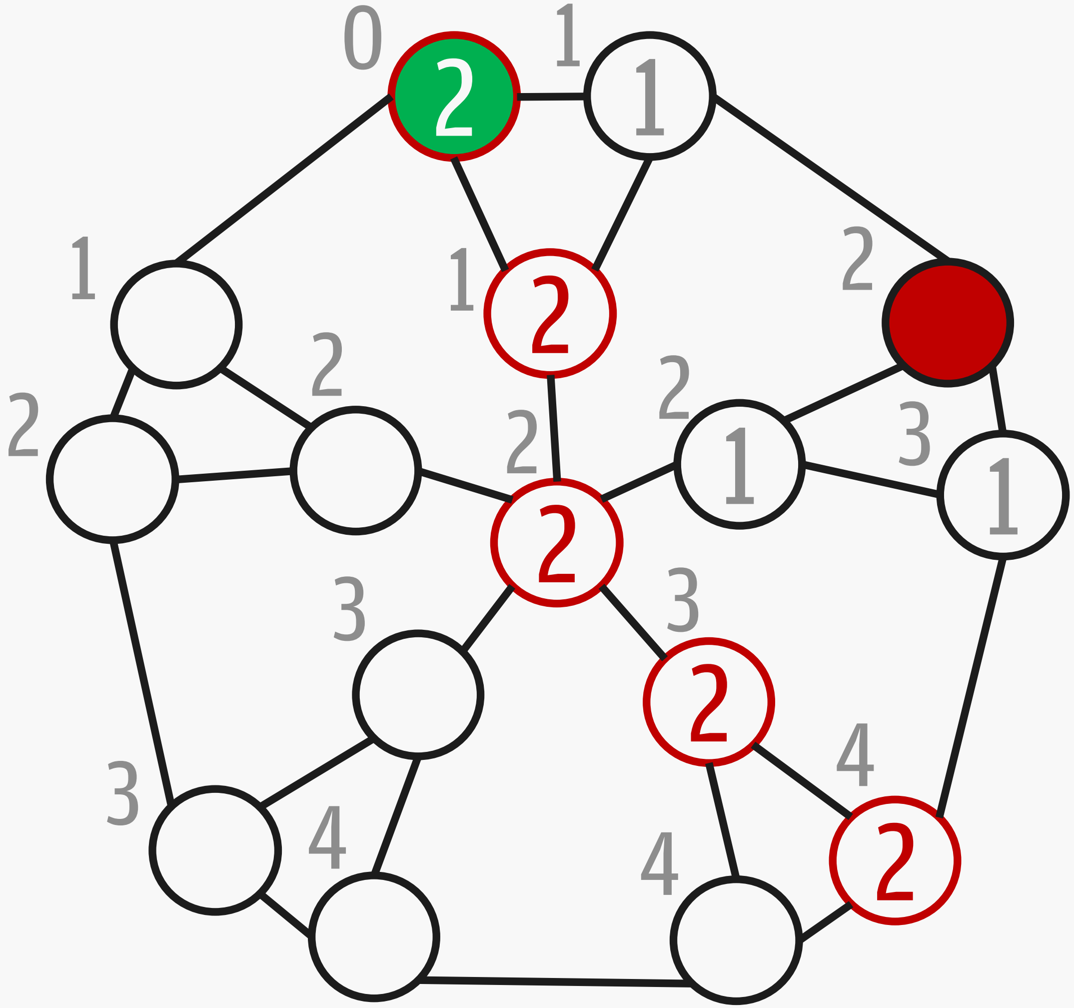

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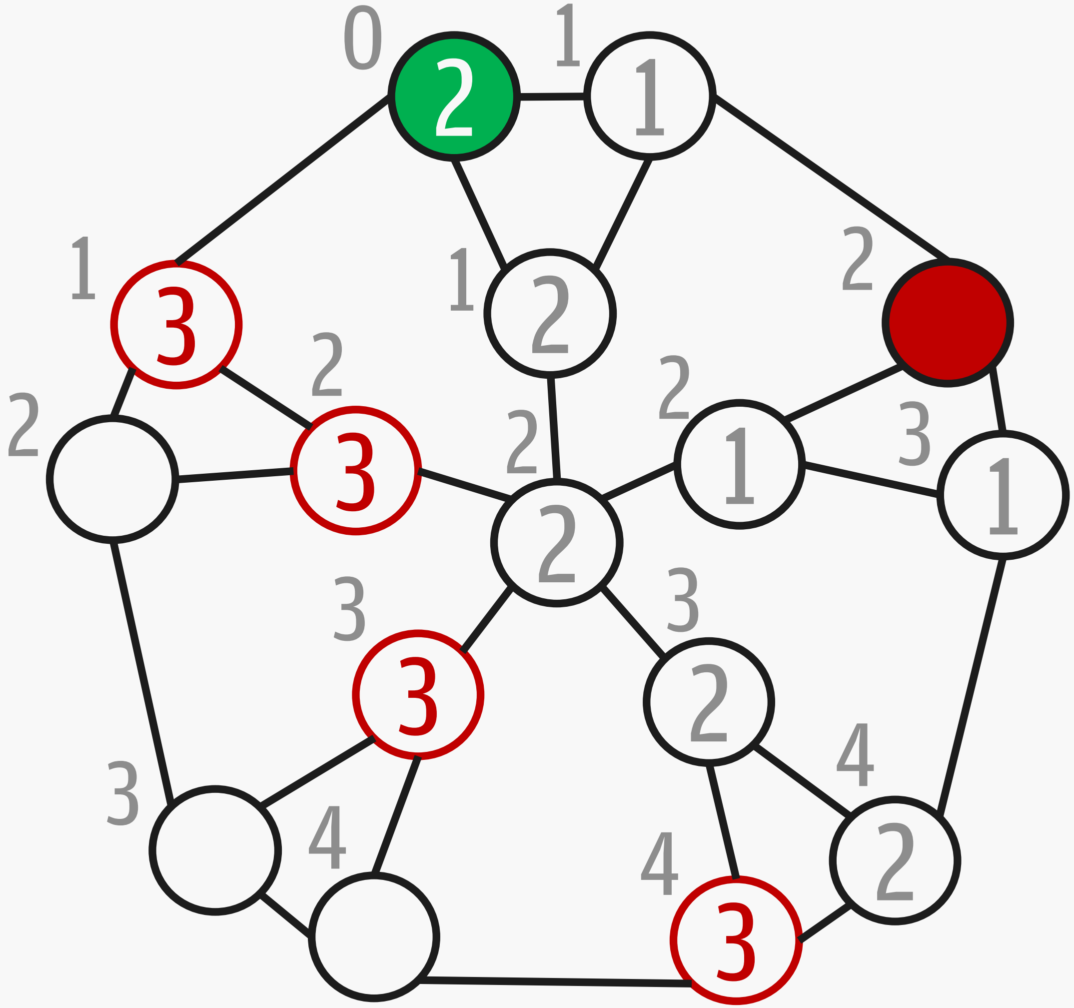
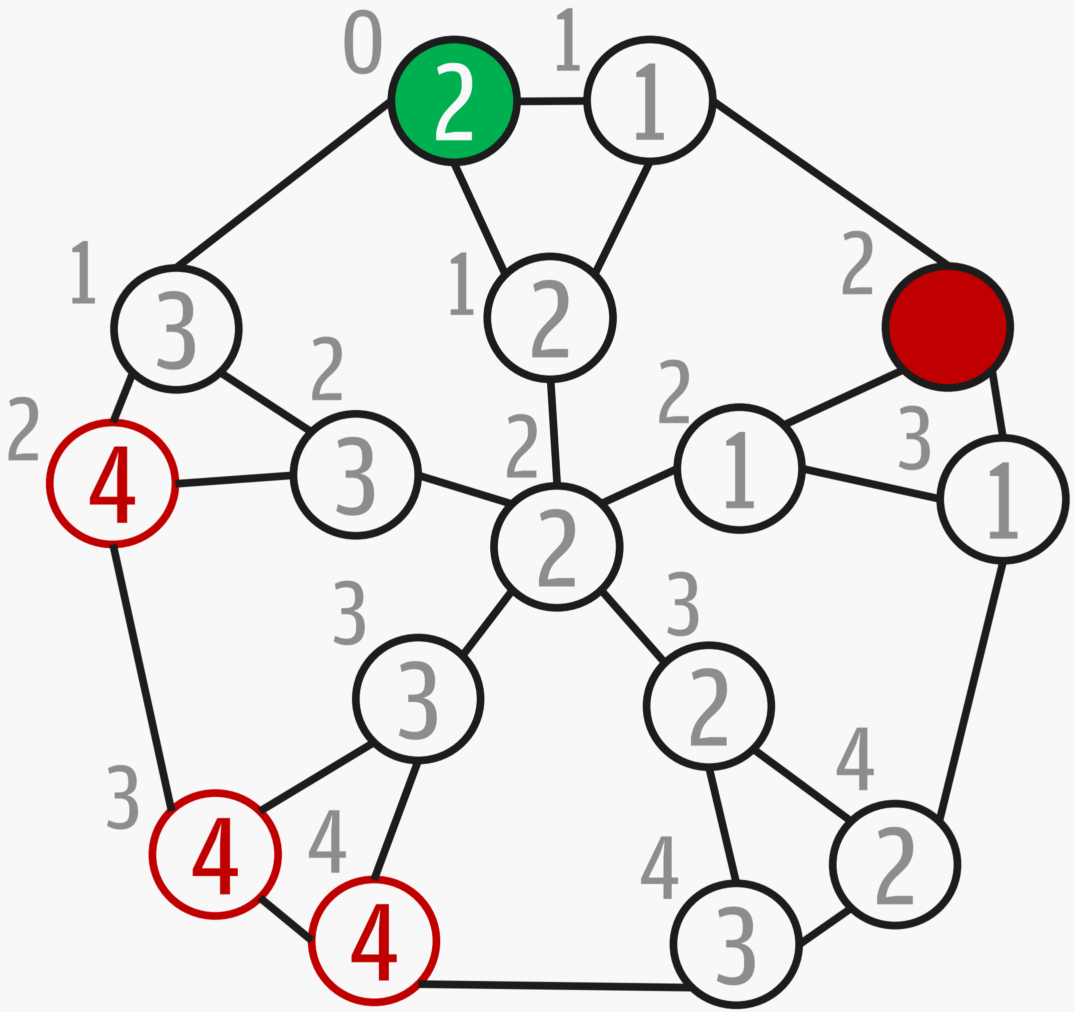
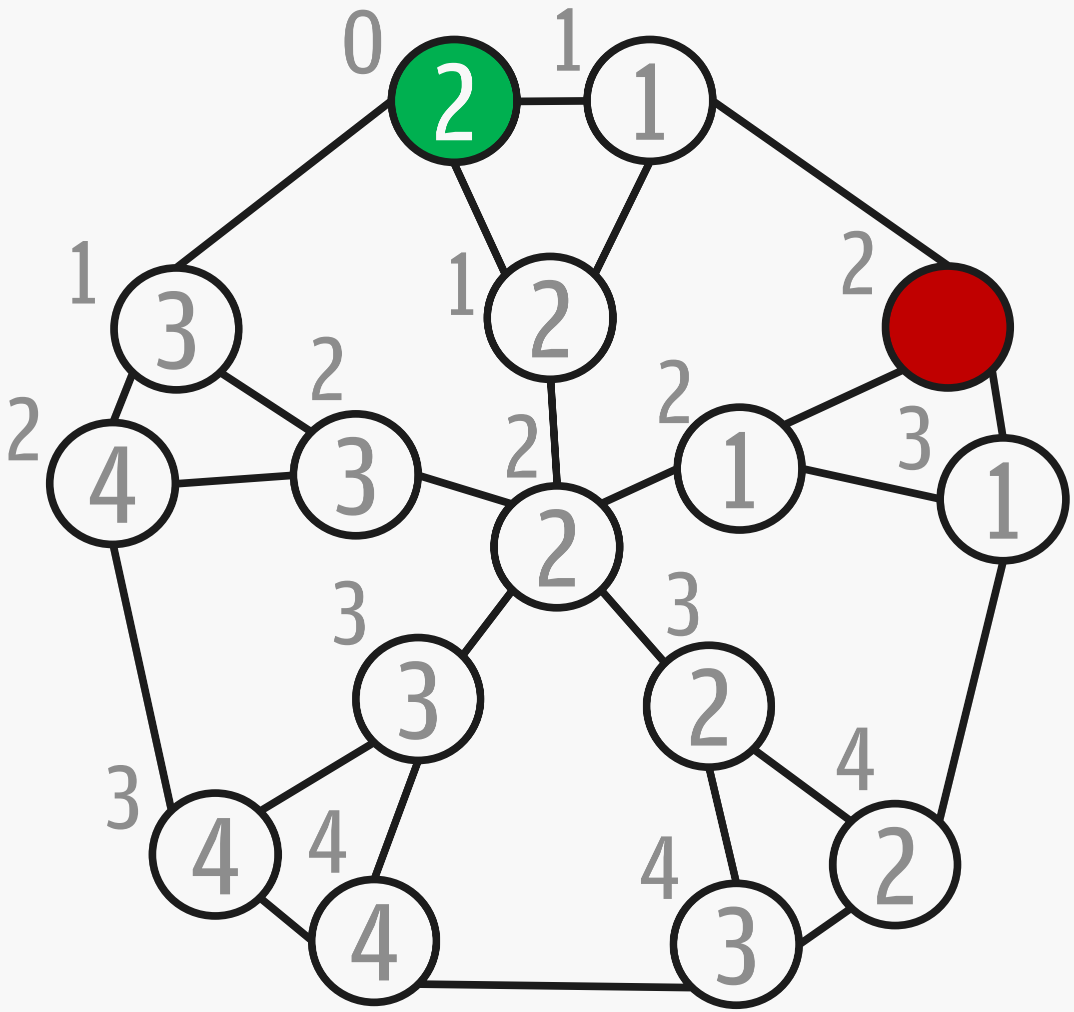


Put these distances aside from now.

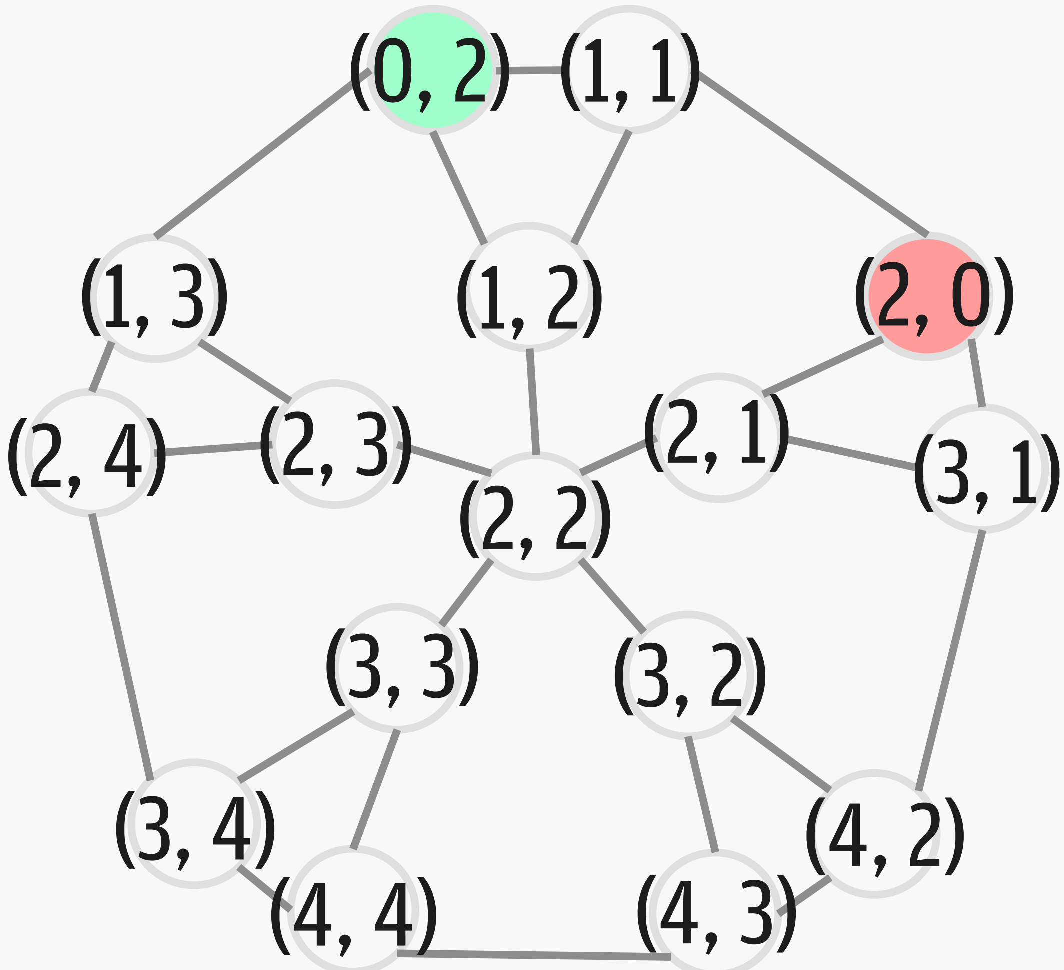
Let's label distances from the second satellite.



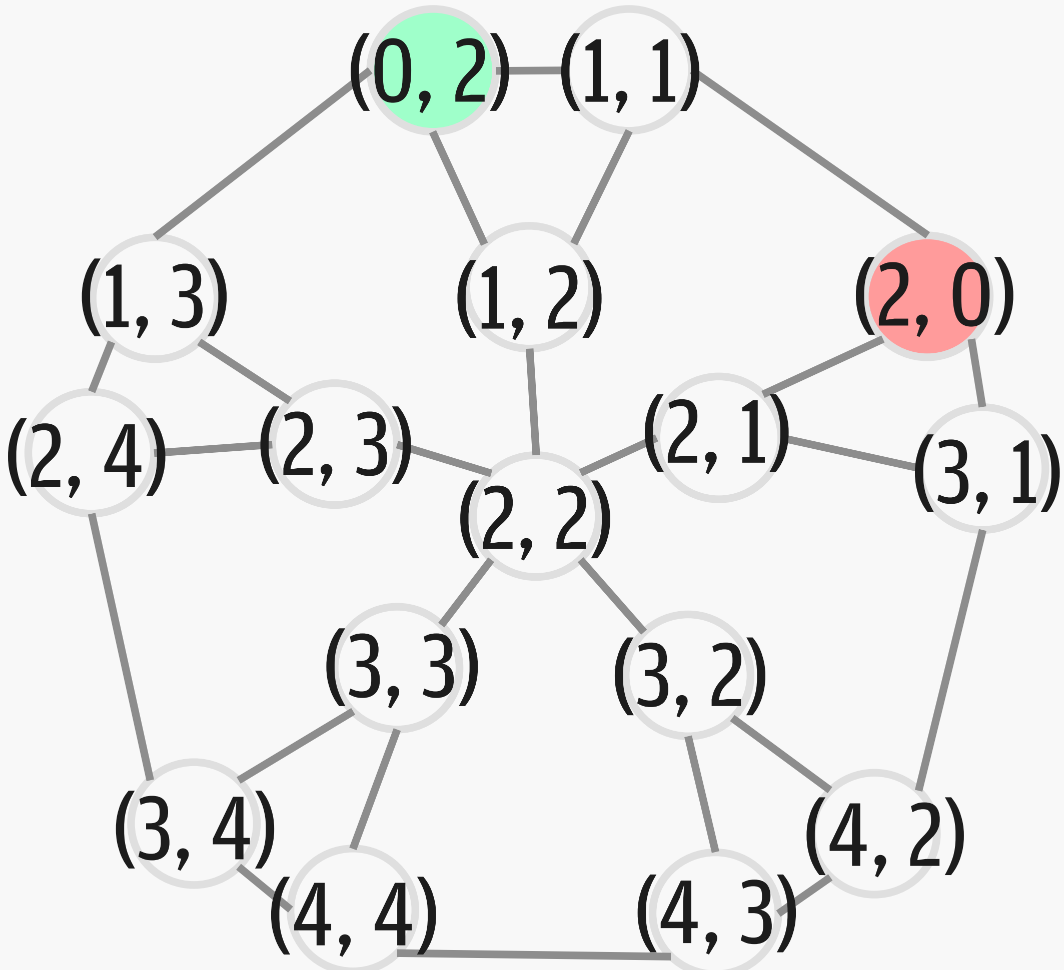
Put these distances aside from now.
Let's label distances from the second satellite.



