Parallel sum

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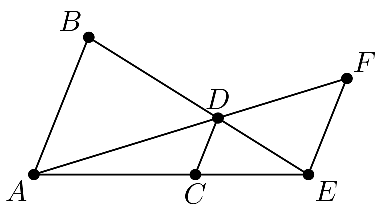
Warmup

1. (Optic equation) A lens of negligible thickness and focal length f has distance a to an object and b to the object's image. Prove that $\frac{1}{f} = \frac{1}{a} + \frac{1}{b}$.
2. Two resistors, of resistance $A\,\Omega$ and $B\,\Omega$, are connected in parallel. Let T be their equivalent overall resistance. Prove that $\frac{1}{T} = \frac{1}{A} + \frac{1}{B}$.
3. (Crossed ladders) In the diagram, $AB \parallel CD \parallel EF$. Prove that $\frac{1}{CD} = \frac{1}{AB} + \frac{1}{EF}$.



4. Pump A can fill a pool in a hours, and pump B can fill the same pool in b hours. If both pumps are used at the same time, it takes t hours. Prove that $\frac{1}{t} = \frac{1}{a} + \frac{1}{b}$.

Definition

Let $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. The **parallel sum** operator (also known as *harmonic sum*, *reduced sum*, *reciprocal sum*, or the same terms but with *addition* instead of *sum*), over $\mathbb{R}^*$, is defined by

$$a \parallel b = \frac{ab}{a+b}.$$

In other words,

$$\frac{1}{a \parallel b} = \frac{1}{a} + \frac{1}{b}.$$

In the order of operations, we'll take parallel to be between multiplication and addition, so $ab \parallel c = (ab) \parallel c$ and $a + b \parallel c = a + (b \parallel c)$.

It's called "parallel" because of resistors. Resistors connected *in parallel* have total resistance equal to their parallel sum. We can also think of how, in crossed ladders, we had three *parallel* line segments, and the length of the center segment was the parallel sum of the other two lengths. Or how, with pumps working *in parallel*, the time taken is the parallel sum of the individual times.

Around the first third of problems in this set are algebraic properties, and the rest are some random, kinda unrelated, appearances in geometry. It's not clear to me whether they're related—if you figure something out, email me: `cj@cjquines.com`.

Problems

1. Which of the following properties about $\parallel$ are true? Can you "fix" the false properties so that they're true?
   1.1 It's commutative: $a \parallel b = b \parallel a$.
   1.2 It's associative: $a \parallel (b \parallel c) = (a \parallel b) \parallel c$.
   1.3 Repeated parallel sum is multiplication: $\underbrace{a \parallel a \parallel \cdots \parallel a}_{b \text{ times}} = ab$.
   1.4 There's an identity: $a \parallel \infty = a$.
   1.5 There's inverses: $a \parallel -a = \infty$.
   1.6 Zero is absorbing: $a \parallel 0 = 0$.
   1.7 Multiplication distributes: $a(b \parallel c) = ab \parallel ac$.
   1.8 Exponentiation distributes: $a^{b \parallel c} = a^b \parallel a^c$.
   1.9 Parallel sum of logarithms: $\log_b a \parallel \log_c a = \log_{bc} a$.
   1.10 The binomial theorem works: $(a \parallel b)^2 = a^2 \parallel 2ab \parallel b^2$.
   1.11 Polynomials work: $x^2 \parallel -5x \parallel 6 = (x \parallel -2)(x \parallel -3) = 0$ has solutions $x = 2$ and $x = 3$.
2. As a Diophantine equation, it's the optic equation:
   2.1 Given $a \parallel b = c$, prove that $(a - c)(b - c) = c^2$.
   2.2 Find all pairs of positive integers (a, b) such that $a \parallel b = 25$.
   2.3 (BMO2 2005/1) Let c be a positive integer. Suppose there are 2005 pairs of positive integers (a, b) such that $a \parallel b = c$. Prove that c is a perfect square.
   2.4 Let a, b, c be positive integers such that $a \parallel b = c$. Prove that $a^2 + b^2 + c^2$ is a perfect square. If $(a, b) = 1$, prove that $a + b$ and abc are perfect squares.
3. Some properties related to transport of structure:
   3.1 Note $2 + 3 = 5$ and $\frac{1}{2} \parallel \frac{1}{3} = \frac{1}{5}$. Similarly, $3 \parallel 6 = 2$ and $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. Is this coincidence?
   3.2 We call this *dualizing*: you can dualize an equation by taking reciprocals, and swapping $+$ and $\parallel$. Convince yourself that dualizing a true equation also gives a true equation.
   3.3 Prove that $1 = \left(\frac{1}{a} + \frac{1}{b}\right)(a \parallel b)$. What do you get by dualizing this?
   3.4 Prove that $ab = (a + b)(a \parallel b)$. As a special case, $a = (a + 1)(a \parallel 1)$. What do you get by dualizing this?
   3.5 Prove that $a + 1 = \sum_{i=0}^{\infty} (a \parallel 1)^i$. What do you get by dualizing this?
   3.6 Let A be the arithmetic mean and H the harmonic mean, so for example, $A(a, b) = \frac{a + b}{2}$. How are $H(a, b)$ and $a \parallel b$ related? What about $H(a, b, c)$ and $a \parallel b \parallel c$? Why is $\parallel$ also called the *harmonic sum*?
   3.7 If you know what a field is: Let $K = \mathbb{R}^* \setminus \{0\}$. Convince yourself that K is a field, with addition $\parallel$ and multiplication $\cdot$. Why do we need to remove 0? Prove that this field is isomorphic to $\mathbb{R}$.

