Non-standard MMC problems

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1 Algebra

1. (15S/9B/E6) A quadratic function $f(x)$ satisfies $f(0) = 30$ and $f(2) = 0$. Determine all the zeros of $f(x)$. [2 and 15]

2. (15S/IVB/E6) What is the smallest positive integer n which satisfies $n^{20} \geq 5^{30}$? [12]

3. (15S/IVB/E9) The cube of a number x equals 15^{15}. What is the product of x and the square of 15^{14}? [15^{33}]

4. (15S/9B/E10) If x and y are positive numbers such that $x^2 - 3y^2 = 2xy$, find $\frac{x}{y}$. [3]

5. (16S/9B/E10) If $4^a = 5, 5^b = 6, 6^c = 7$, and $7^d = 8$, find the value of $abcd$. [$\frac{3}{2}$]

6. (15S/9B/A4) If $f(2x + 1) = 4x^2 + 2x - 6$, what are the zeros of $f(x)$? [$-2, 3$]

7. (16R/10B/A4) Two opposite vertices of a square lie at $(0, 4)$ and $(7, 3)$. Find the other vertices. [$(3, 0), (4, 7)$]

8. (15S/IVB/A5) A linear function f is such that $f(2015) - f(2005) = 100$. What is $f(2051) - f(2015)$? [360]

9. (15S/7B/D2) If n is a constant such that $||x - 2| - 3| = n$ has exactly 3 distinct roots, what are these roots? [$-4, 2, 8$]

10. (16R/10B/D2) Find the point on the x-axis equidistant to the points $(8, 1)$ and $(0, 5)$. [$(\frac{5}{2}, 0)$]

11. (16R/7B/D3) If the sum of fifty consecutive integers that come after n is 6525, what is the sum of fifty consecutive integers before n? [3975]

12. (15S/9B/D3) For what value(s) of a will the quadratic equations $x^2 - ax + 2 = 0$ and $x^2 - 2x + a = 0$ have a common real solution? [-3]

13. (16S/10B/D3) What are the coordinates of the point on $(x - 1)^2 + (y - 2)^2 = 5$ that is closest to the point $(7, -1)$? [$(3, 1)$]

14. (16R/10B/D3) A third-degree polynomial satisfies $P(0) = -3$ and $P(1) = 4$. When $P(x)$ is divided by $x^2 + x + 1$ the remainder is $2x - 1$. In the same division, what is the quotient? [$3x - 2$]

15. (16S/10B/D5) If $a_1 = 3, a_2 = 5$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 3$, what is a_{16}? [$1 + 2^{16}$]

16. (15S/IVB/D6) For $n \geq 2$, the nth term of the sequence equals the sum of all the terms before it. If the eighth term is 320, what is the first term of the sequence? [5]

17. (16S/8B/C3) What is the area of the region bounded by the graph of $|x+y|+|x-y|=4$? [16 sq. units]

2 Geometry

1. (15S/9B/E8) On square $ABCD$, one side measures 1 cm, E is the midpoint of AB and F is the point of intersection of CE and the diagonal BD. How long is FB? [$\frac{\sqrt{2}}{3}$ cm]

2. (16S/8B/E10) Find the radius of the circle inscribed in a triangle of sides $20, 21$ and 29 cm. [6 cm]

3. (15R/7B/A1) Each side of square $ABCD$ is 10 cm long. Inside it a point E is chosen. Find the sum of the areas of triangle ABE and triangle CDE. [50 cm^2]

4. (15R/7B/A5) In equilateral triangle ABC, the segment from A to B is extended to a point Q so that triangle BQC has the same area as triangle ABC. Find angle AQC. [30°]

5. (15S/9B/A5) Three identical equilateral corners of an equilateral triangle with side of length 1cm is to be cut off to reduce the area by one-half. How long should be the side of the cut-off corners? [$\frac{\sqrt{6}}{6}$ cm]

6. (16R/8B/D1) Point D is inside triangle ABC such that BD and CD are angle bisectors of angle ABC and angle ACB, respectively. BD is equal to CD. Find angle BDC if angle BAC is $32°$. [163°]

7. (15S/IVB/D2) Evaluate the sum $\sin 30° + \sin 60° + \sin 90° + \cdots + \sin 510° + \sin 540°$. [$2+\sqrt{3}$]

8. (16S/9B/D4) In a $13-14-15$ triangle, find the length of the altitude to the base of 14 units. [12 units]