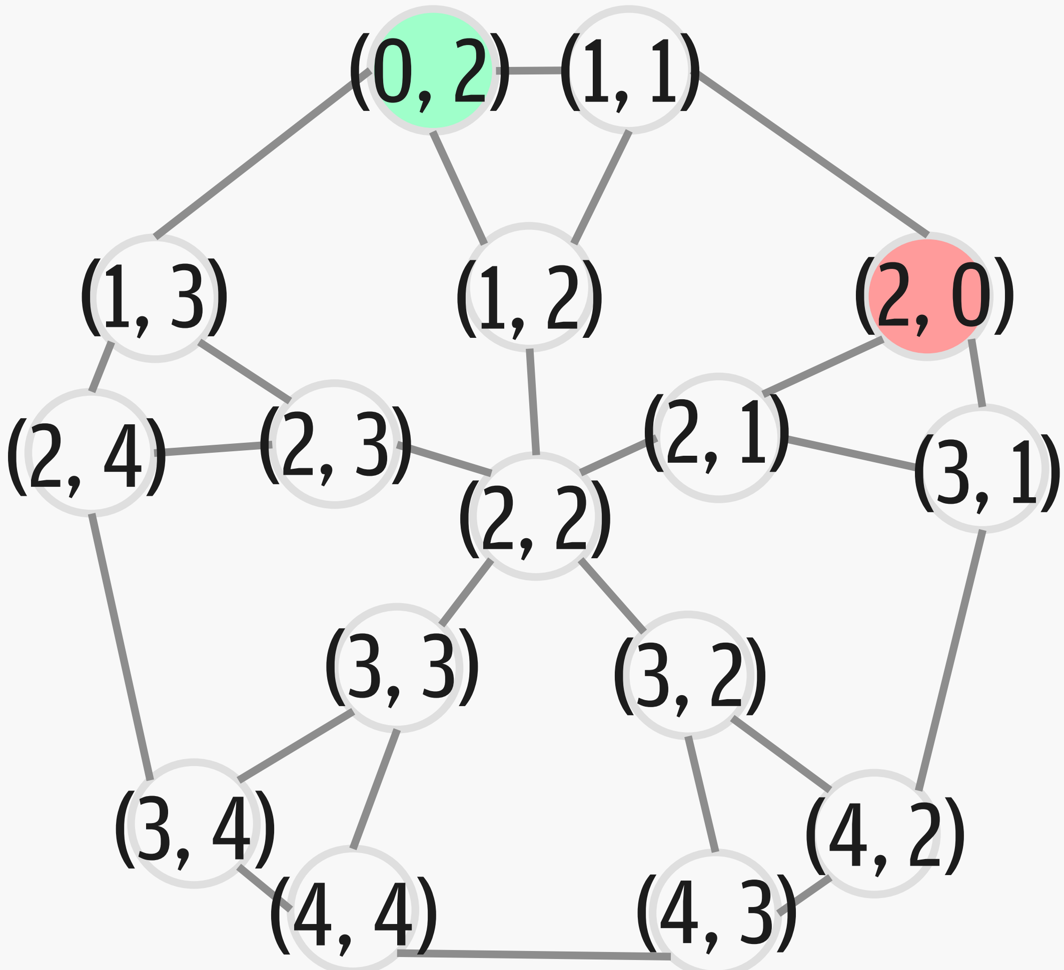
Now, let's label each vertex with a pair (m, n), where m is the distance to the first satellite, and n is the distance to the second satellite.



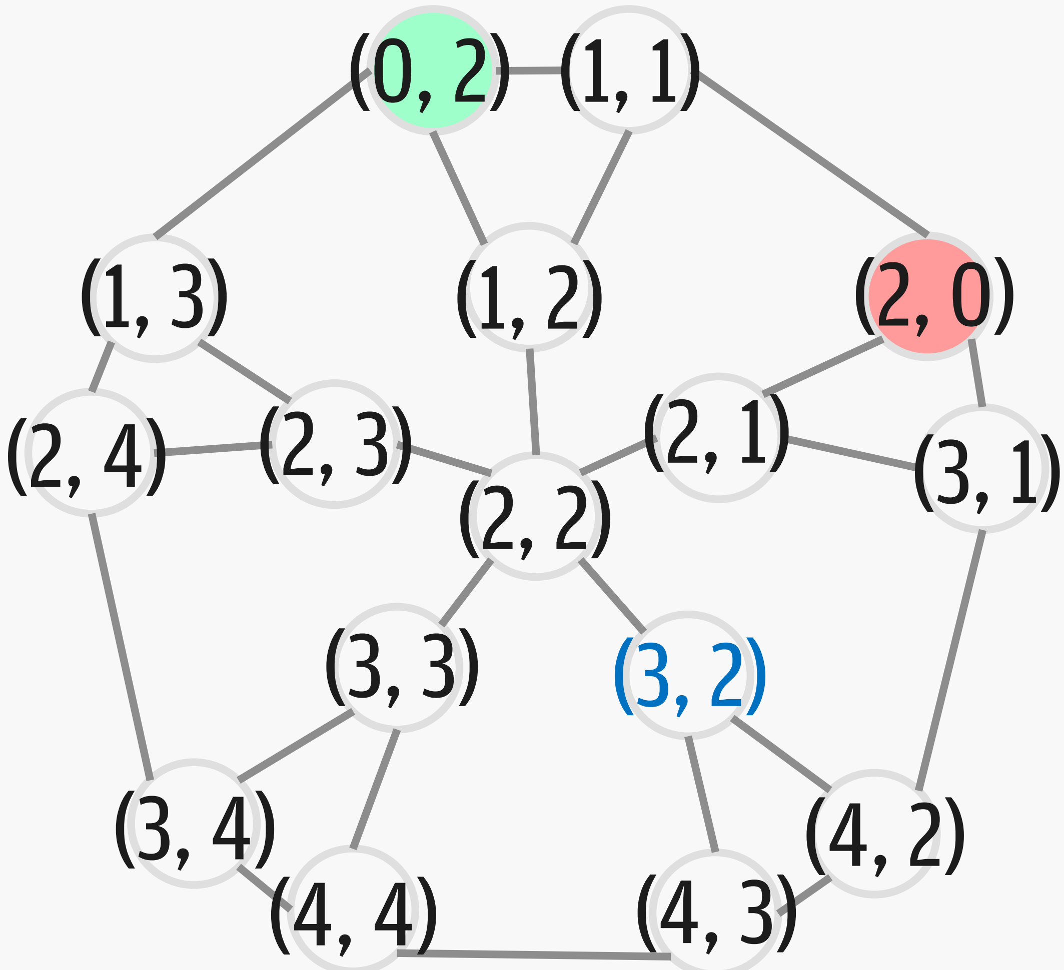
Now, let's label each vertex with (m, n),
where m is the distance to the first satellite,
and n is the distance to the second satellite.



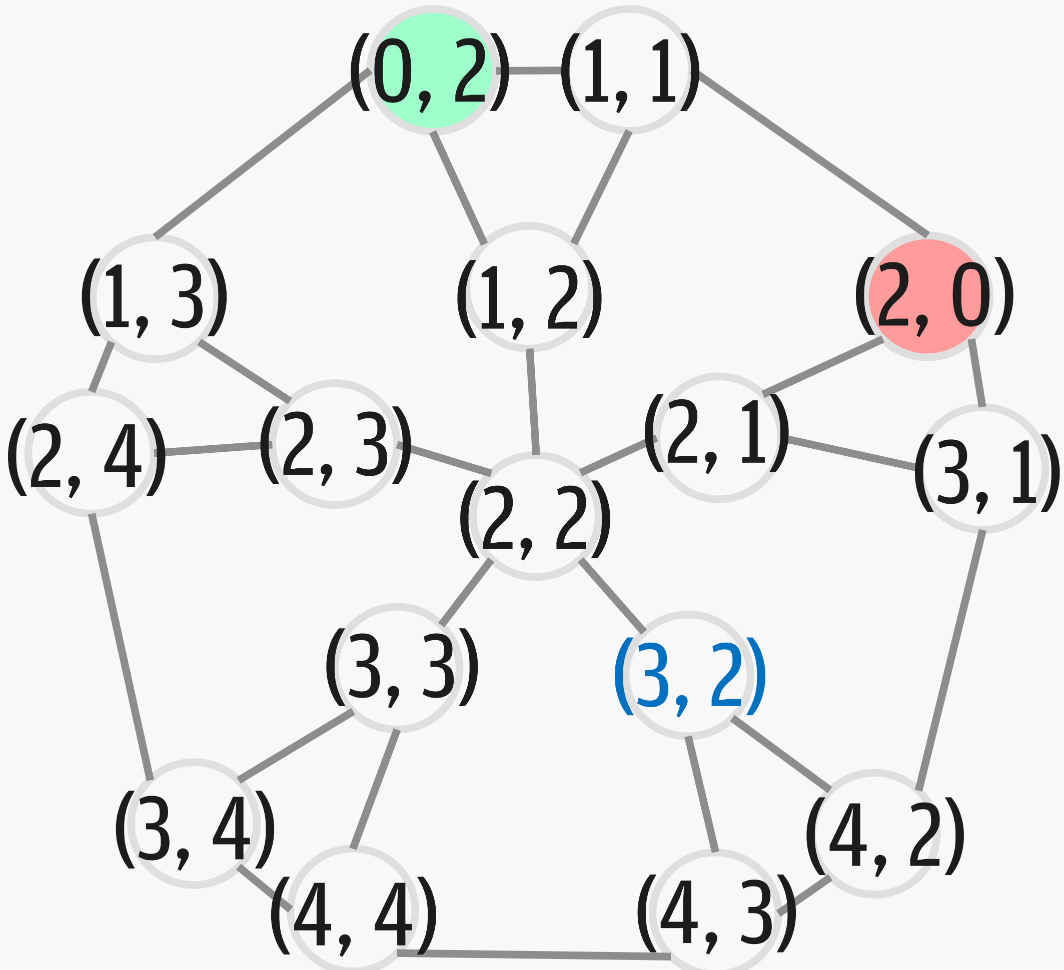
We can use these labels to find vertices.



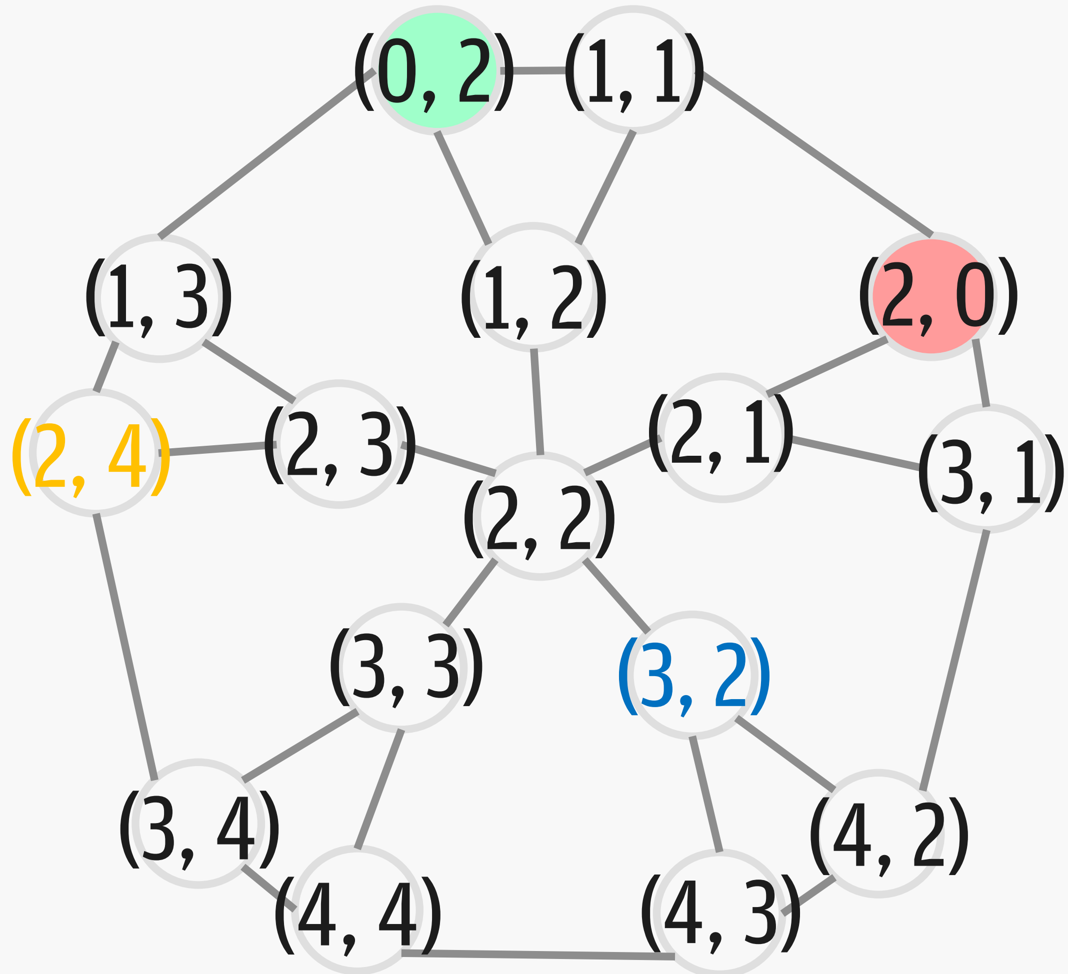
We can use these labels to find vertices.
If I am at distances 3 and 2,



We can use these labels to find vertices.
If I am at distances 3 and 2, I am at (3, 2).

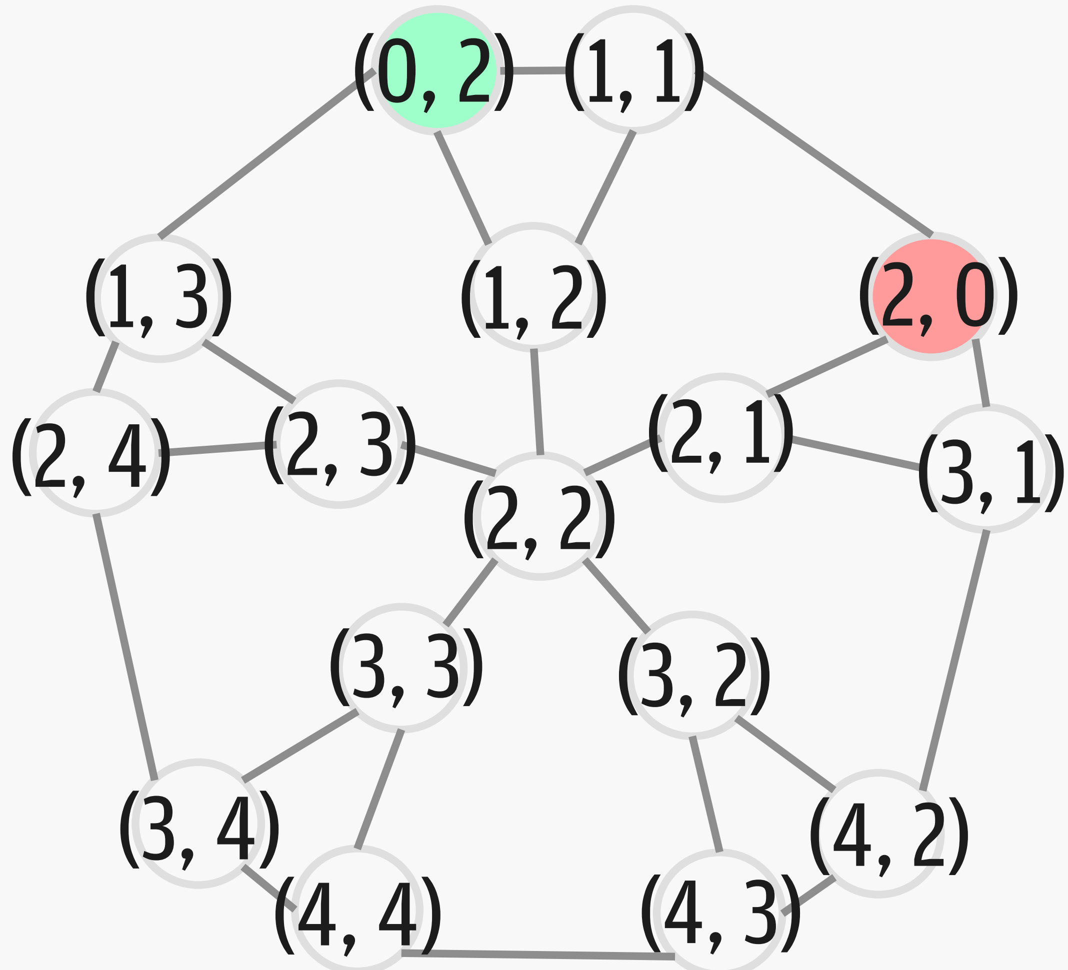


We can use these labels to find vertices.
If I am at distances 2 and 4,

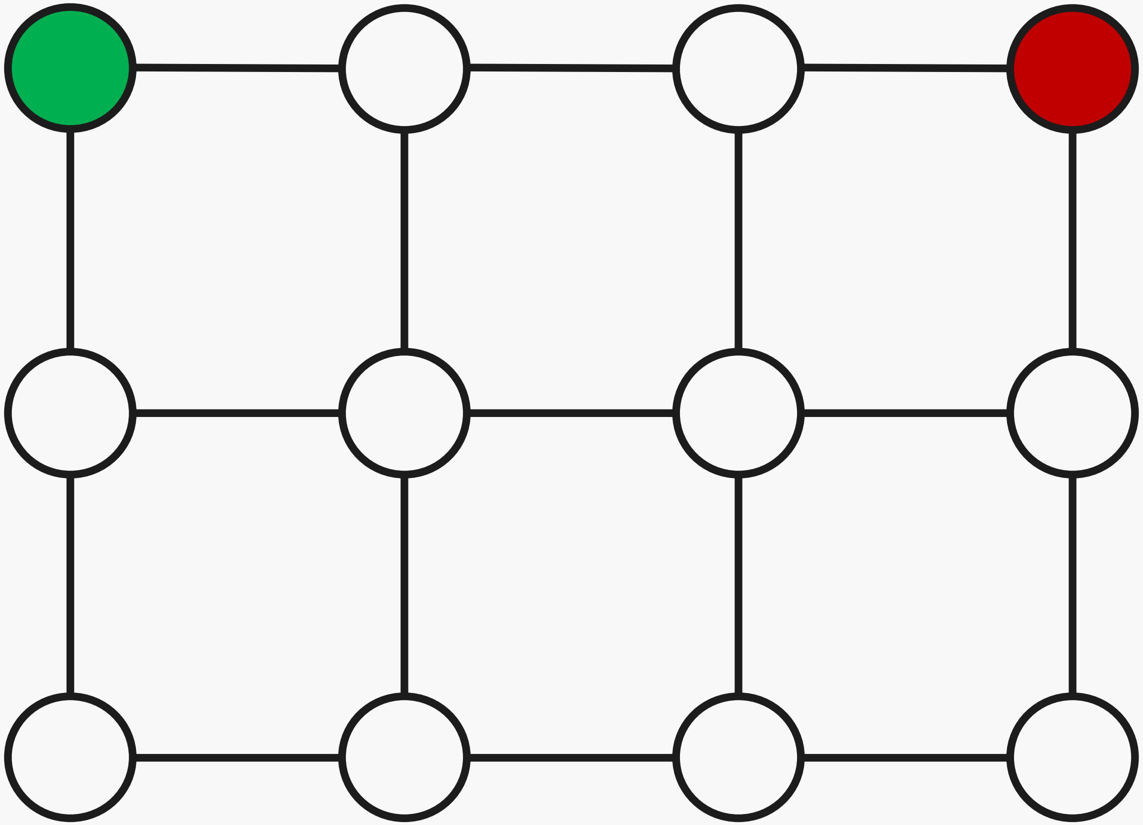


We can use these labels to find vertices.

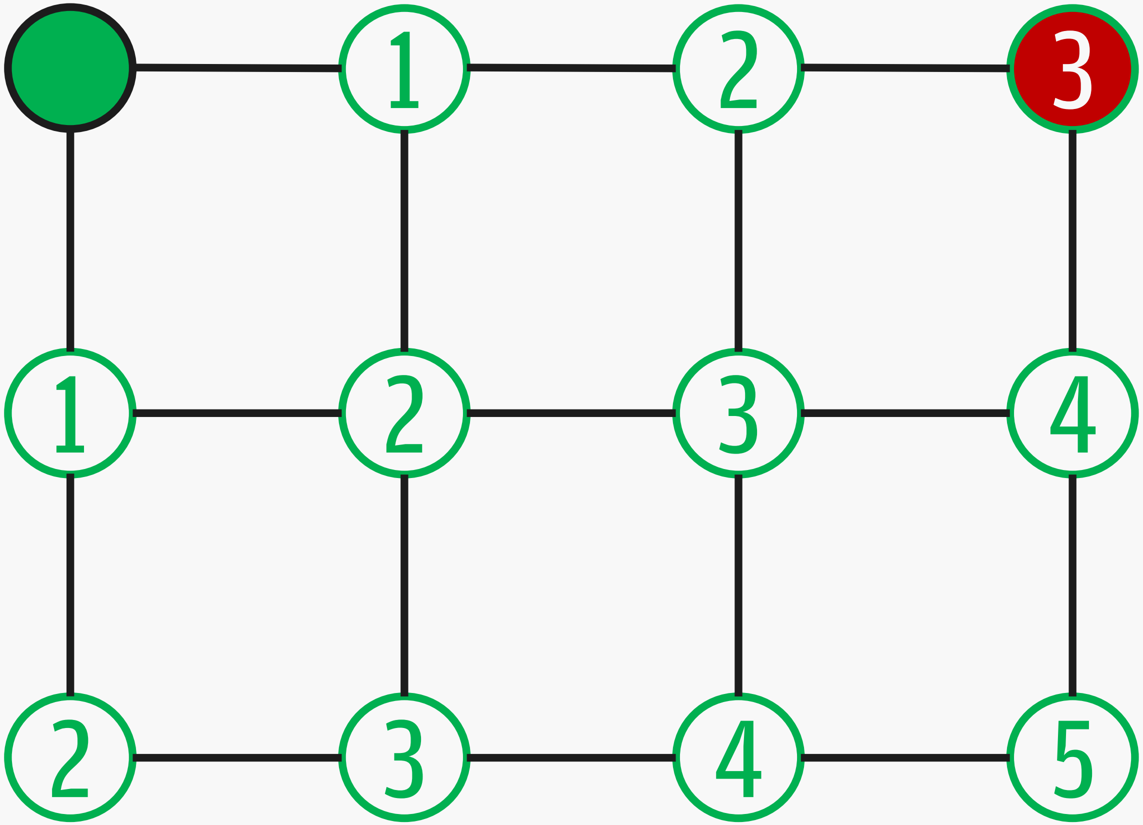
If I am at distances 2 and 4, I am at (2, 4).