4. Lehman's inequality:
   4.1 Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $f(x) = x \parallel a$. Show that $f(x+y) \geq f(x) + f(y)$.
   4.2 Show that $(a+b) \parallel (c+d) \geq (a \parallel c) + (b \parallel d)$. When does equality hold?
   4.3 Show that
   $$(a_1 + \cdots + a_n) \parallel (b_1 + \cdots + b_n) \geq (a_1 \parallel b_1) + \cdots (a_n \parallel b_n).$$
   Note that this is Minkowski's inequality with an exponent of -1.
   4.4 Lehman's inequality proper: Show that
   $$(a_{11} + \cdots + a_{1n}) \parallel \cdots \parallel (a_{m1} + \cdots + a_{mn}) \geq (a_{11} \parallel \cdots \parallel a_{m1}) + \cdots + (a_{1n} \parallel \cdots \parallel a_{mn}).$$
   The original proof involved an $m \times n$ grid of resistors, if you care about circuits.
5. Inverse Pythagorean theorem:
   5.1 Let ABC be a triangle with $\angle BAC = 90°$. Let d be the distance from A to BC. Prove that $d^2 = AB^2 \parallel AC^2$.
   5.2 Let $ABCD$ be a tetrahedron with $\angle BAC = \angle CAD = \angle DAB = 90°$. Let d be the distance from A to the plane BCD. Prove that $d^2 = AB^2 \parallel AC^2 \parallel AD^2$.
6. Angle bisectors, though some may be hidden:
   6.1 (Rhombus lemma) In $\triangle ABC$, points D and E are on AB and AC such that CD bisects $\angle ACB$ and $BC \parallel DE$. Prove that $EC = BC \parallel AC$.
   6.2 In $\triangle ABC$, points D and E are on AB and AC such that CD bisects $\angle ACB$ and $CD \perp DE$. Prove that $EC = 2(BC \parallel AC)$.
   6.3 In equilateral $\triangle ABC$, let D be a point on (ABC) such that D is on the arc BC not containing A. Let $P = AD \cap BC$. Prove that $DP = BP \parallel CP$.
   6.4 (Nomogram) In $\triangle ABC$, $\angle ABC = \angle ACB = 45°$. Let D be a point on (ABC) such that D is on the arc BC not containing A. Let $P = AD \cap BC$. Prove that $DP = \sqrt{2}(BP \parallel CP)$.
7. Squares and triangles:
   7.1 () Let $ABCD$ be a square, and E a point on CD. Let $F = AE \cap BC$, and $G = AE \cap BD$. Prove that $AG = AE \parallel AF$. Prove that $AB^2 = AE^2 \parallel AF^2$.
   7.2 In $\triangle ABC$ with $\angle BAC = 90°$, points D, E, and F are on sides AB, BC, and CA, respectively, such that $ADEF$ is a square. Prove that the side length of the square is $AB \parallel AC$. Why is this a special case of 6.1?
   7.3 A square is inscribed in $\triangle ABC$, such that one side lies on BC, and the other two vertices lie on AB and CA. Let h be the length of the altitude to BC. Prove that the side length of the square is $h \parallel BC$. Why does this generalize 7.2?
   7.4 In $\triangle ABC$ with $\angle BAC = 90°$, a square is inscribed as in 7.2 with side length s, and a square is inscribed as in 7.3 with side length t. Prove that $s^2 = t^2 \parallel BC^2$.
8. Configuration issues:
   8.1 Choose non-collinear points A, B, C. Choose D such that $CD \parallel AB$. Let $E = AC \cap BD$, and let $F \in AD$ such that $CD \parallel EF$. Is it always true that $AC + CE = AE$?