9. (15S/8B/D5) Let $ABCD$ be a square, and let E, F, G and H be the midpoints of sides CD, AD, AB and BC, respectively. The segments AE, BF, CG and DH create a smaller square inside $ABCD$. If the area of this smaller square is 1.5 sq. units, what is the area of $ABCD$? [7.5 sq. units]

10. (15S/9B/D5) The area of rectangle $ABCD$ is 24 sq. cm and E is the midpoint of CD, F is the point of intersection of the diagonal AC and segment BE. Find the area of triangle EFC. [2 cm^2]

11. (15S/7B/C3) The medians to two legs of a right triangle are 6 and 7 units long. How long is the hypotenuse? [$2\sqrt{17}$]

12. (15S/9B/C3) A point O lies inside equilateral triangle ABC so that $AO^2 + BO^2 = CO^2$. Find the measure of angle AOB. [150°]

13. (15S/9B/DoD) How many non-congruent right triangles have sides of integer lengths and have areas numerically equal to three times their perimeters? [6]

3 Not actually MMC problems

AMC is American Math Competition. NIMO is National Internet Math Olympiad Summer Contest. OMO is Online Math Open. PMO is Philippine Mathematical Olympiad. Time may not be suitable, use with discretion.

3.1 Fifteen-second questions

1. (AMC10 2005) A positive number x has the property that $x\%$ of x is 4. What is x? [20]
2. (AMC10 2005) A liter of paint was used to paint a room. One third of the paint is used on the first day. One third of the remaining paint is used on the second day. How many liters of paint remained for the third day? [$\frac{4}{9}$]
3. (AMC10 2005) One fair die has faces $1, 1, 2, 2, 3, 3$ and another has faces $4, 4, 5, 5, 6, 6$. The dice are rolled and the numbers on the top faces are added. What is the probability that the sum is odd? [$\frac{5}{9}$]
4. (AMC10 2007) All sides of the convex pentagon $ABCDE$ are of equal length, and $\angle A = \angle B = 90^\circ$. What is the degree measure of $\angle E$? [150°]
5. (AMC10 2008) A quadratic equation $ax^2 - 2ax + b = 0$ has two real solutions. What is the arithmetic mean of the solutions? [1]
6. (AMC10 2008) An athlete competes in a triathlon in which the swimming, biking and running segments are all of the same length. The athlete swims at a rate of 3 kilometers per hour, bikes at a rate of 20 kilometers per hour, and runs at a rate of 10 kilometers per hour. What is the athlete's average speed, in kilometers per hour, for the entire race? [$\frac{180}{29}$]
7. (PMO 2011) Find the sum $\cos 1^\circ + \cos 3^\circ + \cos 5^\circ + \cdots + \cos 177^\circ + \cos 179^\circ$. [0]
8. (NIMO 2011) If the answer to this problem is x, find the value of $\frac{x^2}{8} + 2$. [4]
9. (PMO 2013) Find the remainder when $0! + 5! + 10! + \cdots + 100!$ is divided by 100. [1]
10. (NIMO 2013) Find the value of $99(99^2 + 3) + 3 \times 99^2$. [$999,999$]
11. (NIMO 2014) Let n be a positive integer. Determine the smallest possible value of $1 - n + n^2 - n^3 + \cdots + n^{1000}$. [1]
12. (NIMO 2014) How many $2 \times 2 \times 2$ cubes must be added to an $8 \times 8 \times 8$ cube to form a $12 \times 12 \times 12$ cube? [152]
13. (OMO Fall 2014) Suppose that $a_1, a_2, \ldots$, $b_1, b_2, \ldots$, and $c_1, c_2, \ldots$ are terms of three arithmetic progressions. Given $a_1 + b_1 + c_1 = 0$ and $a_2 + b_2 + c_2 = 1$, find $a_{2014} + b_{2014} + c_{2014}$. [2013]
14. (OMO Spring 2015) Find the largest positive integer equal to the sum of its digits. [9]
15. (NIMO 2015) Let $a \diamond b = \frac{a+b}{a-b}$. Find $1008 \diamond 1007$. [2015]

16. (NIMO 2015) On a 30 question test, question 1 is worth one point, question 2 is worth two points, and so on. David takes the test and finds out he answered nine of the questions incorrectly. What is the highest possible score he could have attained? [420]

17. (NIMO 2015) Let $P(t) = a^t + b^t$, where a and b are complex numbers. If $P(1) = 7$ and $P(3) = 28$, compute $P(2)$. [19]

18. (OMO Fall 2015) At a camp, students are being housed in single rooms and double rooms. It is known that 75% of the students are housed in double rooms. What percentage of the rooms occupied are double rooms? [60%]