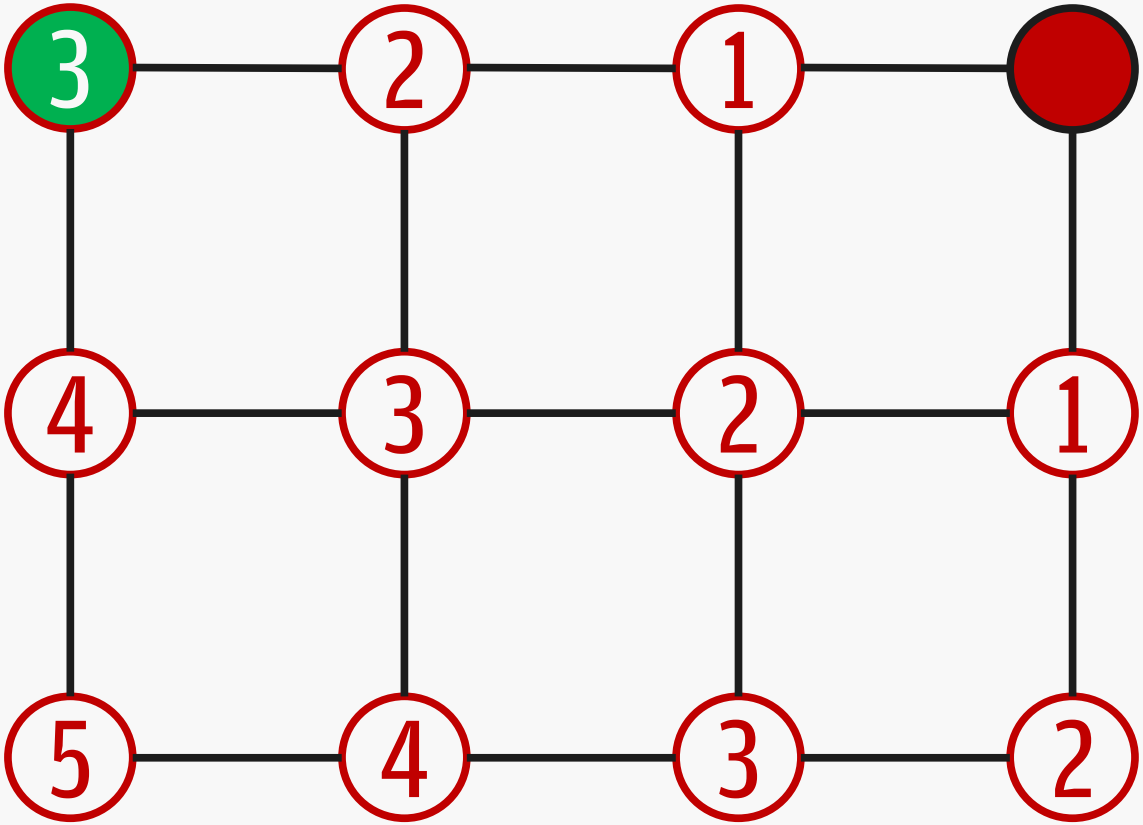
This works because all the labels are different!
So the satellites work if the labels are all different.



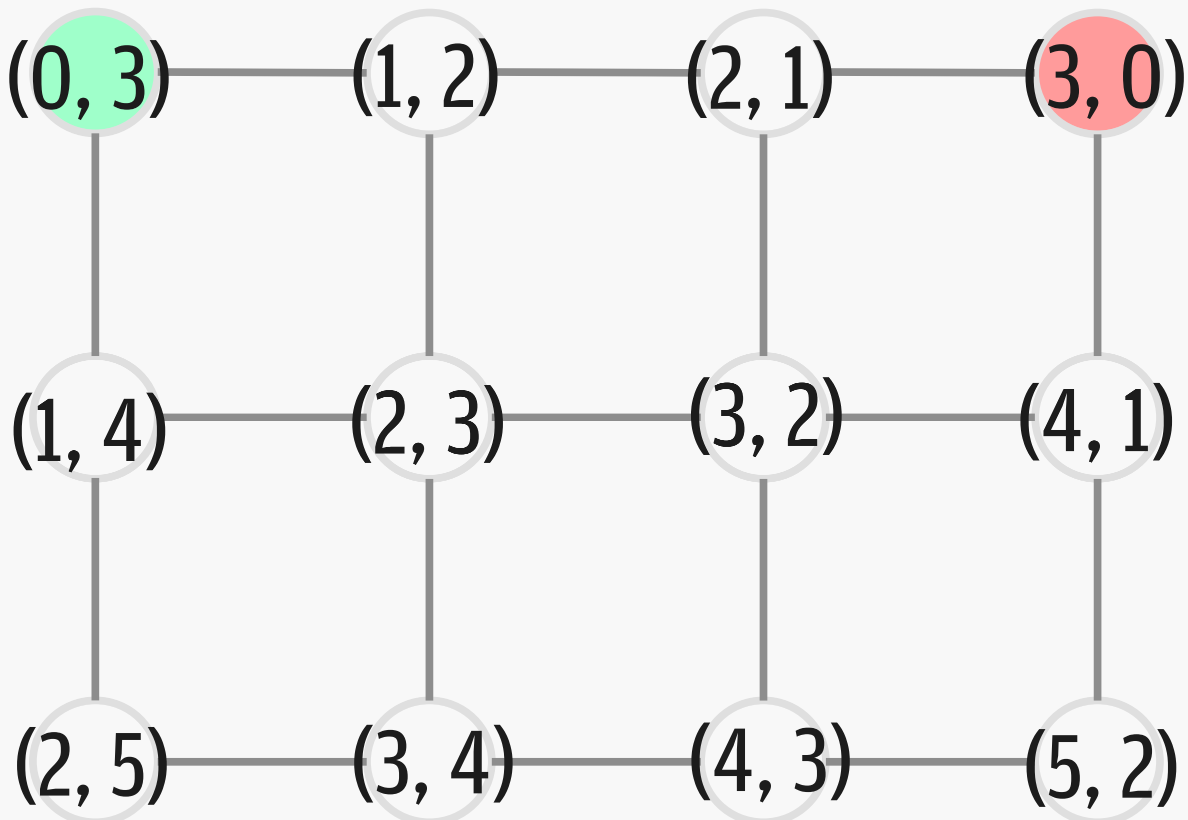
Let's try it on this graph.



Here are the distances to the first satellite.



Here are the distances to the second satellite.



And here are the labels of each vertex.
They are all different, so these satellites work.

November 2016

- We ran out of different kinds of graphs.

November 2016

- We ran out of different kinds of graphs.
- But then we had an idea.

November 2016

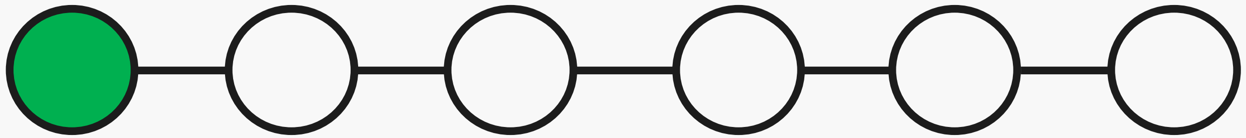
- We ran out of different kinds of graphs.
- But then we had an idea.
- What if we combined **metric dimension** and **planar graphs**?

Metric dimensions of planar graphs

- Problem: planar graphs are too general.

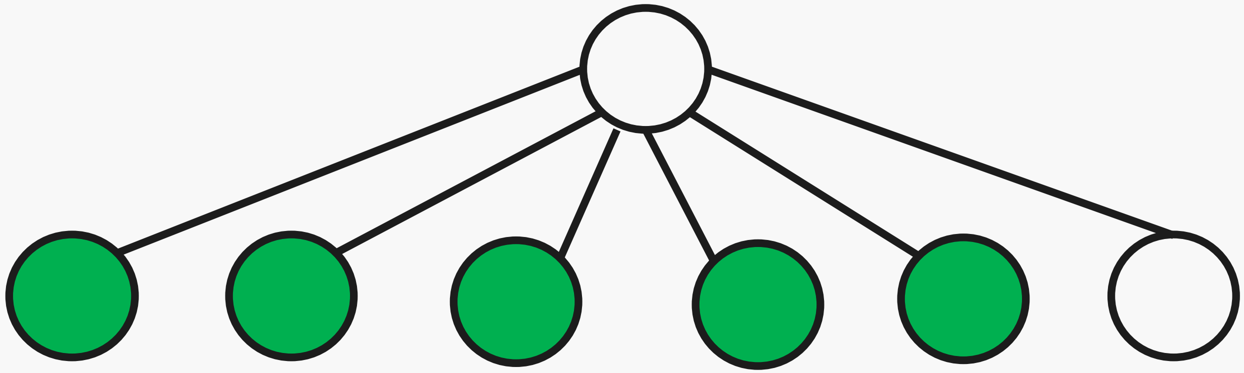
Metric dimensions of planar graphs

- Problem: planar graphs are too general.
- The metric dimension can be 1…



Metric dimensions of planar graphs

- Problem: planar graphs are too general.
- The metric dimension can be 1…
- To almost all the vertices.



Metric dimensions of planar graphs

- Problem: planar graphs are too general.
- The metric dimension can be 1…
- To almost all the vertices.
- We decided to be specific, and consider **maximal planar graphs.**

Metric dimensions of planar graphs

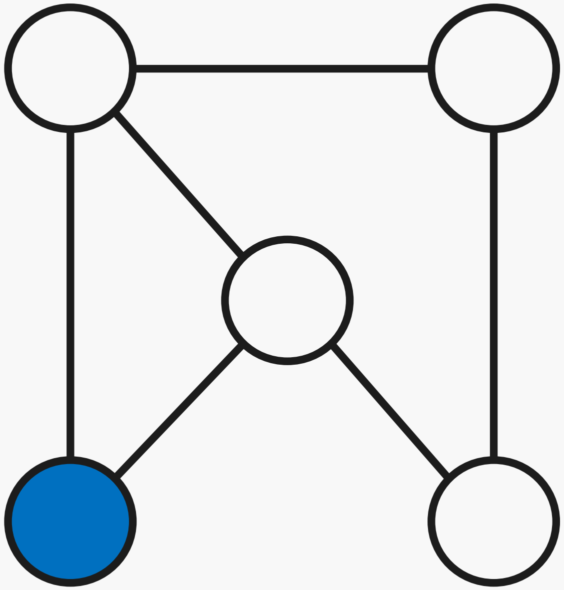
- Problem: planar graphs are too general.
- The metric dimension can be 1…
- To almost all the vertices.
- We decided to be specific, and consider **maximal planar graphs.**
- This is helpful, because all of its faces are triangles, so it's restricted.

Metric dimensions of maximal planar graphs

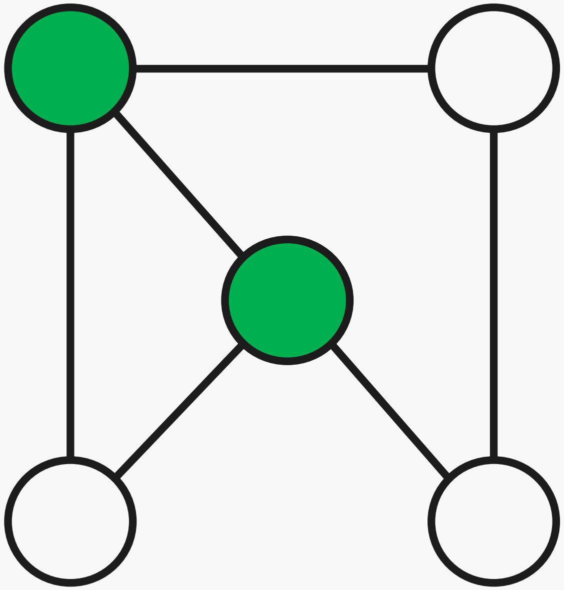
- If a maximal planar graph has N vertices, we will show that its metric dimension is at most 3N/4.

Metric dimensions of maximal planar graphs

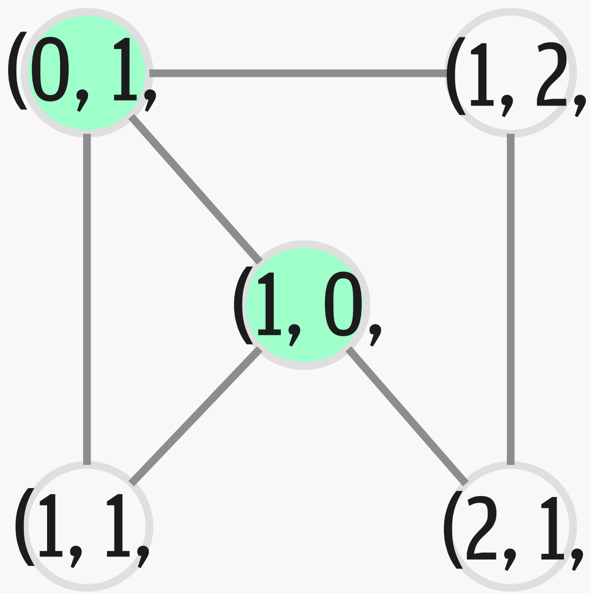
- If a maximal planar graph has N vertices, we will show that its metric dimension is at most 3N/4.
- **Main idea:** If we make all the vertices next to a vertex a satellite, then we're sure we can find that vertex.



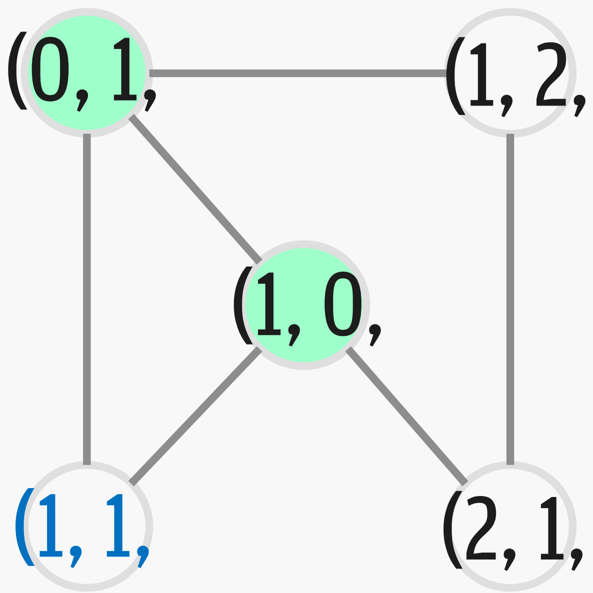
Consider this vertex.



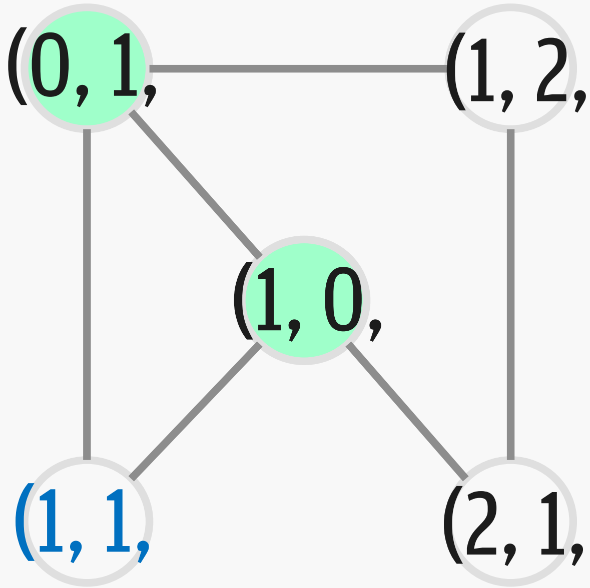
These are the vertices next to it.



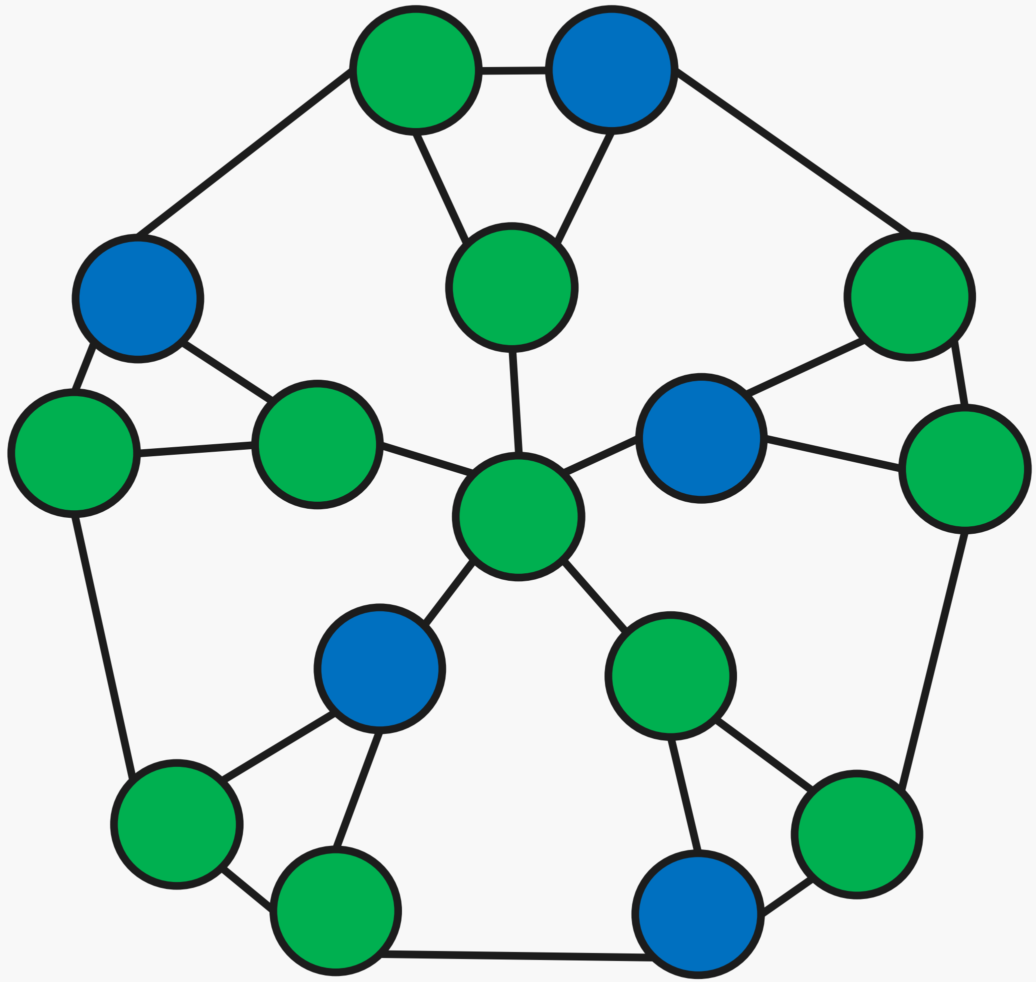
If we replace vertices with distances,



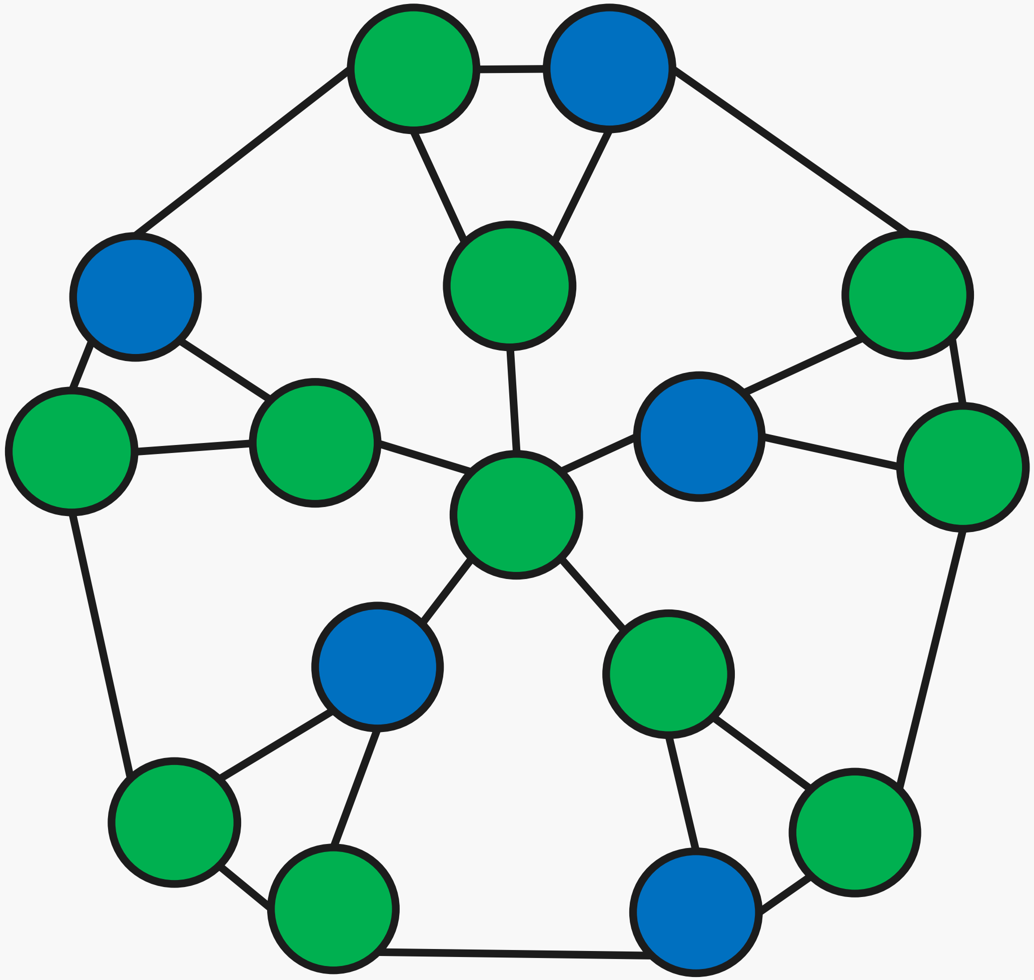
If we replace vertices with distances,
it must be the only vertex that is 1, 1, 1. . . .