8.2 Given parallel segments $AB \parallel CD$, define the directed ratio $\dfrac{AB}{CD}$ to be positive if AB and CD point in the same direction, and negative if they point in opposite directions. In 8.1, prove that $\dfrac{AC}{AE} + \dfrac{CE}{AE} = 1$, where ratios are directed.

8.3 For simplicity, we'll say $AC + CE = AE$, where lengths are directed. We say this only if we can convert the statement to one with directed ratios. In 8.1, prove that $CD = EF \parallel AB$, where lengths are directed.

9. Is there something projective going on?

9.1 (↗) In $\triangle ABC$, points D, E, and F are on BC, CA, and AB respectively, such that $DE \parallel AB$ and $DF \parallel CA$. Let $P = BE \cap DF$ and $Q = CF \cap DE$. Prove that $PQ = BD \parallel DC$.

9.2 (Complete quadrangle ↗) Let $ABCD$ be a quadrilateral, $E = AB \cap CD$, $G = AC \cap BD$, $F = EG \cap AD$, and $H = EG \cap BC$. Prove that $EG = 2(EF \parallel EH)$.

9.3 (Harmonic quadrilaterals ↗) Let ω be a circle and P a point outside it. Choose distinct points A, B, X, Y on ω such that P, A, and B are collinear, and PX and PY are tangent to ω. Let $Q = AB \cap XY$. Prove that $PQ = 2(PA \parallel PB)$.

9.4 (↗) Let G be the centroid of $\triangle ABC$. A line through G intersects BC, CA, and AB at D, E, and F. Suppose that $GD \leq GE \leq GF$. Prove that $GD = GE \parallel GF$.

9.5 (↗) Let $ABCD$ be a quadrilateral, $E = AB \cap CD$, and $F = AD \cap BC$. Let A', C', E', F' be points on line BD such that $AA' \parallel CC' \parallel EE' \parallel FF'$ and $AA' < CC'$. Prove that $AA' = CC' \parallel EE' \parallel FF'$.

10. Tangents and circles:

10.1 (Descartes' theorem ↗) Three pairwise externally tangent circles have a common external tangent. Suppose the circles have radii $a > b > c$. Prove that $\sqrt{c} = \sqrt{a} \parallel \sqrt{b}$.

10.2 (↗) Let AB be a diameter of circle Ω. Let ω be a circle tangent to AB at C and internally tangent to Ω at D. Prove that $CD^2 = 2(AC^2 \parallel CB^2)$.

10.3 (Twin circles ↗) Let A, B, and C be three collinear points. Let Ω be the circle with diameter AC, and ω be the circle with diameter AB. Let D be a point on Ω such that $BD \perp AC$. Let r be the radius of the circle internally tangent to Ω, externally tangent to ω, and tangent to BD. Prove that $r = AB \parallel BC$.

11. The inradius:

11.1 Let $\triangle ABC$ have inradius r and altitudes h_a, h_b, and h_c. Prove that $r = h_a \parallel h_b \parallel h_c$.

11.2 Let $\triangle ABC$ have inradius r and exradii r_a, r_b, and r_c. Prove that $r = r_a \parallel r_b \parallel r_c$.

11.3 (Euler's theorem ↗) Let $\triangle ABC$ have inradius r and circumradius R. Let d be the distance from the incenter to the circumcenter. Prove that $r = (R - d) \parallel (R + d)$.

11.4 (↗) Let $ABCD$ be a quadrilateral with an inscribed and circumscribed circle. Let I be the incenter and r the inradius. Prove that $r^2 = AI^2 \parallel CI^2$.

