Newton interpolation and the umbral calculus

Tiny Explanations 3

Carl Joshua Quines

July 4, 2020

Take a sequence that's determined by a linear polynomial, like $a_n = 3n + 8$ giving $8, 11, 14, 17, 20, \ldots$. (We're starting n from 0.) And then take the sequence formed by taking the difference of each term from the term before it. So $11 - 8$ is 3, $14 - 11$ is 3, and $17 - 14$ is 3, which means this sequence is $3, 3, 3, \ldots$. These are called the *first differences* of this sequence, and for a linear sequence, it's constant.

Now take a sequence determined by a quadratic polynomial, like $a_n = n^2 - n + 2$. This sequence goes $2, 2, 4, 8, 14, \ldots$. The first differences of this are $0, 2, 4, 6, \ldots$. And then if we take the first differences of the first differences, we get $2, 2, 2, \ldots$, and these are what we call the *second differences*. For a quadratic sequence, the second differences are constant.

This is true in general: a sequence whose formula is a kth degree polynomial has constant kth differences. And the converse is also true. If the kth differences of a sequence is constant, then the formula must be a kth degree polynomial. As an example, consider the sequence $1, 5, 12, 22, 35, \ldots$. We can compute its first and second differences by placing them in a table like this:

$$\begin{array}{ccccccccc}
1 & & 5 & & 12 & & 22 & & 35 \\
& 4 & & 7 & & 10 & & 13 & \\
& & 3 & & 3 & & 3 & &
\end{array}$$

If we had a good reason to believe that the second differences are constant, then the formula has to be quadratic. Indeed, it's $a_n = \frac{3}{2}n^2 + \frac{1}{2}n + 1$, and you can check that this works.

1 Newton interpolation

How do we find this formula? Look at the three leftmost numbers in each row. Here, they are 1, 4, and 3. For a quadratic sequence, these three numbers determine every other number. And the way that these numbers work are *independent* of each other—the 1 doesn't care about the 4, nor the 3. Look at what happens when we only take the 1, make the other two numbers 0, and compute the table:

$$\begin{array}{ccccccccc}
1 & & 1 & & 1 & & 1 & & 1 \\
& 0 & & 0 & & 0 & & 0 & \\
& & 0 & & 0 & & 0 & &
\end{array}$$

Then take only the 4, make the other numbers 0, and compute the table:

$$\begin{array}{ccccccccc} 0 & & 4 & & 8 & & 12 & & 16 \\ & 4 & & 4 & & 4 & & 4 & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Finally, if you take only the 3, and make the other numbers 0:

$$\begin{array}{ccccccccc} 0 & & 0 & & 3 & & 9 & & 18 \\ & 0 & & 3 & & 6 & & 9 & \\ & & 3 & & 3 & & 3 & & \end{array}$$

The original sequence is now just the sum of these three sequences!

$$\begin{array}{rrrrr} 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 8 & 12 & 16 \\ 0 & 0 & 3 & 9 & 18 \\ \hline 1 & 5 & 12 & 22 & 35 \end{array}$$

So to find the formula, we only need the formula for each of these three sequences, and then add them together. The top row is easy: it's just 1. The second row is $4n$, which makes sense, because you're adding 4 each time.

The third row is a bit tricky, but it might be easier to recognize if we divide everything by their common factor, 3. We get $0, 0, 1, 3, 6, \ldots$. These are the triangular numbers, and their formula is $\frac{1}{2}n(n-1)$. So multiplying by 3 gives the formula for the third row, which is $\frac{3}{2}n(n-1)$. We get the final formula by adding these together: $\frac{3}{2}n(n-1) + 4n + 1$. We can even rewrite this as $3\binom{n}{2} + 4\binom{n}{1} + 1$.

There is nothing special about the numbers 1, 4, and 3. As we just saw, the last row is just 3 times the triangular numbers, but we could've replaced 3 with anything. If we replaced it with 9, then it'd just be 9 times the triangular numbers. (Think about the finite difference table we had earlier, and multiply all the numbers by 3.) Generally, if these numbers were a, b, and c, then we would get the formula $a + b\binom{n}{1} + c\binom{n}{2}$.

So now we know how to find the formula if the second differences are constant, or if the sequence is quadratic. But what about if the *third* differences were constant, or if it was a cubic? Well, we can use the same trick again, but we can reuse some of our previous work. We already know what it looks like for second differences:

$$\begin{array}{ccccccccccc} 0 & & 0 & & 1 & & 3 & & 6 & & 10 \\ & 0 & & 1 & & 2 & & 3 & & 4 & \\ & & 1 & & 1 & & 1 & & 1 & & \end{array}$$

So to get third differences, we can just *add a row on top*!

$$\begin{array}{ccccccccccccc} 0 & & 0 & & 0 & & 1 & & 4 & & 10 & & 20 \\ & 0 & & 0 & & 1 & & 3 & & 6 & & 10 & \\ & & 0 & & 1 & & 2 & & 3 & & 4 & & \\ & & & 1 & & 1 & & 1 & & 1 & & & \end{array}$$

At this point, you may already recognize the pattern. We're forming Pascal's triangle, but rotated so that the tip is on the lower-left instead. The diagonals in Pascal's triangle become our rows: the leftmost diagonal of all 1s is our bottom row of all 1s. And you might know that the formula for the kth diagonal of Pascal's triangle is $\binom{n}{k}$, which is *exactly* the formula we need!