If we replace vertices with distances,
it must be the only vertex that is 1, 1, 1. . . .
So we can definitely find it.



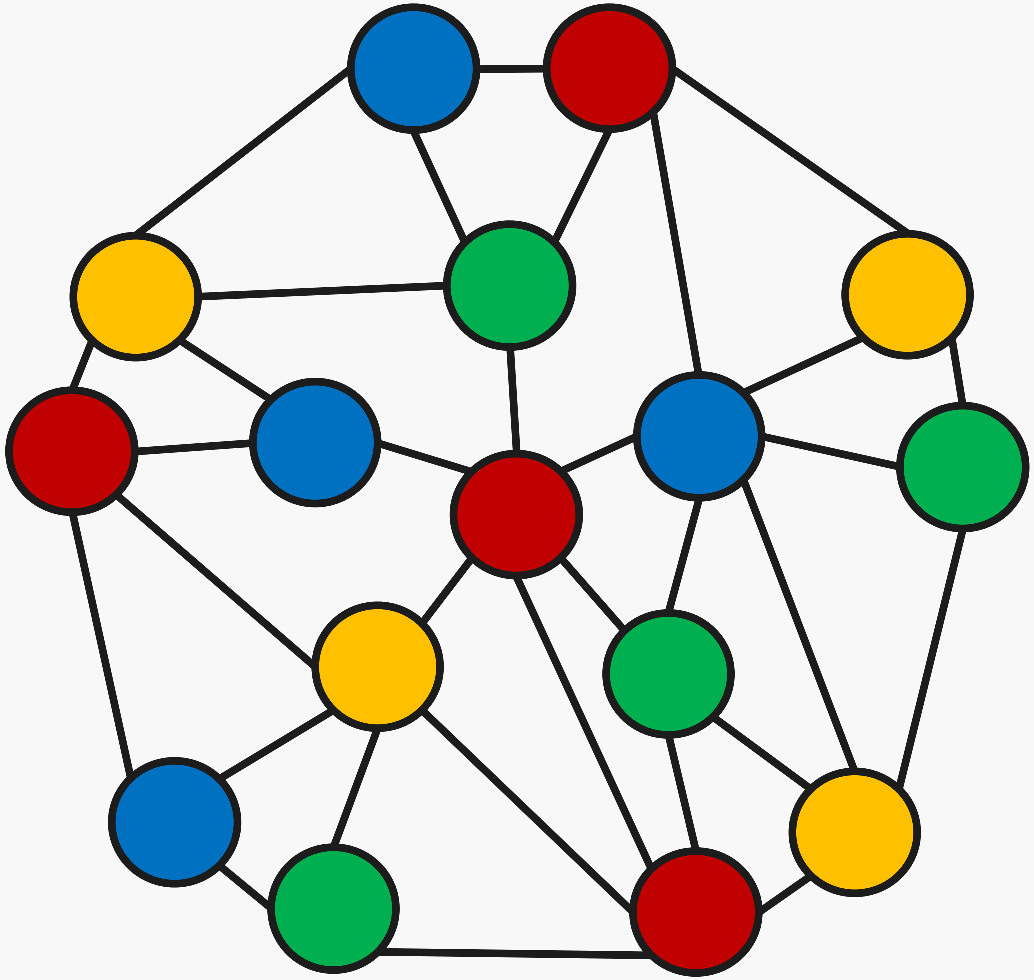
So if you pick satellites such that all other vertices have only satellites next to them, it works.



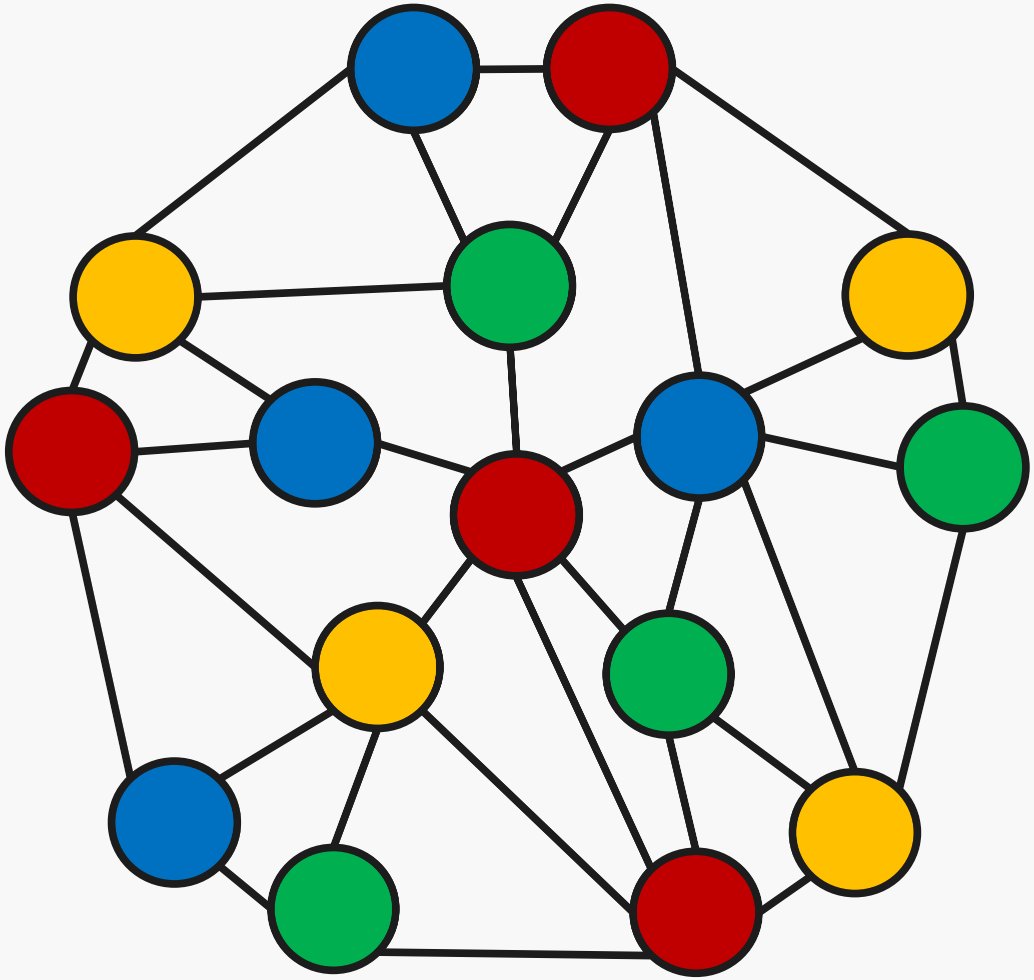
This is the same as none of the other vertices being next to each other.

Four-color theorem

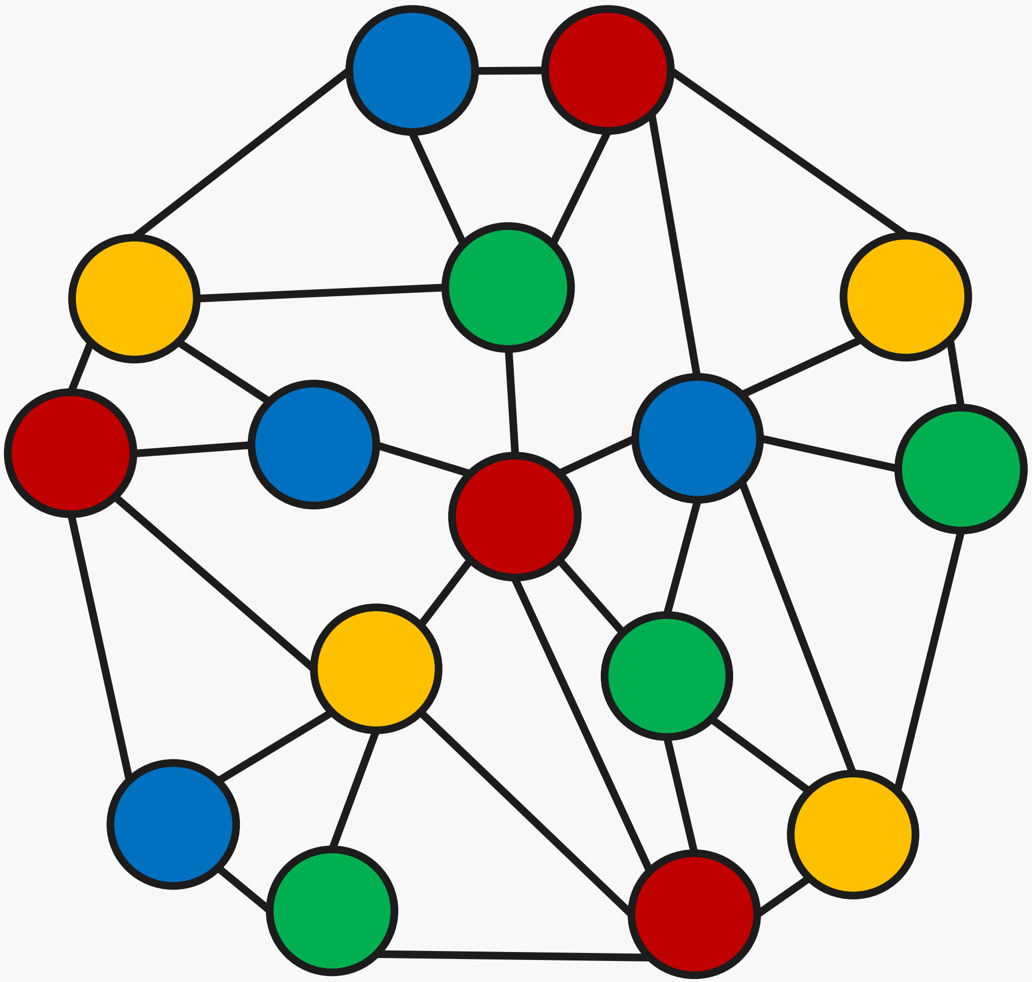
- One way is through **coloring**.
- **Four-color theorem**: In any planar graph, you can color the vertices red, green, yellow and blue, such that no two vertices next to each other are the same color.



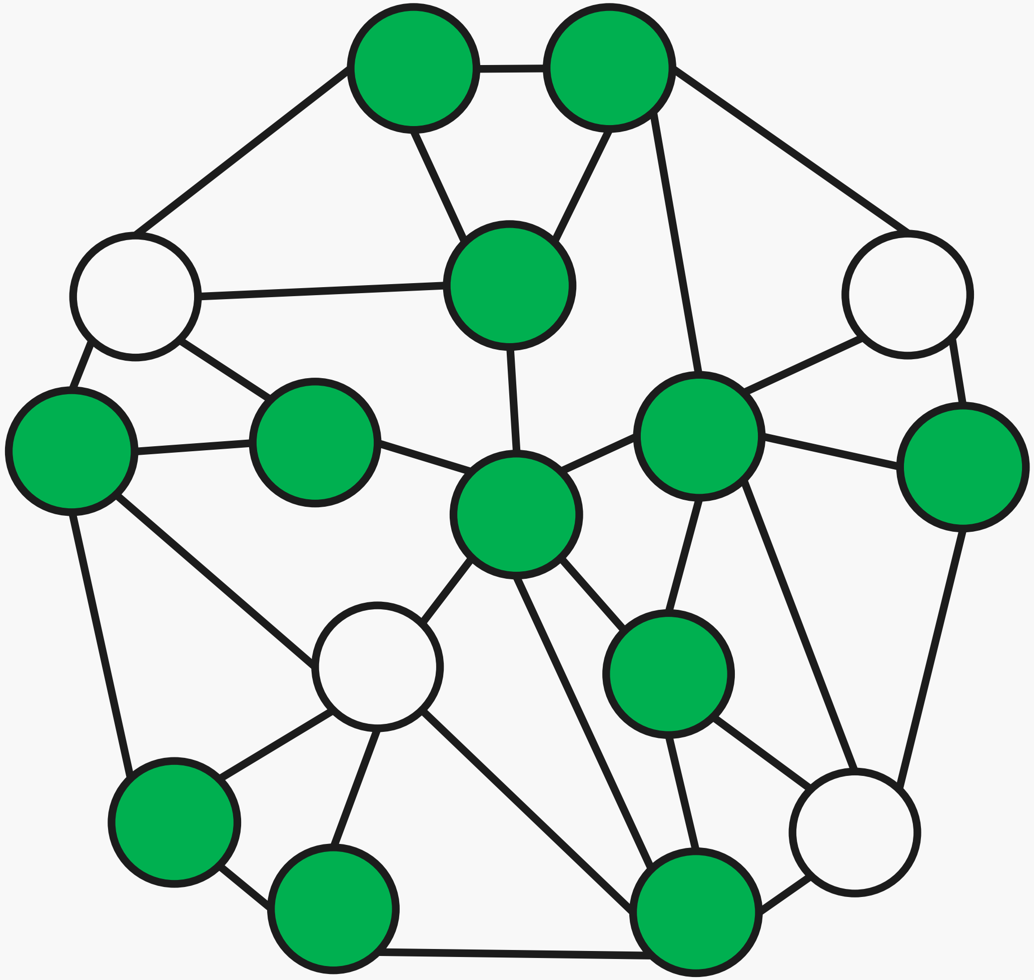
Remember that we need to pick satellites such that all other vertices have only satellites next to them.



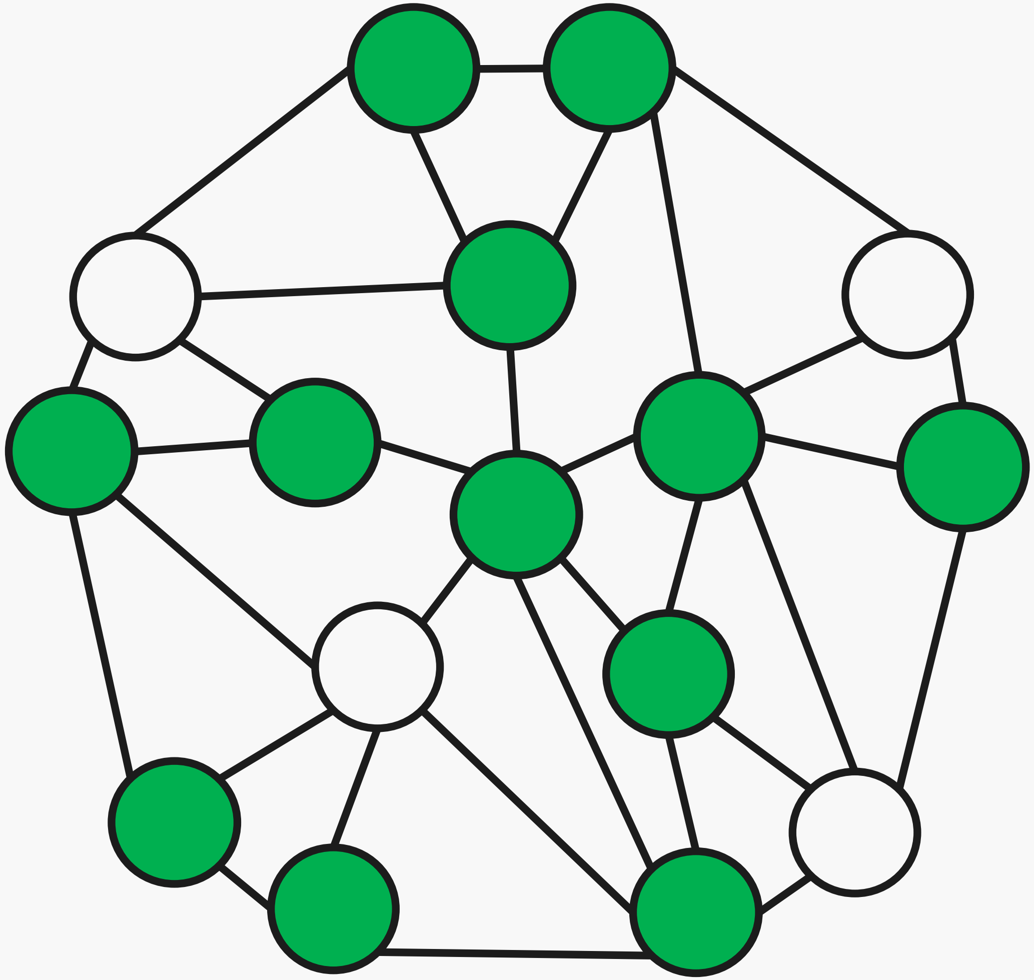
This is the same as choosing non-satellites
so that none are next to each other.



What if we make all colors except one a satellite?
Does this work?



The vertices that aren't satellites can't be next to each other, because they were the same color.



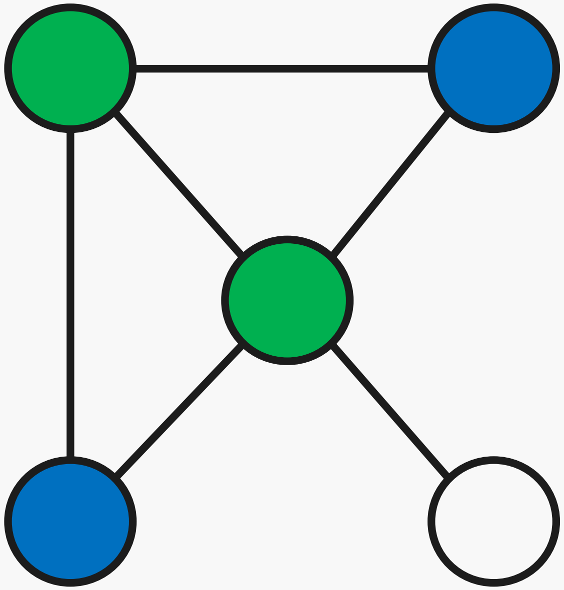
One of the colors must have at least N/4 vertices.
Taking everything except that color gives 3N/4.

The problem

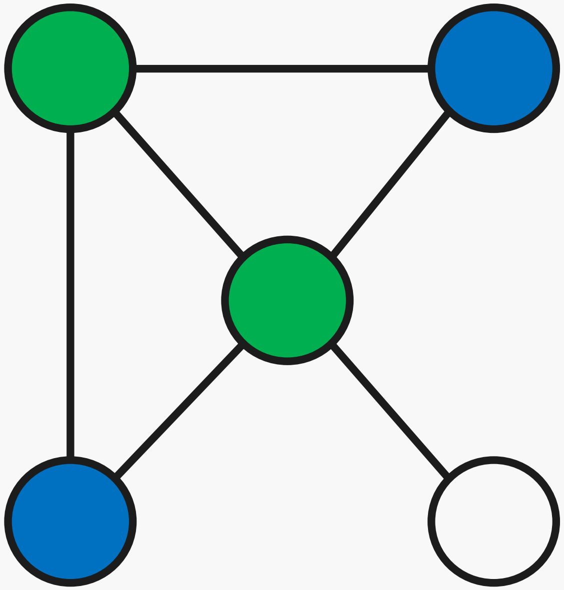
- If non-satellites only have satellites next to them, and that non-satellite is the **only vertex** next to all the satellites, then it works.

The problem

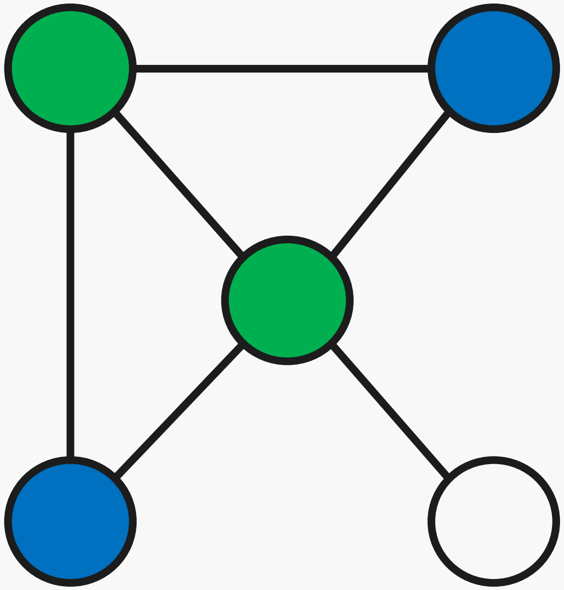
- If non-satellites only have satellites next to them, and that non-satellite is the **only vertex** next to all the satellites, then it works.
- But what if there are two vertices that are next to the same set of vertices? Then we have a problem.