11.5 (Fuss's theorem ↗) Let $ABCD$ be a quadrilateral with an inscribed and circumscribed circle. Let r be the inradius, R the circumradius, and d the distance from the incenter to the circumcenter. Prove that $r^2 = (R - d)^2 \parallel (R + d)^2$.

Selected sketches

2.4 Work with 2.1. Assume $(a,b,c)=1$. If $p \mid c$ then $p \mid a-c$ or $p \mid b-c$. It can't be both, as otherwise $p \mid (a,b,c)$. Thus, $p^2 \mid a-c$ or $p^2 \mid b-c$. This is true for every prime, so $a-c$ and $b-c$ are perfect squares, say $a-c=m^2, b-c=n^2$ and $c=mn$; substituting the parametrization and factoring works.

3.5 By 3.4, $a=(a+1)(a \parallel 1)$, which is $\frac{a}{a+1}=a \parallel 1$. Hence $1-\frac{a}{a+1}=1-a \parallel 1$; take the reciprocal of both sides, and expand the RHS as an infinite geometric series.

3.7 The map $x \mapsto \frac{1}{x}$ is a bijection $\mathbb{R} \to K$, as sets. It preserves addition, multiplication, and sends identities to identities; it follows K must also be a field, and one isomorphic to $\mathbb{R}$. A decent number of properties of $\parallel$ drop from this from field axioms (though you have to prove the 0 case separately).

4.1 The second derivative of f is $-\frac{2a^2}{(x+a)^3}$ so it's concave. (It's only over $\mathbb{R}^+$.) Hence $f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$, then use distributivity.

5.2 Let $C' \in CD$ such that $AC' \perp CD$. Apply 5.1: $AC'^2 = AC^2 \parallel AD^2$. Apply 5.2: $d^2 = AB^2 \parallel AC'^2 = AB^2 \parallel AC^2 \parallel AD^2$. Alternatively, use De Gua's theorem; see Obscure geometry theorems, page 15. (Anything to publicize my own work...)

6.1 Let $F \in BE$ such that $AF \parallel BC$. Angle chase to get $\triangle EDC$ and $\triangle ACF$ are isosceles with $EC=ED$ and $AF=AC$. By crossed ladders, $ED = BC \parallel AF$, so $EC = BC \parallel AC$.

6.3 Angle chase to get $\angle CDP = \angle PDB = 60^\circ$. Construct $B' \in BD$ and $C' \in CD$ such that $\triangle PB'D$ and $\triangle PC'D$ are equilateral. By similarity, $\frac{BP}{BC}=\frac{PB'}{CD}=\frac{PD}{CD}$ and $\frac{CP}{BC}=\frac{PC'}{BD}=\frac{PD}{BD}$. Add to get $\frac{BP+CP}{BC}=\frac{PD}{CD}+\frac{PD}{BD}$. LHS is 1, rearrange to get $DP = BP \parallel CP$.

7.1 Let $H \in AD$ such that $GH \perp AD$. By similarity, AG, AE, and AF are proportional to HG, DE, and AB, but by crossed ladders $HG = DE \parallel AB$.

7.3 Construct square $BCDE$ such that A and D are on the same side of BC. Let $F \in BC$ such that $AF \perp BC$. Let $B' = AB \cap EF$ and $C' = AC \cap DF$. There's a homothety centered at F taking DE to $C'B'$, which takes the square $BCDE$ to the desired square. By crossed ladders, its side length is $AF \parallel BE = h \parallel BC$. (If we constructed the square external to $\triangle ABC$, there's a similar homothety centered at A—does this give another solution?)

9.1 Construct G such that $BDGF$ is a parallelogram. Prove that $PQ \parallel BC$. Then by crossed ladders, $PQ = FG \parallel DC = BD \parallel DC$.

9.5 This is a purely affine statement, so take an affine transformation sending C to infinity. Then $CC' = \infty$, and the statement reduces to crossed ladders.

10.2 Let $E = AD \cap (BCD)$ and $F \in AD$ such that $CF \perp AD$. Angle chase: $\triangle ACE$, $\triangle ECB$, $\triangle CFD$ are all right isosceles. By 5.1: $CF^2 = AC^2 \parallel CE^2$, but as $CF\sqrt{2} = CD$, $CD^2 = 2(AC^2 \parallel CB^2)$.