This is the main idea behind *Newton interpolation*! If the "leftmost numbers" of each row were called $d_0, d_1, d_2, \ldots$, then the formula for the sequence would be

$$a_n = d_0\binom{n}{0} + d_1\binom{n}{1} + d_2\binom{n}{2} + d_3\binom{n}{3} + \cdots .$$

There are infinitely many terms in this formula, but that's fine. If this is the formula for a kth degree polynomial, then we know that d_k would be constant, so $d_{k+1}, d_{k+2}, d_{k+3}, \ldots$ would all be 0. Some contest math applications are in PRIME 2016 S8 under Sequences and Engineering under Sequence guessing skills.

2 Umbral calculus

Let's rewrite this previous formula in a slightly different way. We're going to introduce the *falling factorial* notation:

$$n^{\underline{k}} = n(n-1)(n-2)\cdots(n-(k-1))\,.$$

It's kind of like the factorial, except it cuts off after k terms. Or it's like the exponential n^k, except the factors are smaller: instead of all of them being n, they go n, $n-1$, $n-2$, and so on. This lets us rewrite $\binom{n}{k}$ as $\frac{1}{k!}n^{\underline{k}}$, giving

$$a_n = \frac{d_0}{0!}n^{\underline{0}} + \frac{d_1}{1!}n^{\underline{1}} + \frac{d_2}{2!}n^{\underline{2}} + \frac{d_3}{3!}n^{\underline{3}} + \cdots .$$

If you're familiar with the Taylor series, notice how it's similar to that:

$$f(n) = \frac{f(0)}{0!}n^0 + \frac{f'(0)}{1!}n^1 + \frac{f''(0)}{2!}n^2 + \frac{f'''(0)}{3!}n^3 + \cdots .$$

Here, we have the differences $d_0, d_1, d_2, \ldots$ turning into derivatives $f(0), f'(0), f''(0), \ldots$, and the falling factorial into exponentiation. And suddenly, we got the Taylor series!

Let's think about this analogy more. Consider how, in normal calculus, the first derivative of x^3 is $3x^2$. If we "convert back" to the differences version, we want to find the first differences of $x^{\underline{3}}$. But the first differences are

$$\begin{aligned}(x+1)^{\underline{3}} - x^{\underline{3}} &= (x+1)(x)(x-1) - (x)(x-1)(x-2) \\ &= [(x+1)-(x-2)]\,(x)(x-1) \\ &= 3x(x-1) \\ &= 3x^{\underline{2}}.\end{aligned}$$

This phenomenon isn't entirely coincidence. Think about what happens when you take the derivative of the Taylor series several times and then plug in $x = 0$. And then in the differences version, think about what happens when we take the first difference several times, and then plug in $n = 0$.

This is one of the observations that led to the discovery of *umbral calculus*. Applying umbral calculus is more subtle than just changing everything into the umbral versions, and explaining it in more detail wouldn't fit a tiny explanation.

But there's one quick example I can show you, which comes from the binomial theorem. Here's the regular binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}.$$

The umbral version of the binomial theorem is pretty much the same:

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}.$$

And already this is something that's pretty surprising. Even for $n = 2$, this isn't as obvious as the regular binomial theorem:

$$(x+y)(x+y-1) = y(y-1) + 2xy + x(x-1).$$

Let's rewrite this in a slightly nicer version, by converting the falling factorials back to binomial coefficients. Remember that $n^{\underline{k}} = k!\binom{n}{k}$, so:

$$\begin{aligned} (x+y)^{\underline{n}} &= \sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}, \\ n!\binom{x+y}{n} &= \sum_{k=0}^{n} \binom{n}{k} k!\binom{x}{k} (n-k)!\binom{y}{n-k}, \end{aligned}$$

and conveniently enough, expanding $\binom{n}{k}$,

$$\begin{aligned} n!\binom{x+y}{n} &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} k!\binom{x}{k} (n-k)!\binom{y}{n-k}, \\ \binom{x+y}{n} &= \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}, \end{aligned}$$

which is the Chu–Vandermonde identity. Note that this is a polynomial identity. Here, x and y are *variables*; we don't require x and y to be positive, or even integers! As something to think about, does a similar trick work for $(x+y+z)^{\underline{n}}$?