19. (OMO Spring 2016) If $x + y + z = 20$ and $x + 2y + 3z = 16$, what is $x + 3y + 5z$? [12]

20. (NIMO 2016) Evaluate $\left(9 + \frac{9}{9}\right)^{9-9/9} - \frac{9}{9}$. [$99,999,999$]

21. (NIMO 2016) Nine people sit in three rows of three chairs each. Find the probability that two of them sit next to each other in the same row. [$\frac{1}{6}$]

3.2 Thirty-second questions

1. (own) What is the sum of the reciprocals of the roots of the equation $\frac{2015}{2016}x^2 + x + 1 = 0$? [$-1$]

2. (own) There are 3 glahps in 10 gleeps, 5 gleeps in 6 gloops, and 3 gloops in 2 gluhps. How many glahps are there in 4 gluhps? [$\frac{3}{2}$]

3. (NIMO 2011) Find the number of ordered pairs of integers (a, b) that satisfy the inequality $1 < a < b + 2 < 10$. [28]

4. (NIMO 2011) In equilateral triangle ABC, the midpoint of BC is M. If the circumcircle of triangle MAB has area 36π, find the perimeter of the triangle. [36]

5. (NIMO 2011) Let $P(x) = x^2 - 20x - 11$. If a and b are positive integers such that a is composite, the greatest common divisor of a and b is 1, and $P(a) = P(b)$, compute the value of ab. [99]

6. (NIMO 2012) Let $f(x) = (x^4 + 2x^3 + 4x^2 + 2x + 1)^5$. Find the prime p satisfying $f(p) = 418,195,493$. [2]

7. (NIMO 2012) Compute the number of positive integers n satisfying the inequality $2^{n-1} < 5^{n-3} < 3^n$. [5]

8. (NIMO 2012) Find the last two digits of $1! + 2! + 3! + \cdots + 2012!$. [13]

9. (NIMO 2013) Let ABC and DEF be two triangles such that $AB = DE = 20, BC = EF = 13$, and $\angle A = \angle D$. If $AC - DF = 10$, determine the area of triangle ABC. [126]

10. (NIMO 2013) Find the sum of the real roots of the polynomial

$$(x^2 - 11x + 1)(x^2 - 11x + 2) \cdots (x^2 - 11x + 100).$$

[330]

11. (PMO 2014) In a sequence, the average of the first and second terms is 1, the average of the second and third terms is 2, the average of the third and fourth terms is 3, and so on. Find the average of the first and one hundredth terms. [50]

12. (NIMO 2014) In a five-digit positive integer N, we select every pair of digits of N, keeping them in order, to obtain ten numbers: $33, 37, 37, 37, 38, 73, 77, 78, 83, 87$. Find N. [37837]

13. (NIMO 2015) A list of integers with average 89 is split into two disjoint groups. The average of the integers in the first group is 73 while the average of the integers in the second group is 111. What is the smallest possible number of integers in the original list? [19]

14. (OMO Spring 2015) A geometric progression of positive integers has n terms: the first term is 10^{2015} and the last term is odd. How many possible values of n are there? [8]

15. (PMO 2016) What is the fifth largest divisor of the number $2,015,000,000$? [$251,875,000$]

16. (NIMO 2016) A positive integer n is nice if $2n+1, 3n+1$, and $4n+1$ are all composite numbers. Find the smallest nice number. [16]

17. (NIMO 2016) If $\sin a + \sin b = 1$ and $\cos a + \cos b = \frac{3}{2}$, find $\cos(a-b)$. [$\frac{5}{8}$]

18. (NIMO 2016) Consider all 1001-element subsets of the set $\{1, 2, 3, \ldots, 2015\}$. From each subset we choose the median. Find the arithmetic mean of all these medians. [1008]

19. (OMO Spring 2016) A store offers packages of 12 pens for 10 pesos and packages of 20 pens for 15 pesos. Find the greatest number of pens 173 pesos can buy at this store. [224]

20. (OMO Spring 2016) Given that x is a real number, find the minimum value of $|x+1| + 3|x+3| + 6|x+6| + 10|x+10|$. [54]

3.3 Sixty-second questions

1. (own) In triangle ABC, points D and E are chosen on AC and BC, respectively, such that $AD : DC = 3 : 7$ and $BE : EC = 1 : 2$. If the intersection of BD and AE is F and G is a point on AB such that C, F, and G are collinear, find the ratio of $CF : FG$. [$13 : 3$]