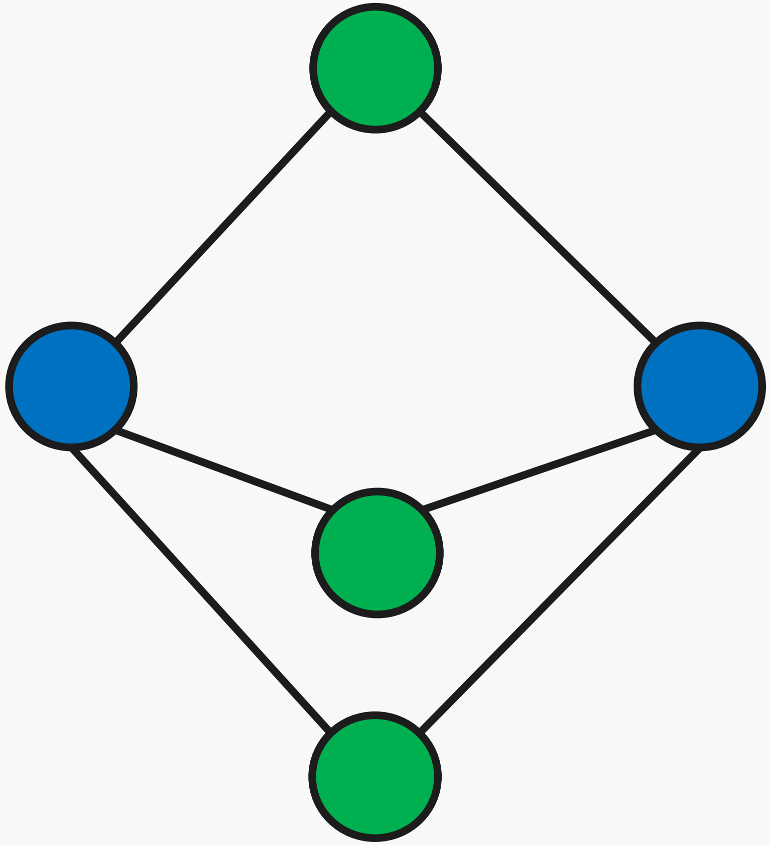
The problem is when there are several vertices that have the same vertices next to them.



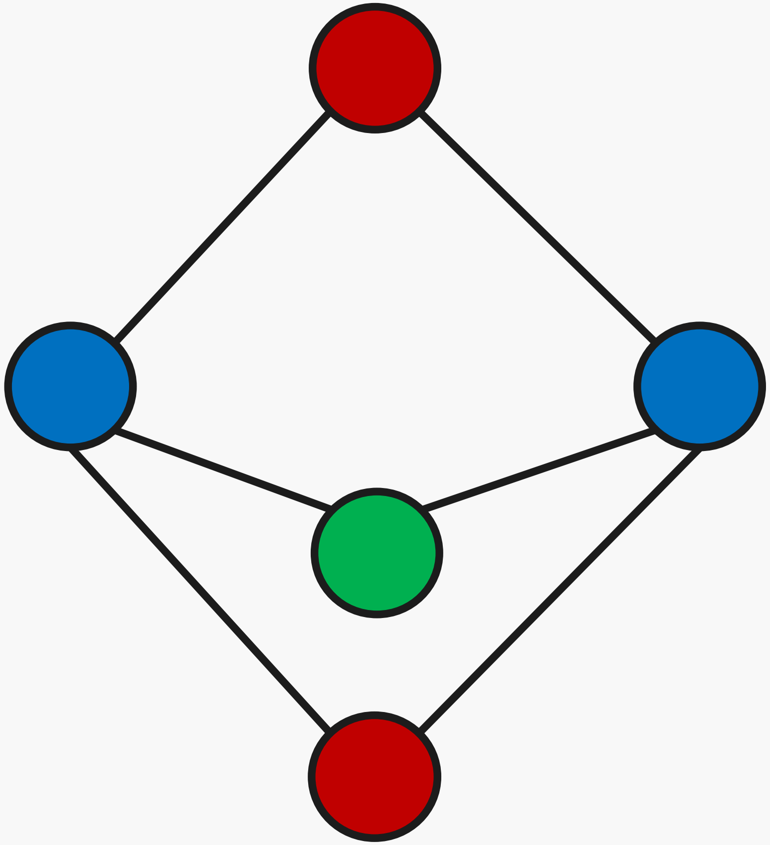
We will show that this problem almost never happens for maximal planar graphs.



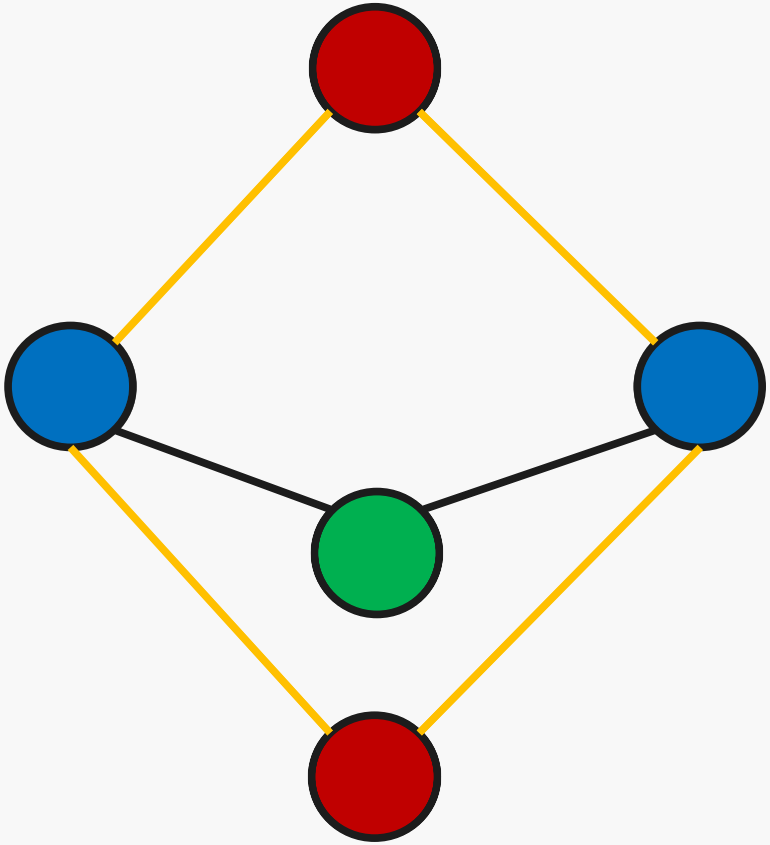
We will show that this problem almost never happens for maximal planar graphs.
We use the fact all its faces are triangles.



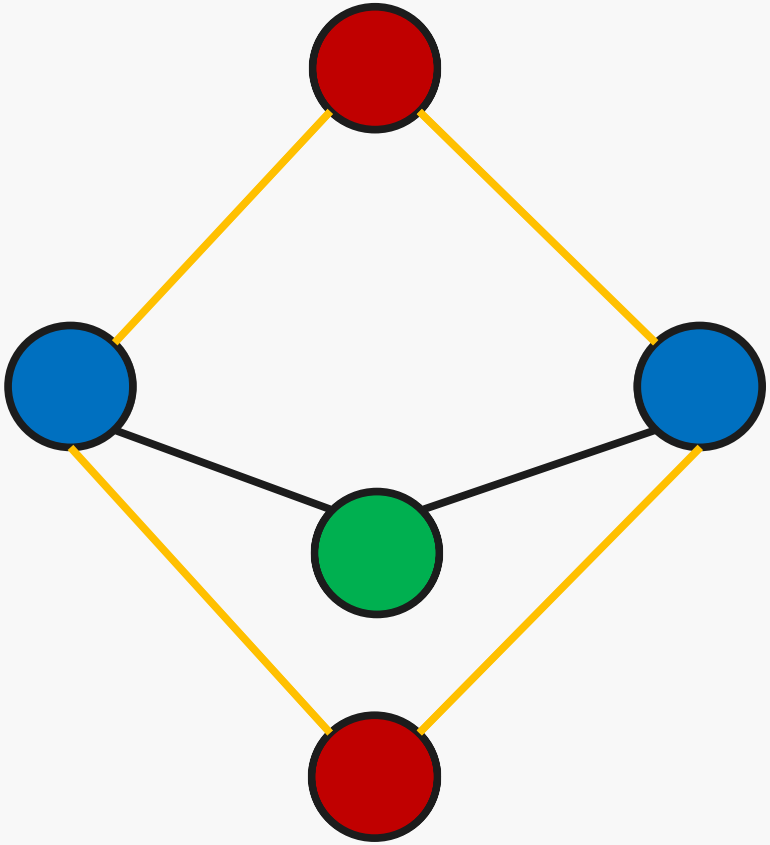
Suppose a maximal planar graph has two vertices with the same vertices next to them.



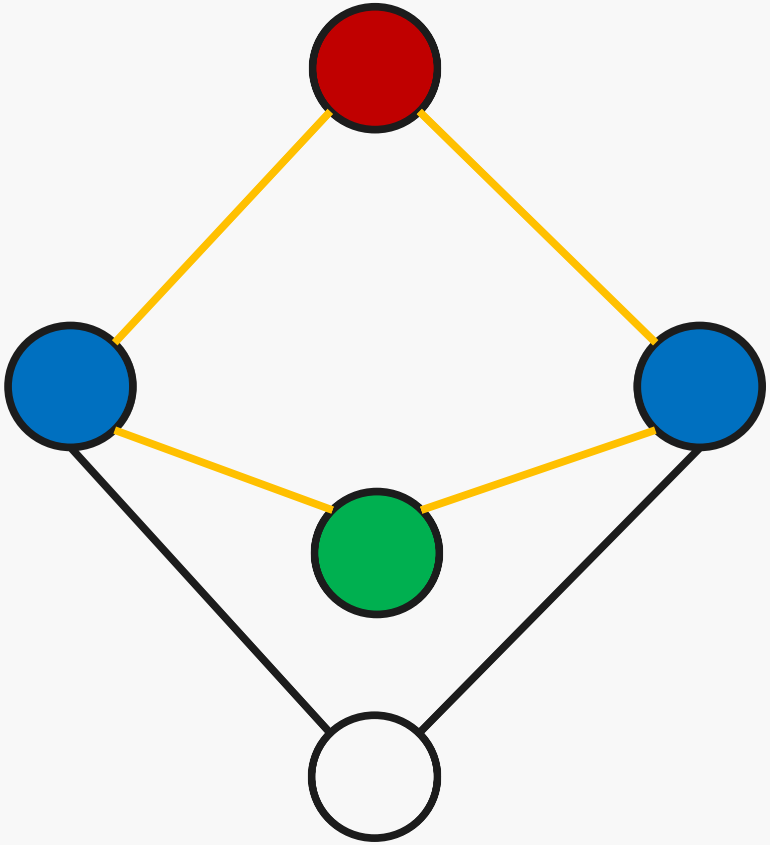
Choose two of the vertices next to them.



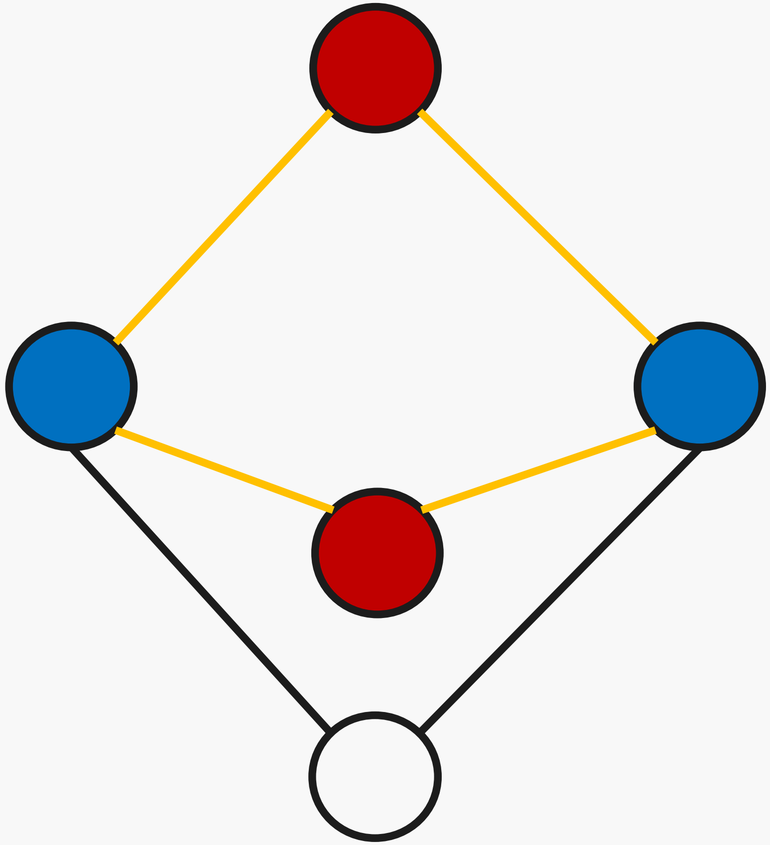
Consider the square formed by the two vertices and the two vertices next to them.



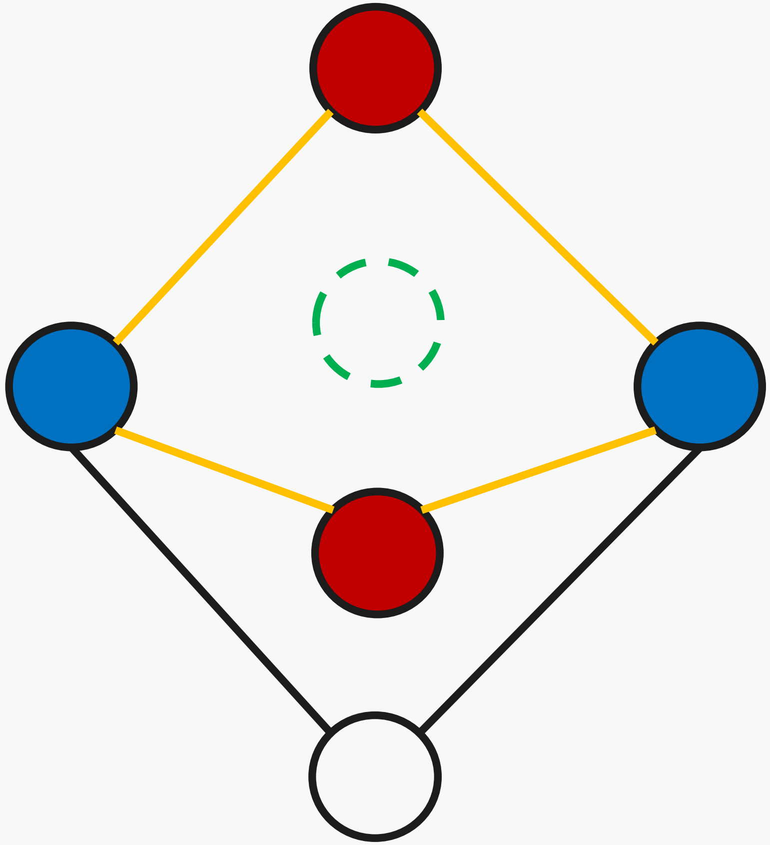
If the square has a vertex that is next to both…



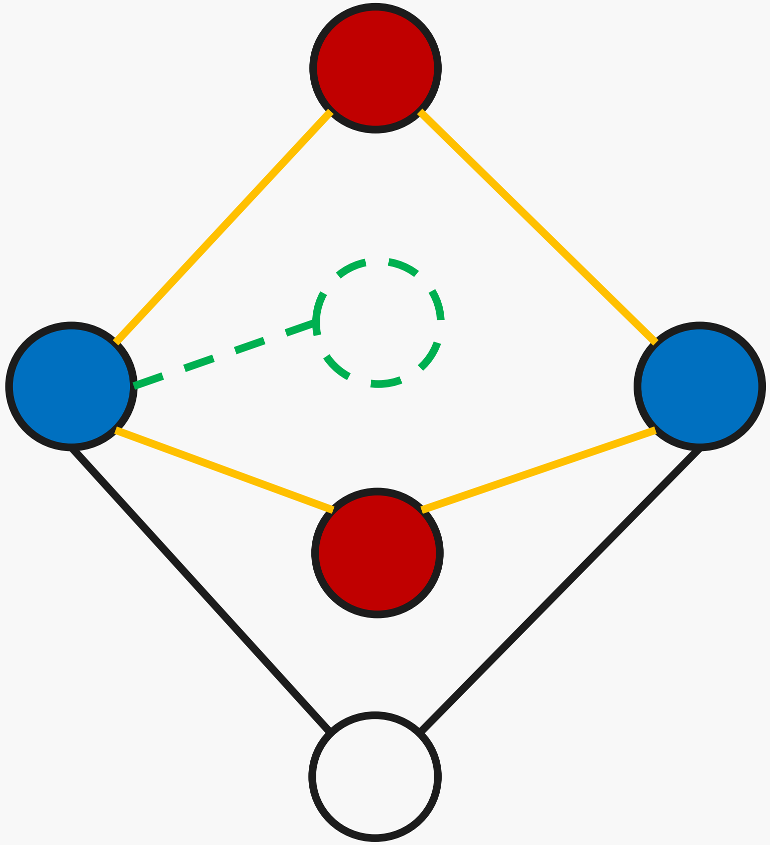
Pick that vertex instead.



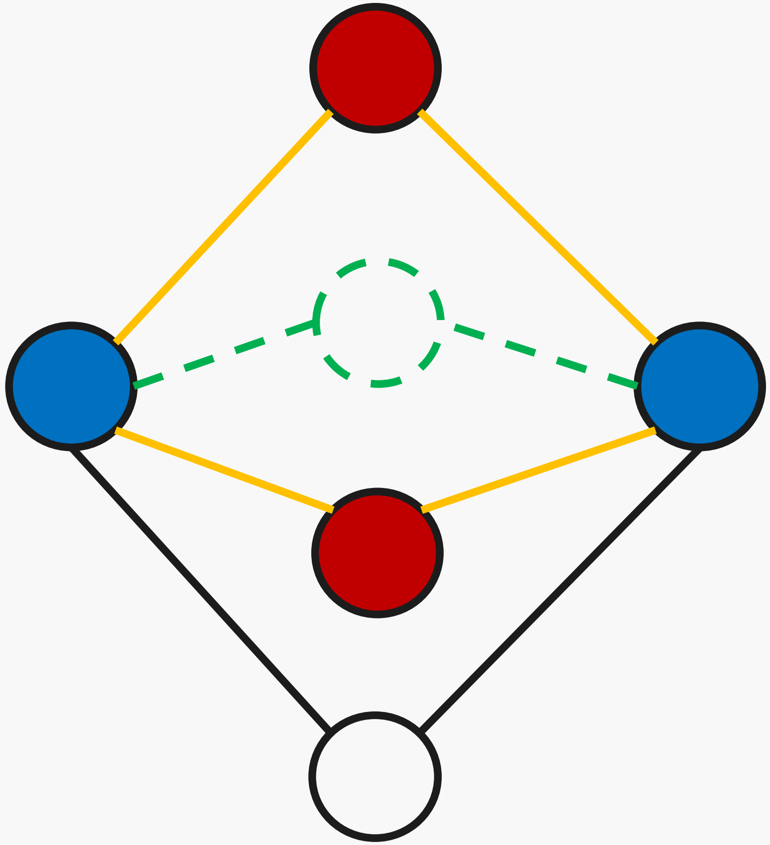
If we keep doing this, the square will have no vertices next to both.



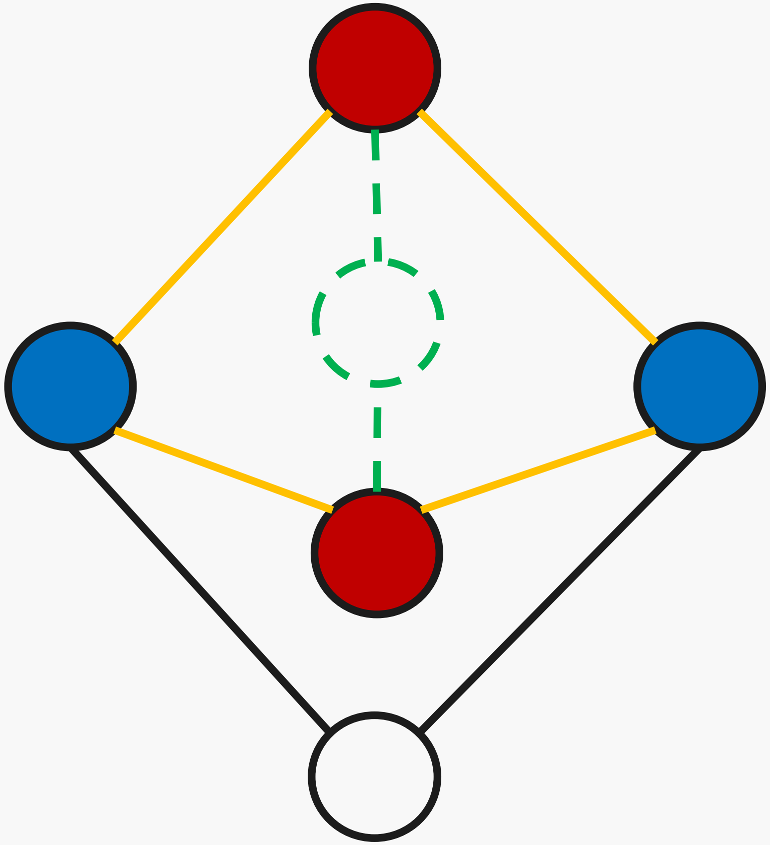
Can we have a vertex that is **not** next to both inside?



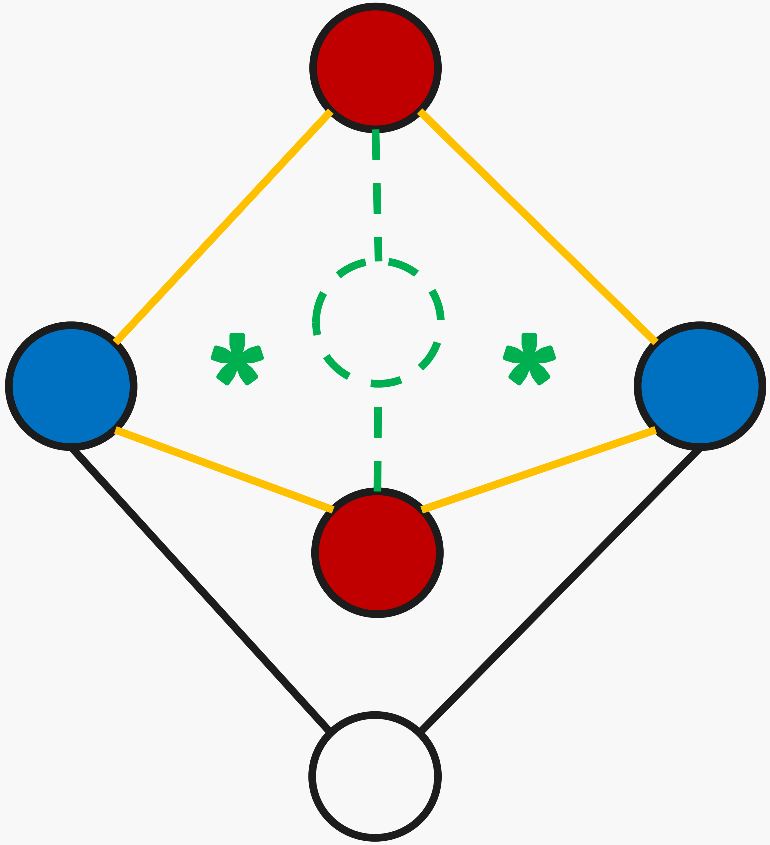
It can’t be next to one of them,



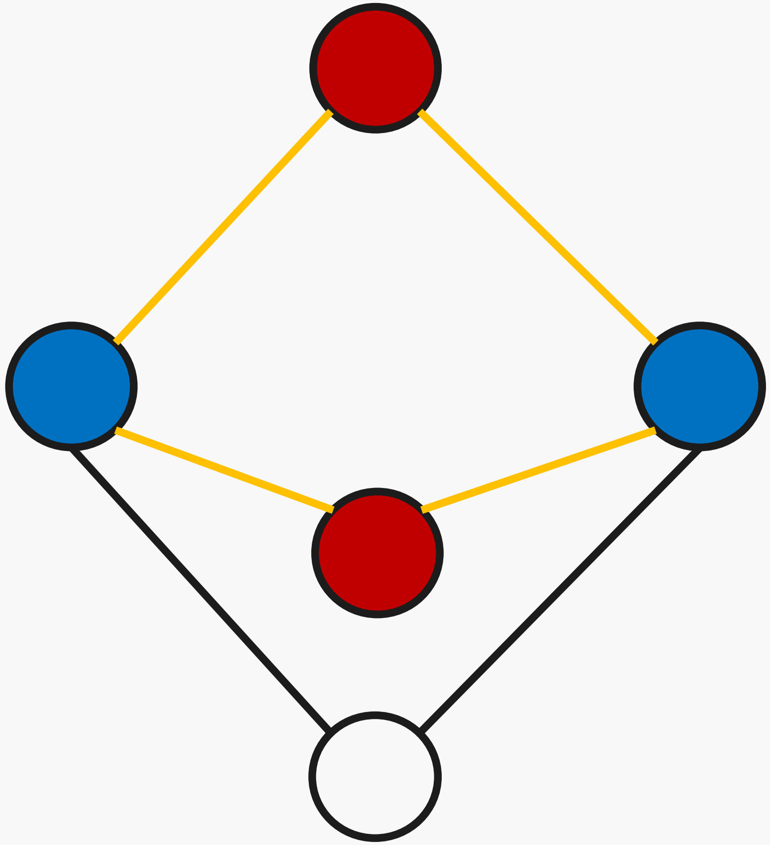
It can't be next to one of them,
because it has to be next to the other.



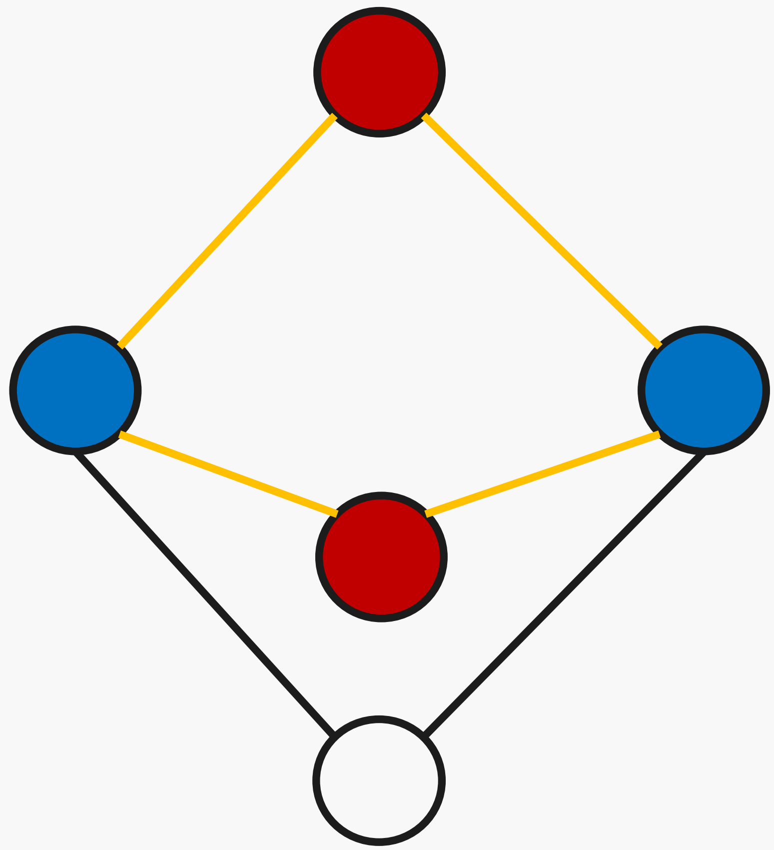
So it has to be connected like this.



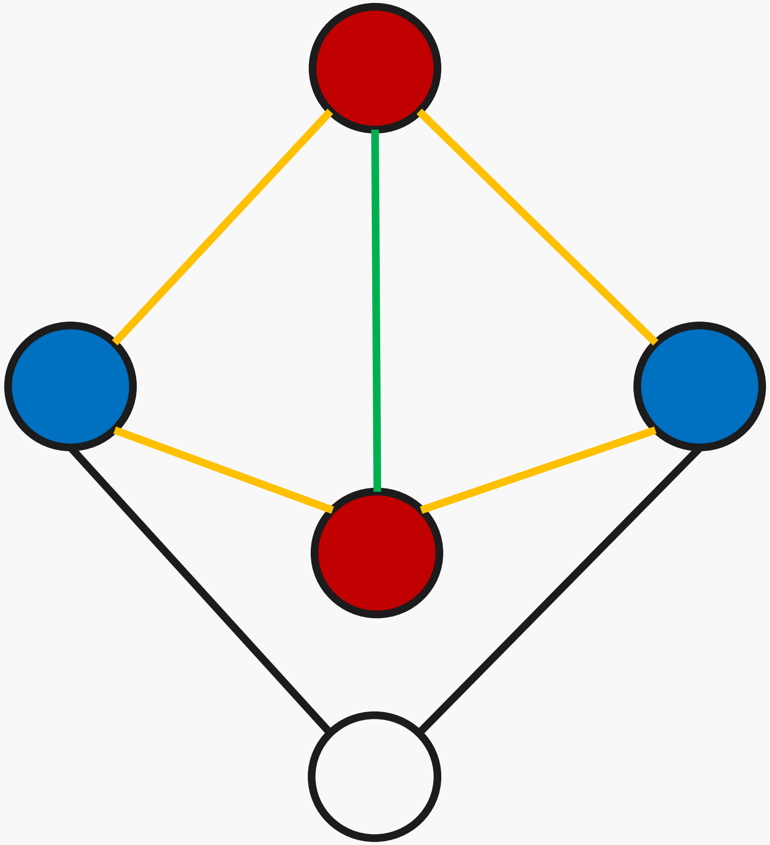
But then the starred faces wouldn't be triangles!



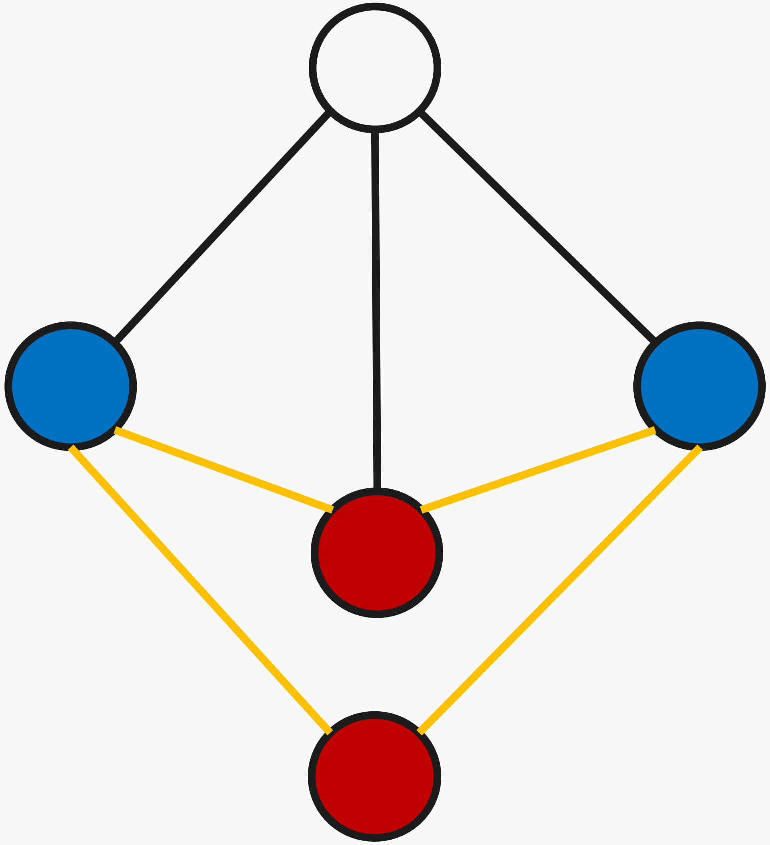
So the square doesn’t have any more vertices.



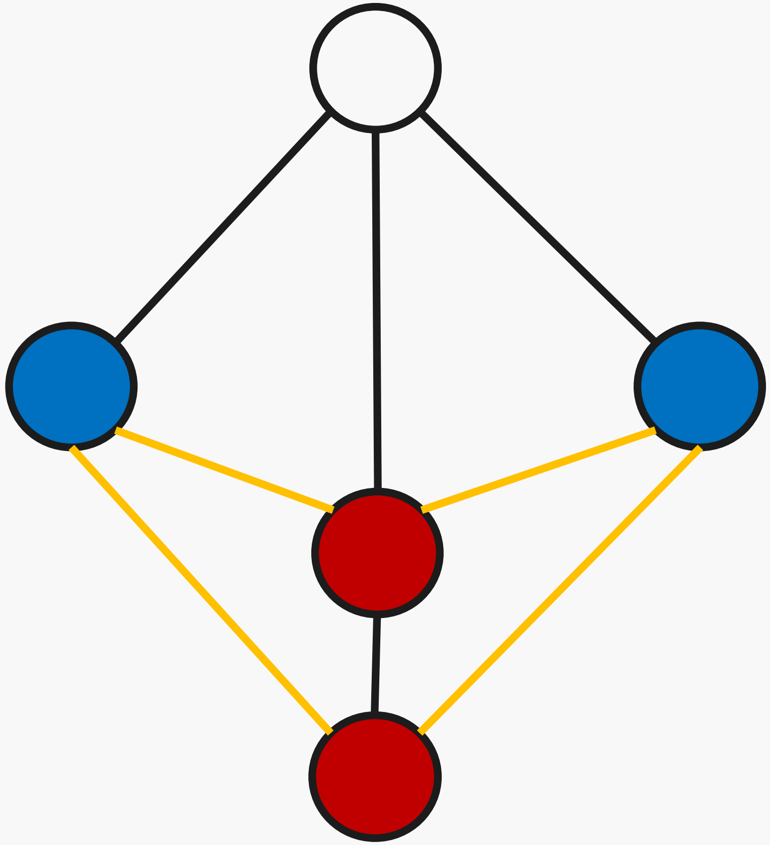
Since it is maximal planar, all faces are triangles.



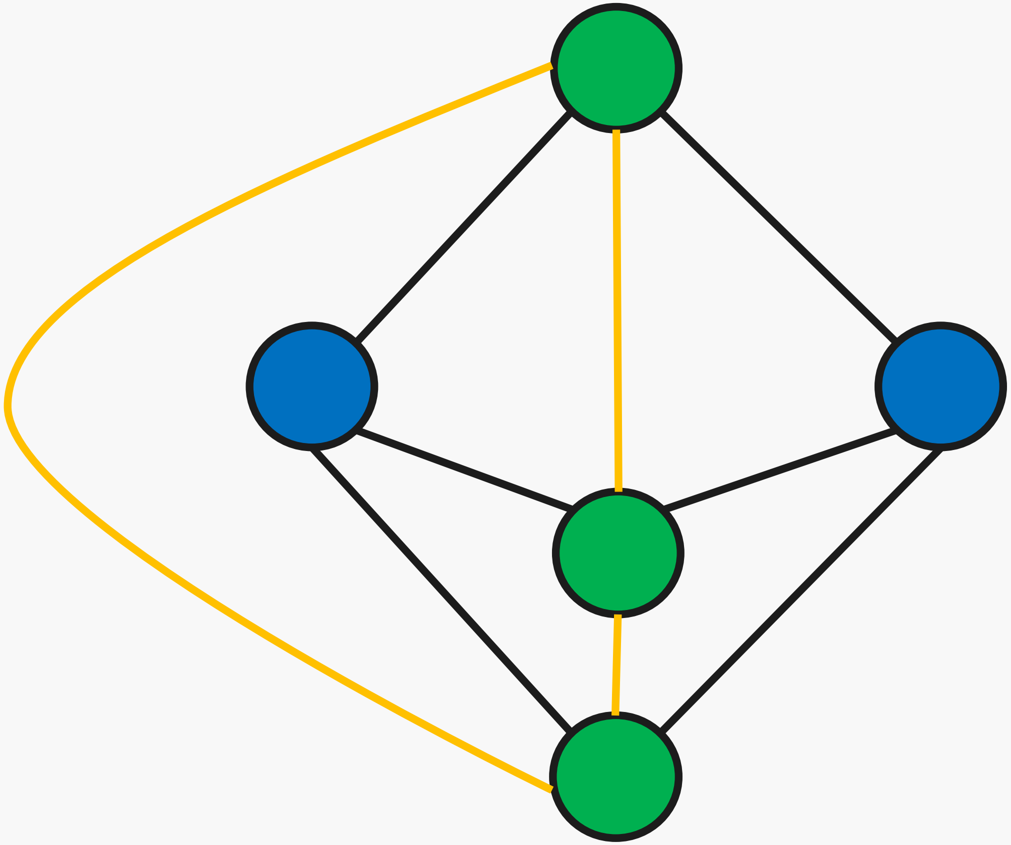
So the square has to have this edge.



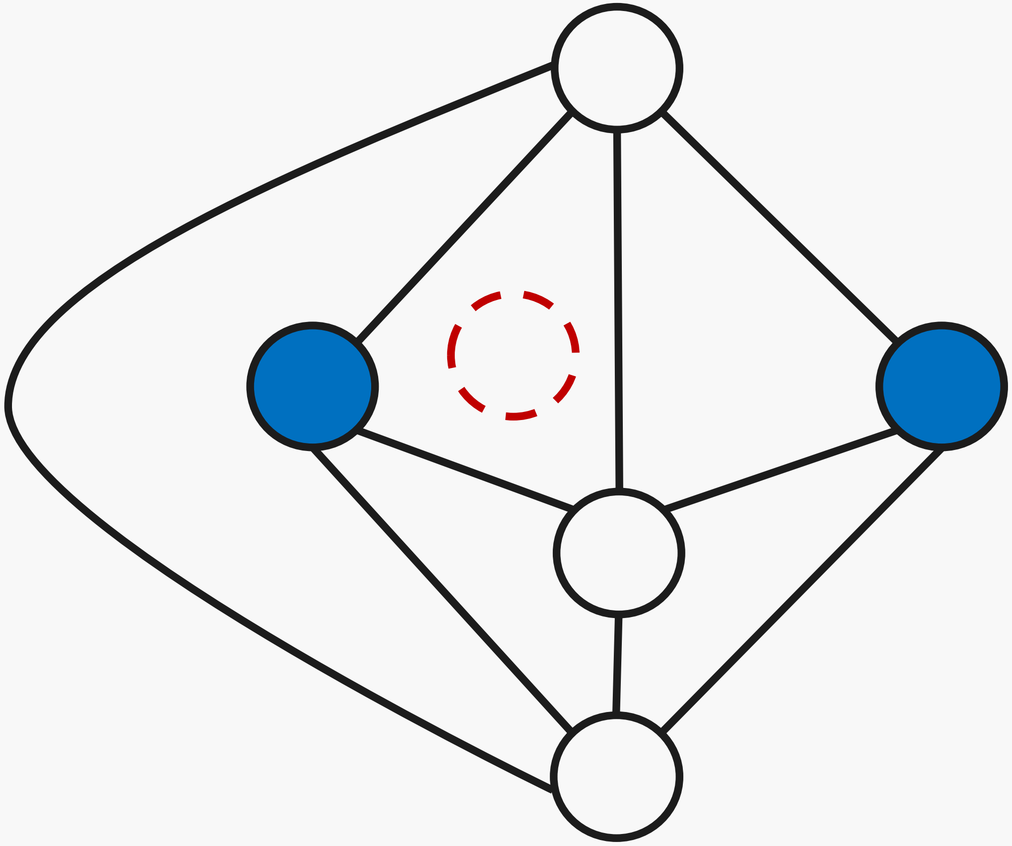
If we do this for other squares,



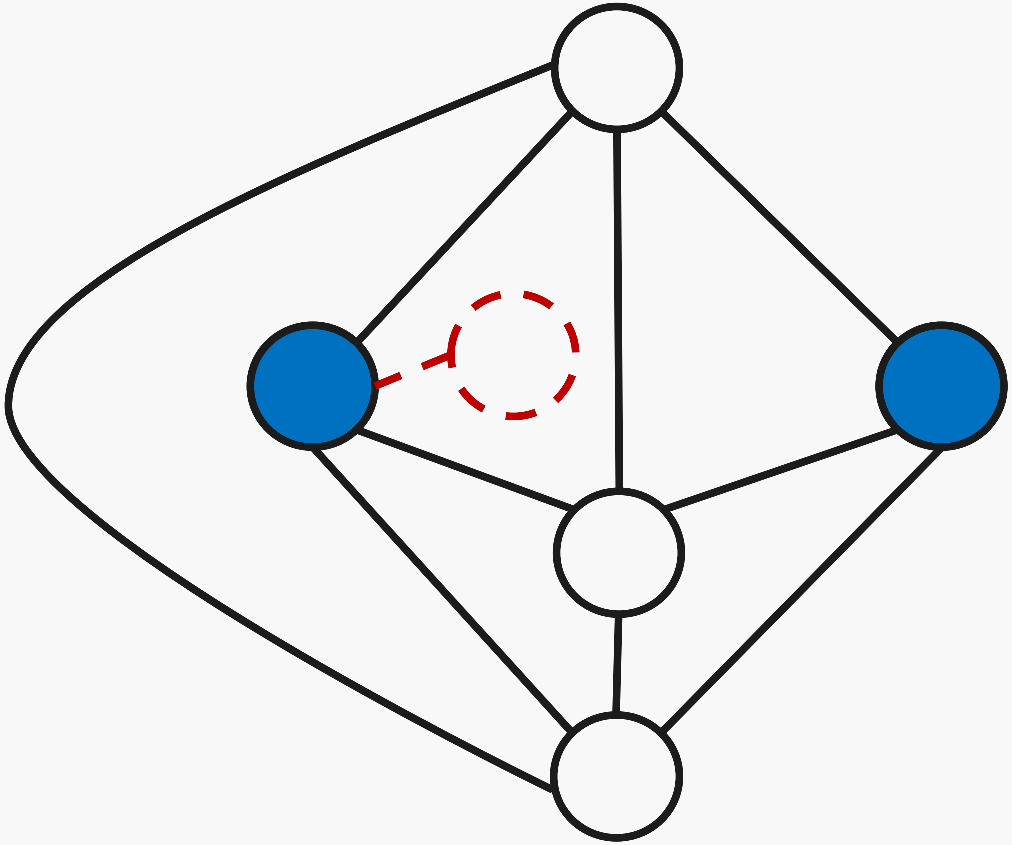
If we do this for other squares,
then these vertices have to be connected too.