2. (NIMO 2011) The roots of the polynomial $P(x) = x^3 + 5x + 4$ are r, s, and t. Evaluate $(r+s)^4(s+t)^4(t+r)^4$. [256]

3. (NIMO 2012) Let a and b be two positive integers satisfying $20\sqrt{12} = a\sqrt{b}$. Find the sum of all distinct products ab. [10800]

4. (NIMO 2012) A quadratic polynomial $p(x)$ with integer coefficients satisfies $p(41) = 42$. For some integers $a, b > 41, p(a) = 13$ and $p(b) = 73$. Find the value of $p(1)$. [2842]

5. (NIMO 2012) The degree measures of the angles of nondegenerate hexagon $ABCDEF$ are integers that form a non-constant arithmetic sequence in some order, and $\angle A$ is the smallest angle. Compute the sum of all possible degree measures of $\angle A$. [1300]

6. (NIMO 2012) When Eva counts, she skips all the numbers containing a digit divisible by 3. For example, the first ten numbers she counts are $1, 2, 4, 5, 7, 8, 11, 12, 14, 15$. What is the 100th number she counts? [255]

7. (NIMO 2013) A point (a, b) in the plane is called *sparkling* if it also lies on the line $ax + by = 1$. Find the maximum possible distance between two sparkling points. [2]

8. (NIMO 2013) Let $P(x)$ be a fourth-degree polynomial such that $P(165) = 20, P(42) = P(69) = P(96) = P(123) = 13$. Find the value of $P(1) - P(2) + P(3) - P(4) + \cdots + P(165)$. [20]

9. (OMO Spring 2014) The integers $1, 2, \ldots, n$ are written in order on a long slip of paper. The slip is then cut into five pieces, so that each piece consists of some non-empty consecutive set of integers. The averages of the numbers on the five slips are $1234, 345, 128, 19$, and 9.5 in some order. Compute n. [2014]

10. (OMO Fall 2014) Let $f(x, y, z) = x^{y^z} - x^{z^y} + y^{z^x} - y^{x^z} + z^{x^y}$. Evaluate $f(1, 2, 3) + f(1, 3, 2) + f(2, 1, 3) + f(2, 3, 1) + f(3, 1, 2) + f(3, 2, 1)$. [24]

11. (OMO Fall 2014) Let a, b, c be positive real numbers such that $\frac{5}{a} = b + c$, $\frac{10}{b} = c + a$ and $\frac{13}{c} = a + b$. Find the value of $a + b + c$. [$\frac{49}{6}$]

12. (NIMO 2015) Let S_0 be the empty set, and let $S_n = \{S_0, S_1, \ldots, S_{n-1}\}$. Find the number of elements in the set $(S_{10} \cap S_{20}) \cup (S_{30} \cap S_{40})$. [30]

13. (NIMO 2015) Let ABC be a triangle whose angles measure A, B, C, respectively. Suppose $\tan A, \tan B, \tan C$ form a geometric sequence in that order. If $1 \leq \tan A + \tan B + \tan C \leq 2015$, find the number of possible integer values for $\tan B$. [11]

14. (NIMO 2015) It is given $4^{11} + 1$ is divisible by some prime greater than 1000. Determine this prime. [2113]

15. (OMO Fall 2015) How many integers between 123 and 321 have exactly two digits that are 2? [18]

16. (OMO Fall 2015) In a sequence, $a_0 = 0, a_1 = 1$, and $a_i = 2a_{i-1} - a_{i-2} + 2$ for all $i \geq 2$. Find a_{1000}. [$1,000,000$]

17. (OMO Fall 2015) Let $s_1, s_2, \ldots$ be an arithmetic progression of positive integers. Suppose that $s_{s_1} = x + 2, s_{s_2} = x^2 + 18$ and $s_{s_3} = 2x^2 + 18$. Find the value of x. [16]

18. (OMO Fall 2015) Let $P(x) = x^3 - 10x^2 + x - 2016$ have roots a, b, c. The cubic polynomial $Q(x)$ has leading coefficient 1 and roots $bc - a^2, ca - b^2, ab - c^2$. What is the sum of the coefficients of $Q(x)$? [$2,016,000$]

19. (OMO Spring 2016) Compute the number of ordered quadruples of positive integers (a, b, c, d) such that $a! \times b! \times c! \times d! = 24!$. [28]

20. (NIMO 2016) Compute the number of permutations (a, b, c, x, y, z) of $(1, 2, 3, 4, 5, 6)$ which satisfy $a < b < c, x < y < z, a < x, b < y$, and $c < z$. [5]