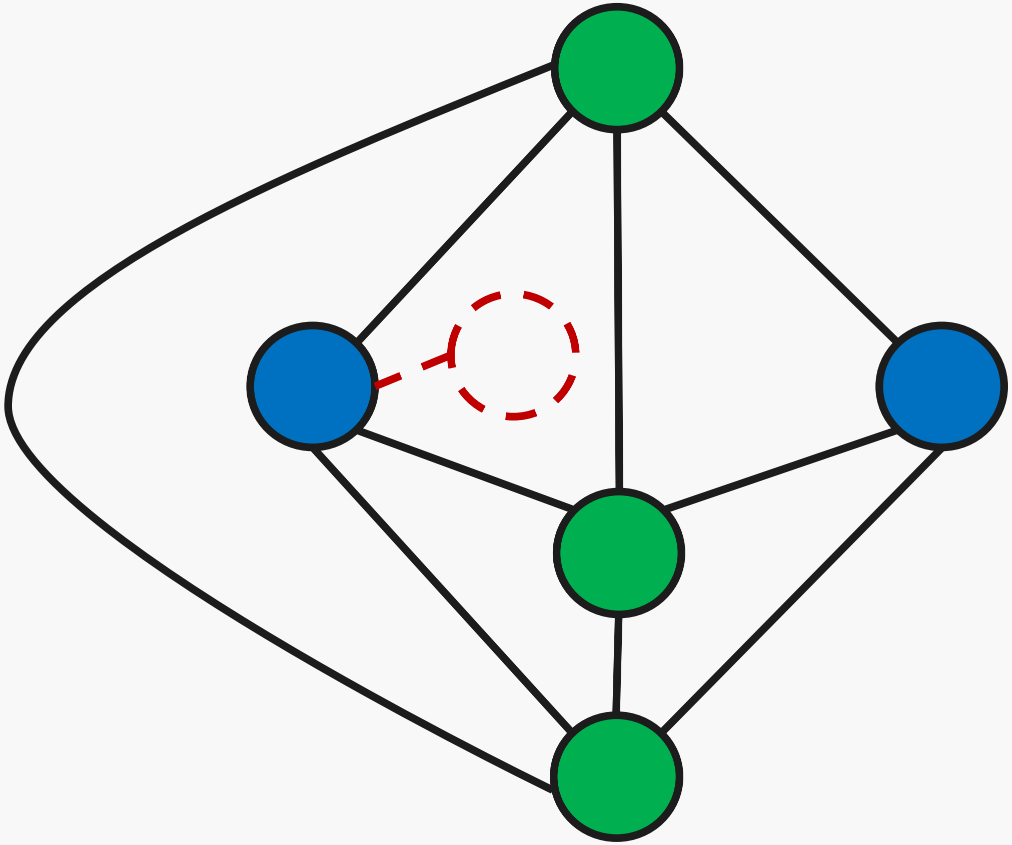
So all of the vertices next to both are in a cycle.



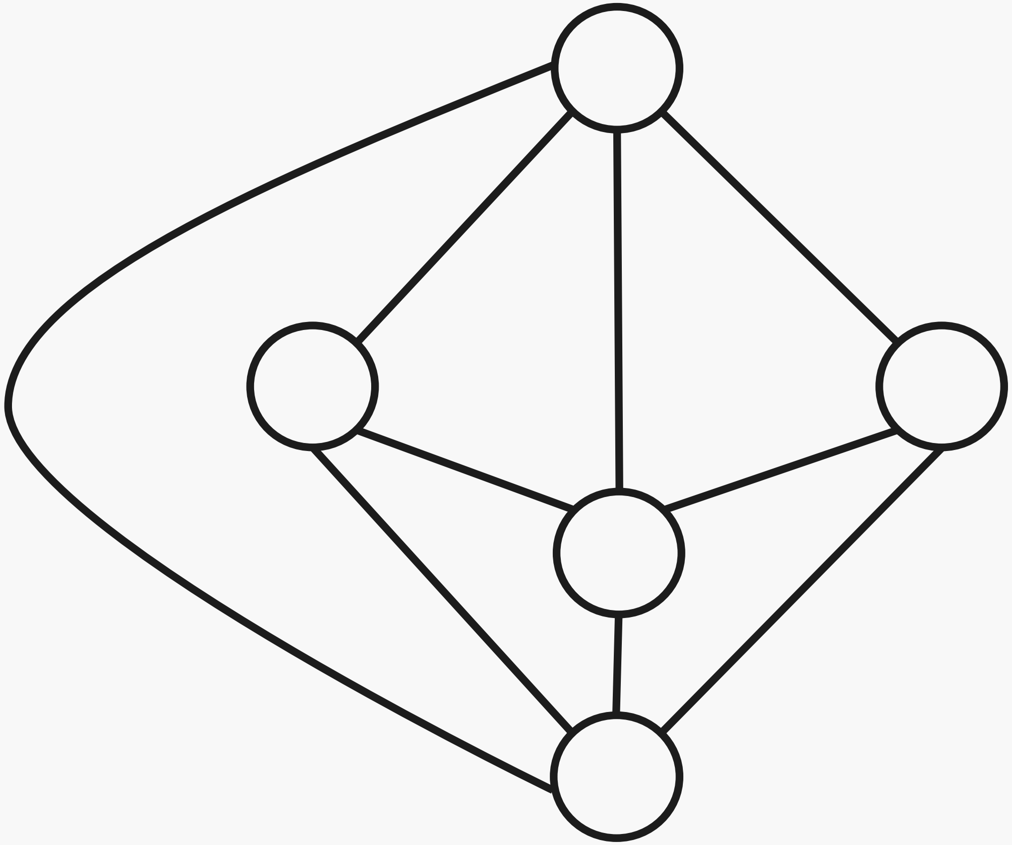
Can we have any other vertices?



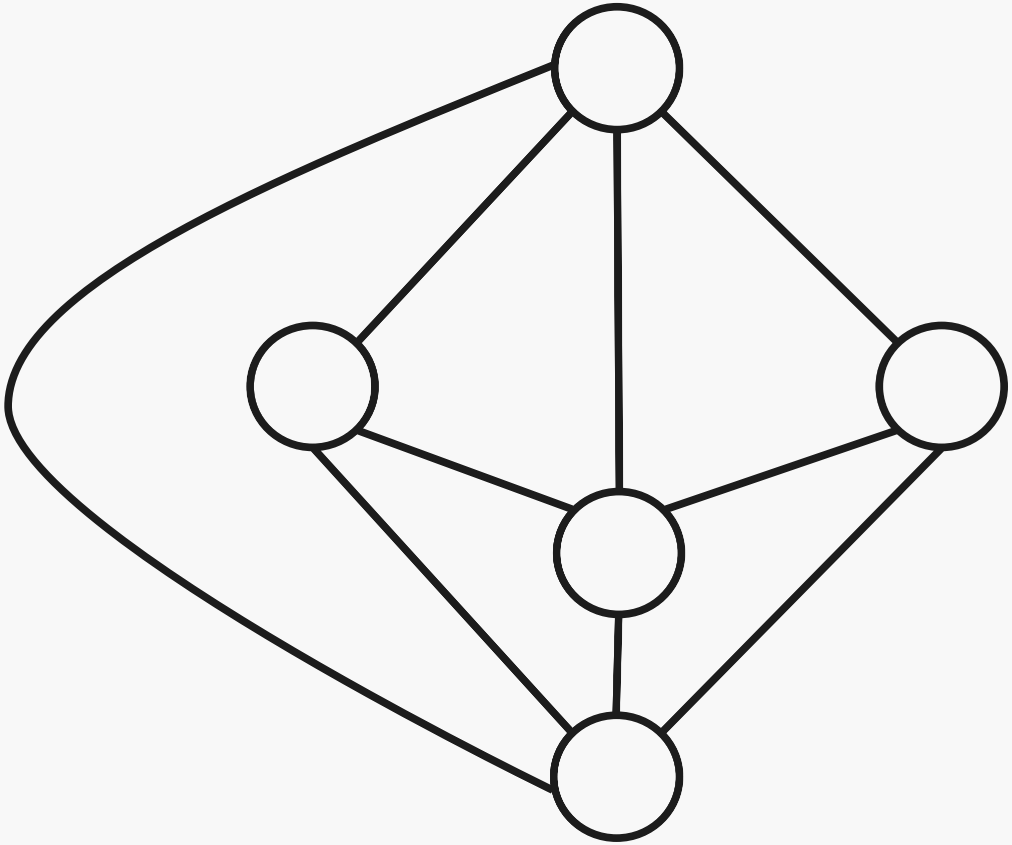
Can we have any other vertices?
No, because it has to be connected to one.



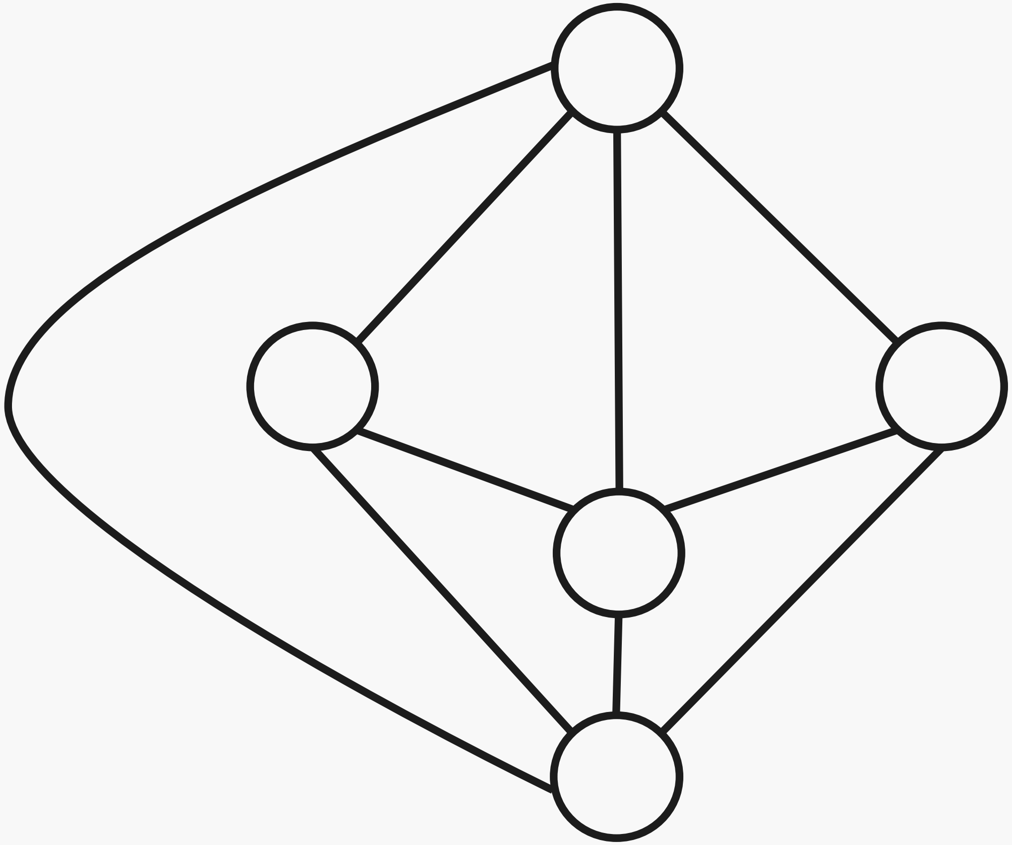
But it's not part of the vertices next to both!



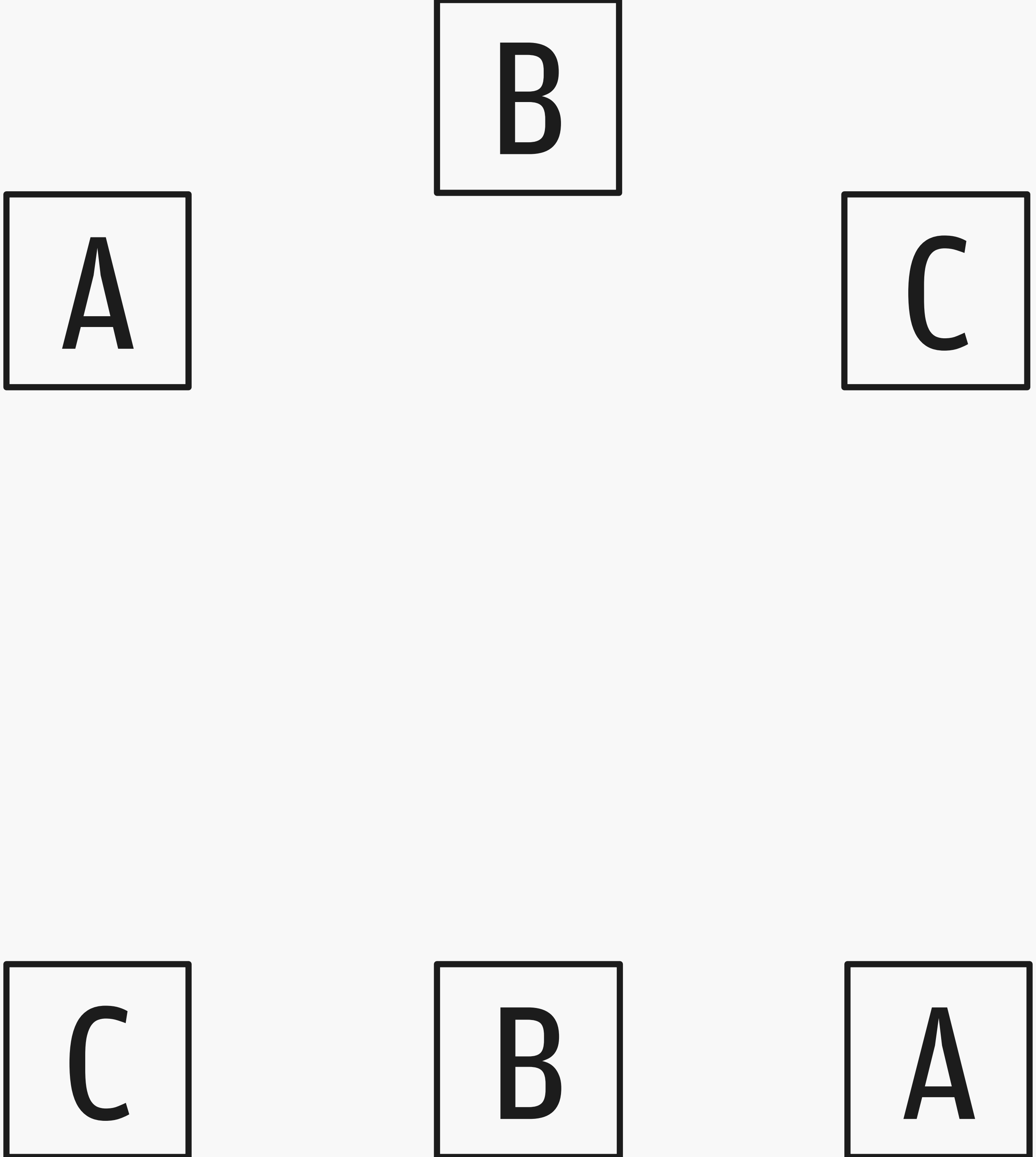
Only graphs like these can have vertices with the problem.

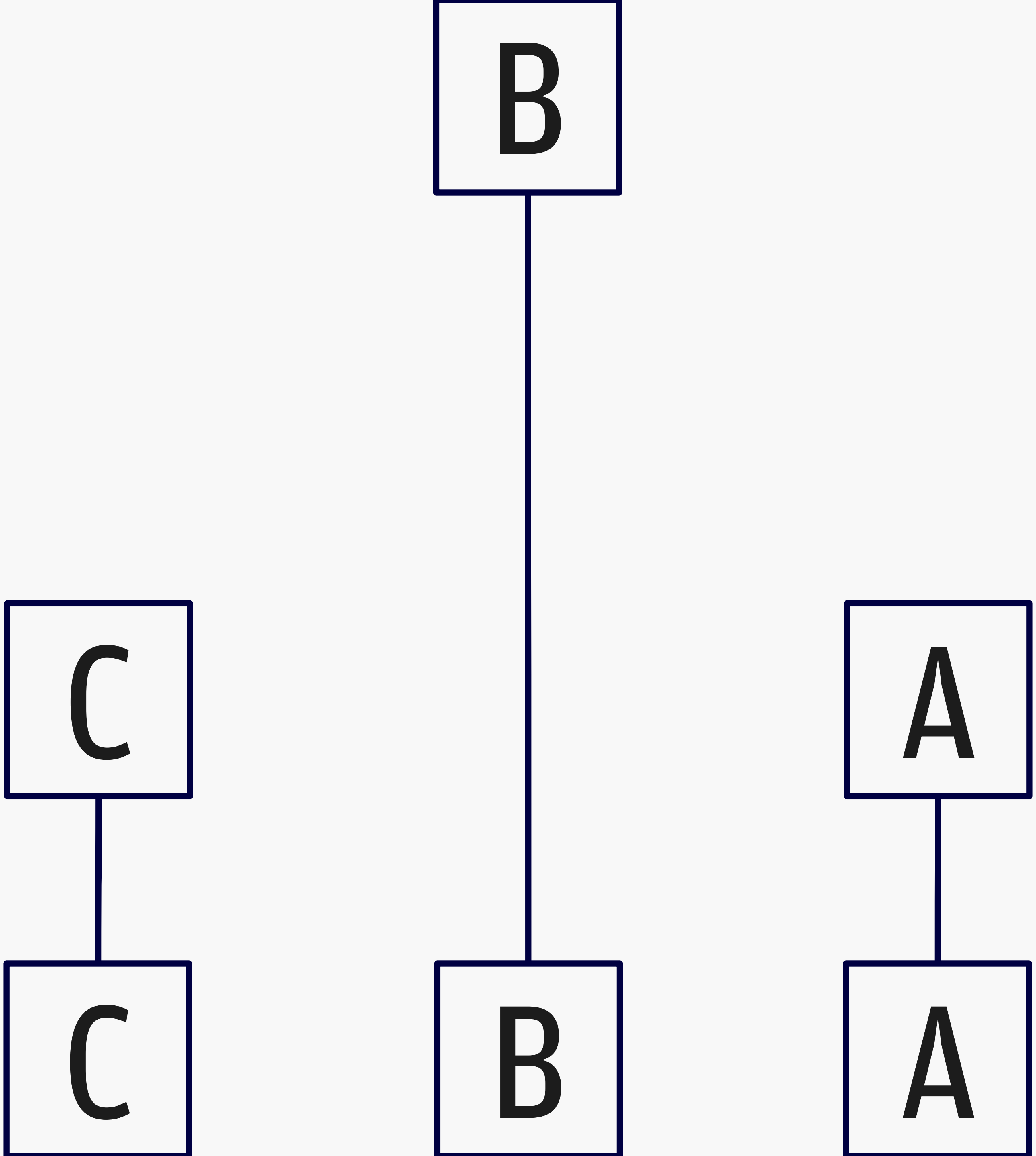


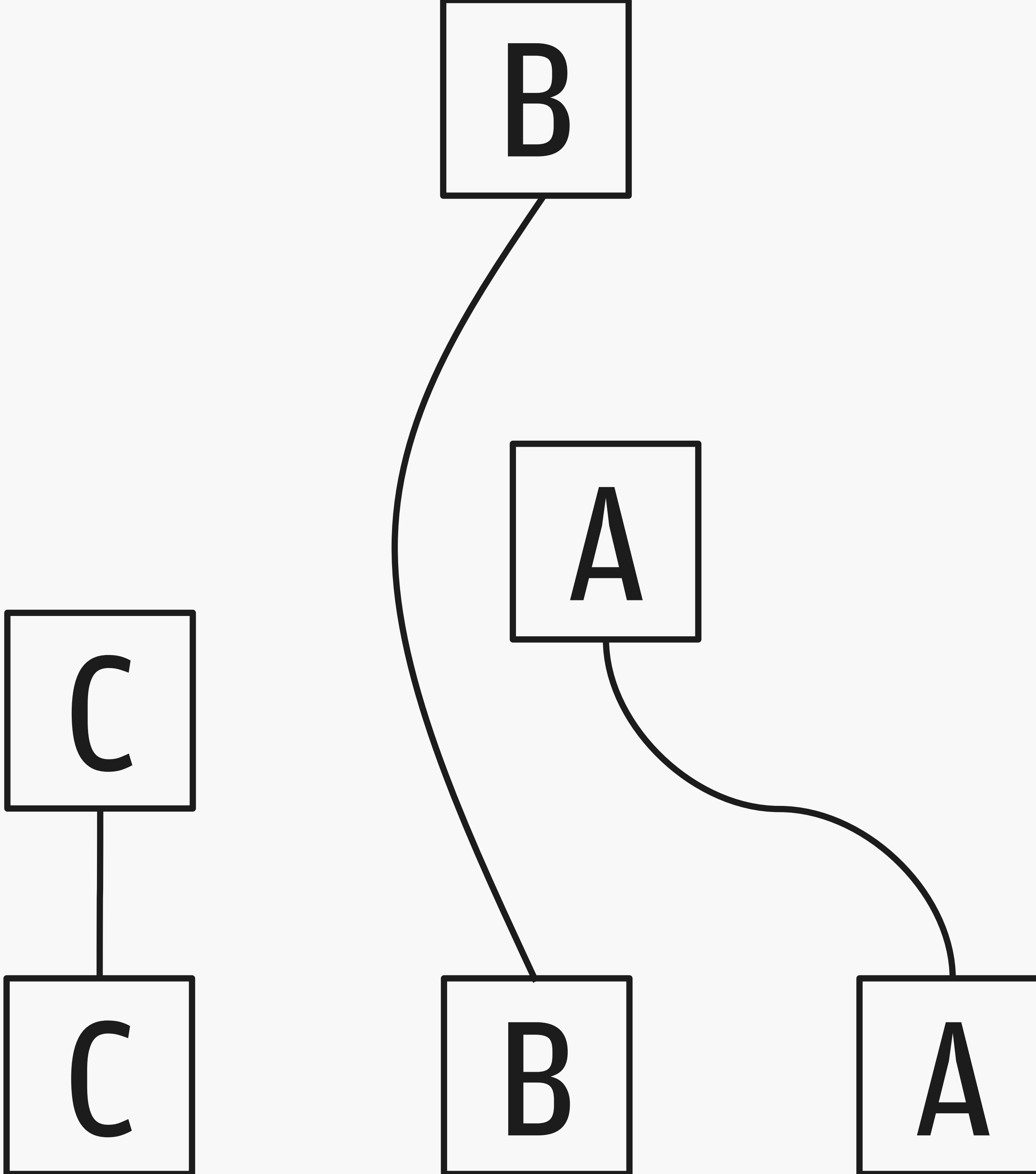
In fact, graphs like these have metric dimension of 2N/5, which is smaller than 3N/4!

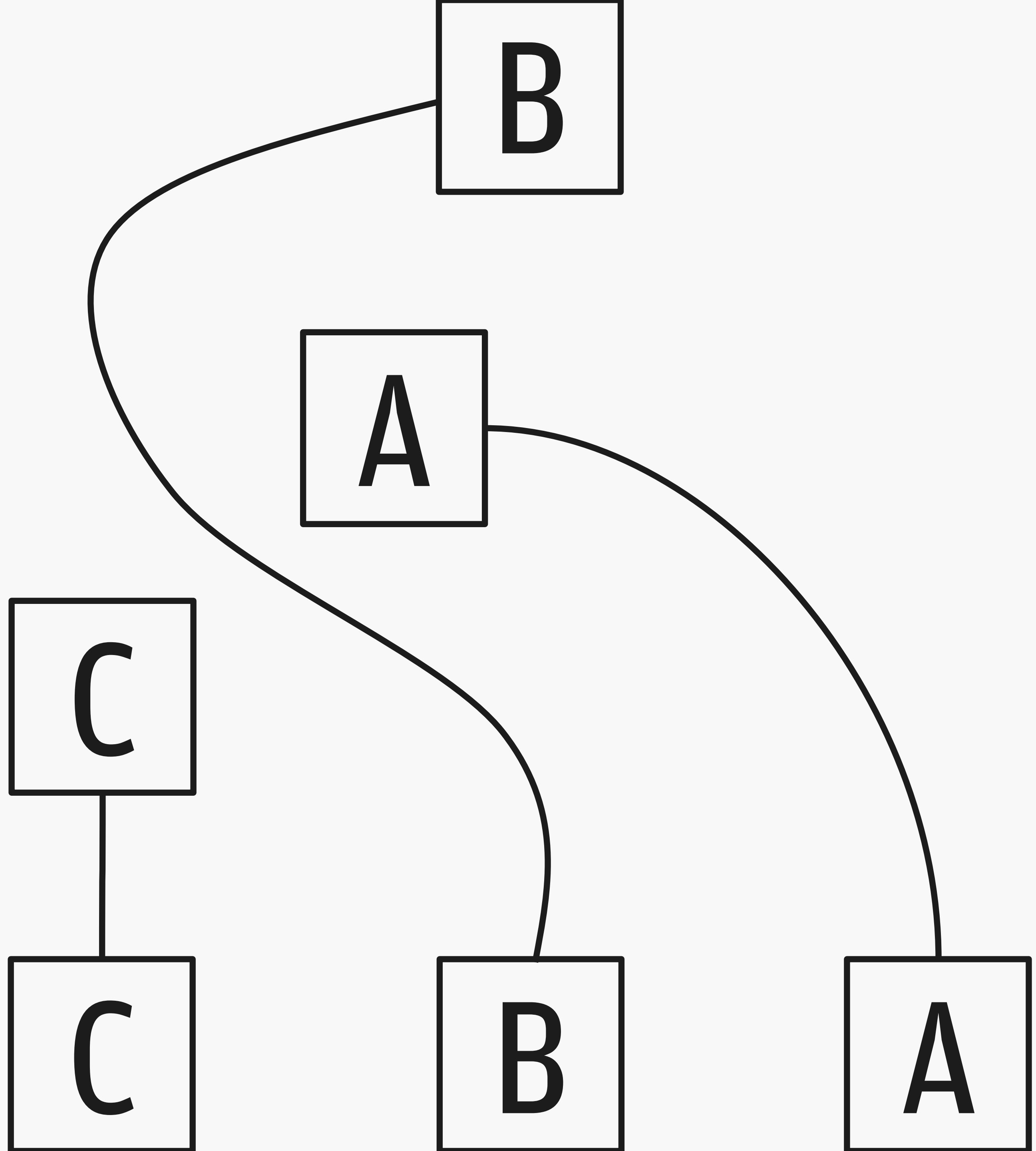


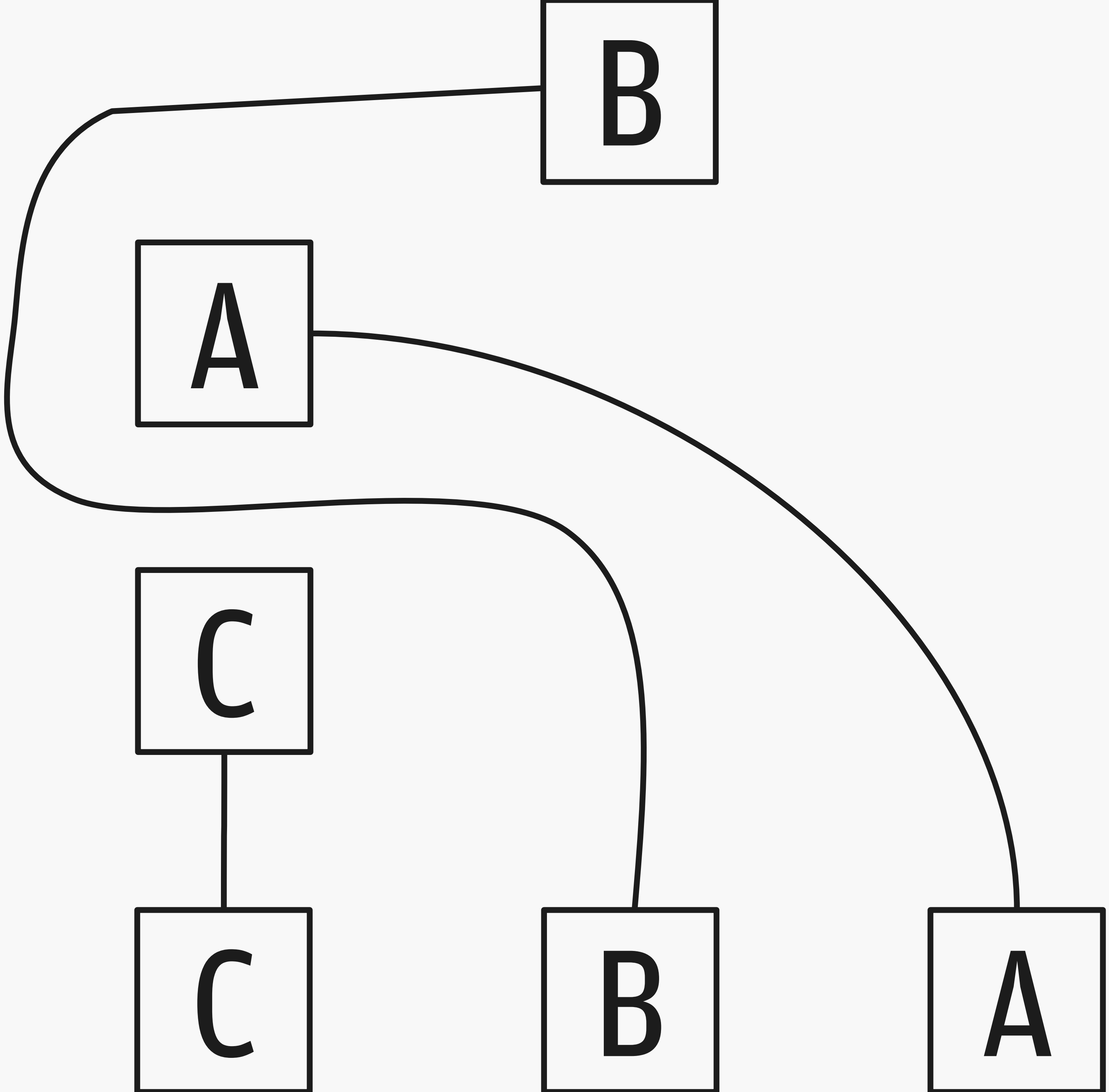
So all maximal planar graphs with N vertices have a metric dimension less than 3N/4.

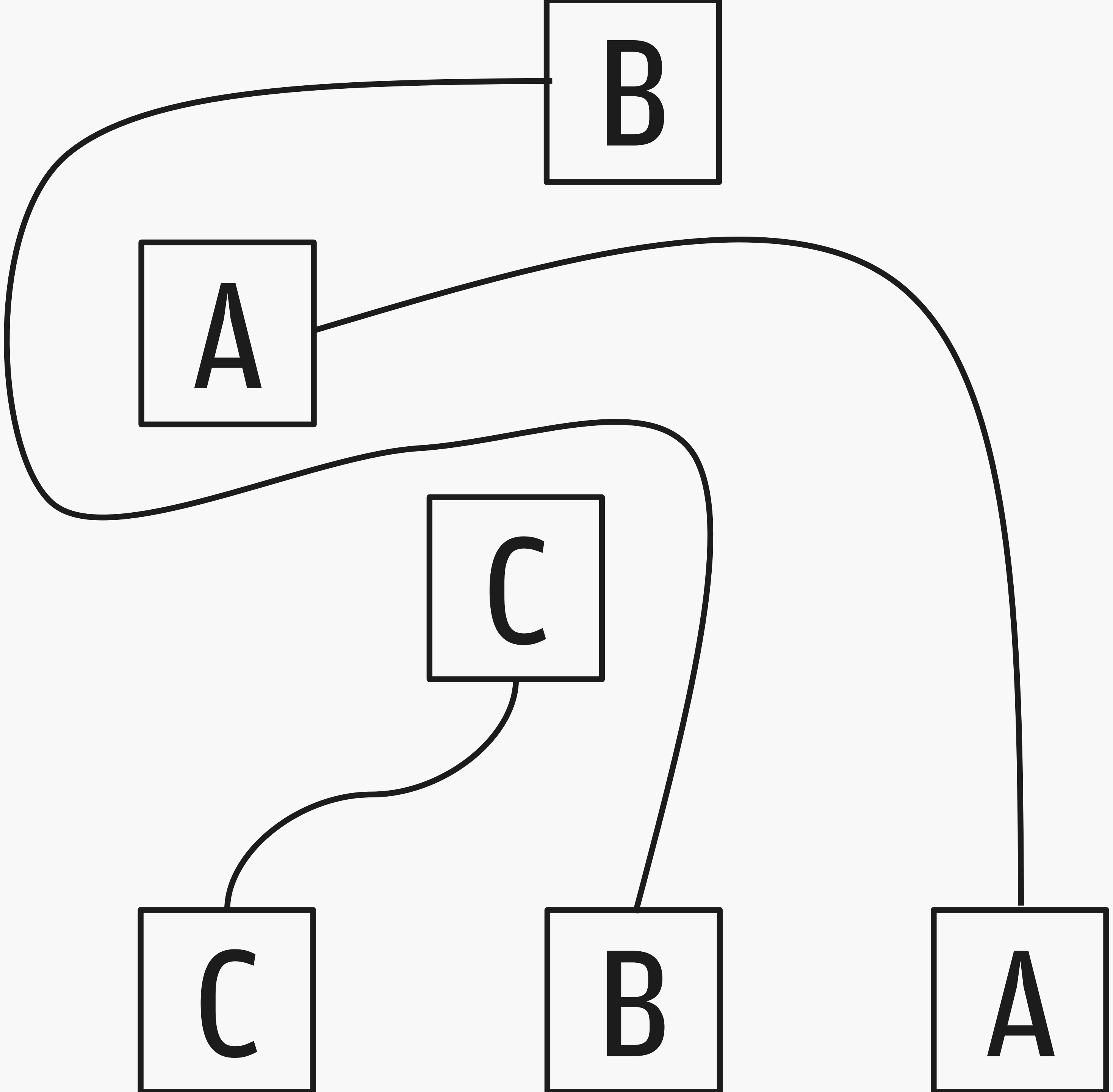

B
A
C
C
B
A

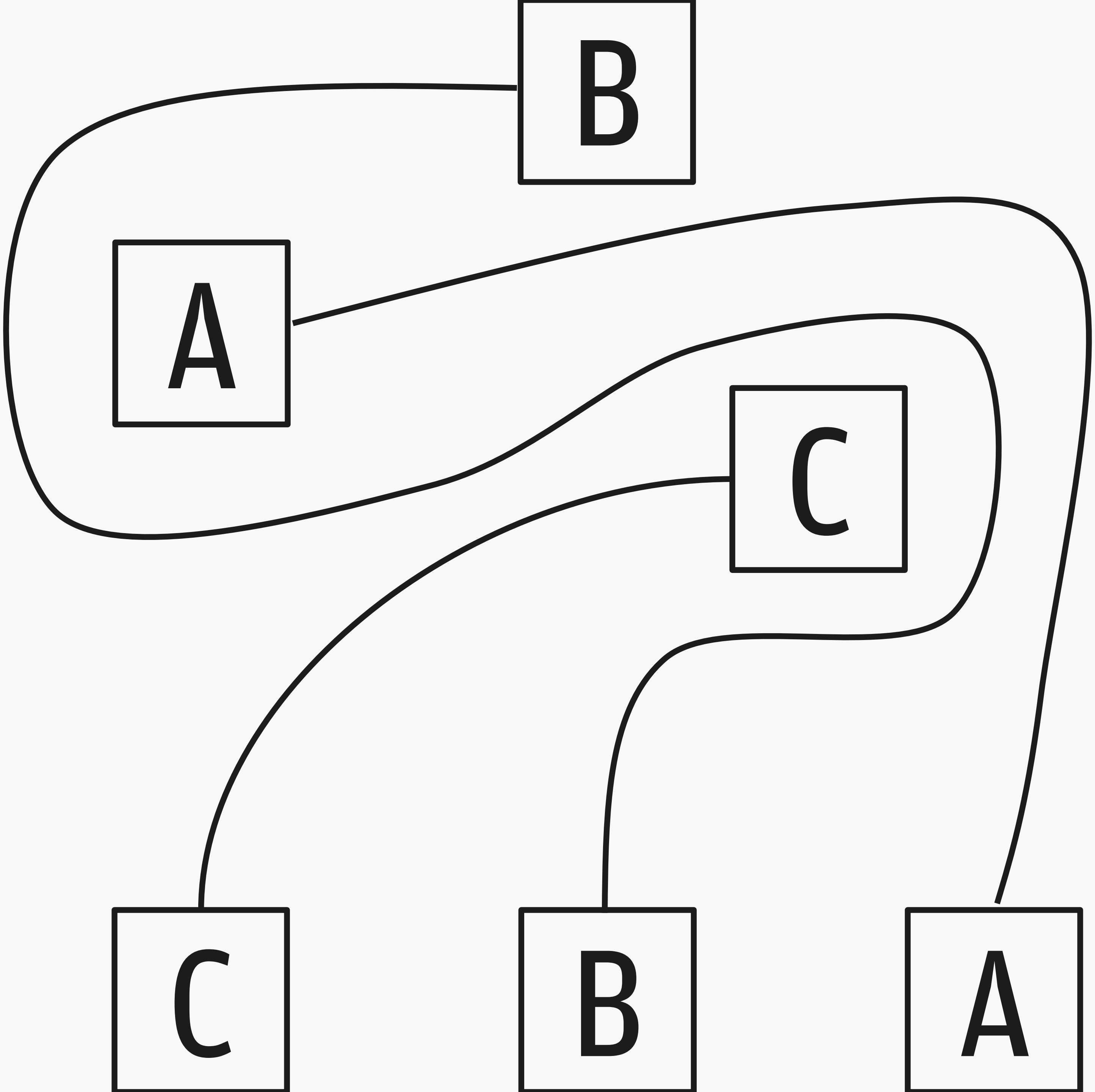

B
C
A
C
B
A

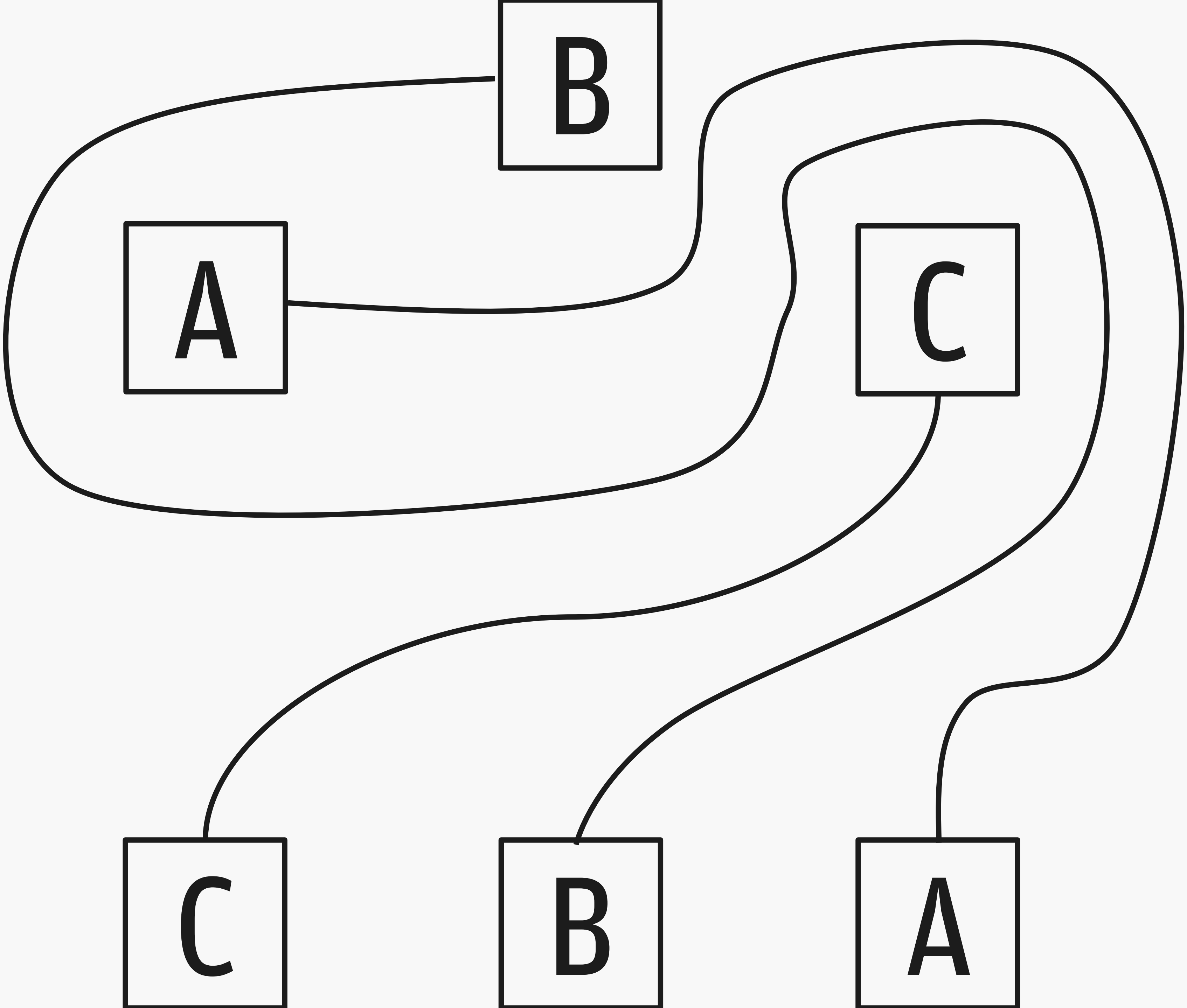

B
A
C
C
B
A


B
A
C
C
B
A


B
A
C
C
B
A


B
A
C
C
B
A


B
A
C
C
B
A


B
A
C
C
B
A