12^{th} Philippine Mathematical Olympiad

Qualifying Stage

24 October 2009

Part I. Each correct answer is worth two points.

1. If $\underbrace{2009 + 2009 + \cdots + 2009}_{\text{2009 terms}} = 2009^x$, find the value of x.

(a) 2 (b) 3 (c) 2009 (d) 2010

2. What is the least positive difference between two three-digit numbers if one number has all the digits 2, 4, and 6, while the other has all the digits 1, 2, and 9?

(a) 72 (b) 54 (c) 48 (d) 27

3. Which of the following numbers is closest to one of the roots of the equation $x^2 - 10000x - 10000 = 0$?

(a) 10001 (b) 5000 (c) 100 (d) 25

4. The ratio of the areas of two squares is $3 : 4$. What is the ratio of the lengths of their corresponding diagonals?

(a) $1 : 2$ (b) $3 : 4$ (c) $2 : 3$ (d) $\sqrt{3} : 2$

5. In $\triangle ABC$, let P be a point on segment BC such that $BP : PC = 1 : 4$. Find the ratio of the area of $\triangle ACP$ to that of $\triangle ABC$.

(a) $1 : 4$ (b) $1 : 5$ (c) $3 : 4$ (d) $4 : 5$

6. In how many ways can three distinct numbers be selected from the set $\{1, 2, 3, \ldots, 9\}$ if the product of these numbers is divisible by 21?

(a) 15 (b) 16 (c) 17 (d) 18

7. If $|2x - 3| \leq 5$ and $|5 - 2y| \leq 3$, find the least possible value of $x - y$.

(a) -5 (b) 0 (c) -1 (d) 5

8. Define the operations $\clubsuit$ and $\heartsuit$ by

$$a \clubsuit b = ab - a - b \quad \text{and} \quad a \heartsuit b = a^2 + b - ab.$$

What is the value of $(-3 \heartsuit 4) - (-3 \clubsuit 4)$?

(a) 38 (b) 12 (c) -12 (d) -38

9. Solve for x in the following system of equations:

$$\begin{cases} \log x + \log y &= 2 \\ \log y + \log z &= 7 \\ \log z + \log x &= 3. \end{cases}$$

(a) 10 (b) 1 (c) 0.1 (d) 0.01

10. If $a + 1 = b - 2 = c + 3 = d - 4$, which is the smallest among the numbers a, b, c, and d?

(a) a (b) b (c) c (d) d

11. Solve for x in the inequality $5^x \geq 25^{2x}$.

(a) $x \leq 1$ (b) $x \leq 0$ (c) $x \geq 0$ (d) $x \geq 1$

12. Today, 24 October 2009, is a Saturday. On what day of the week will 10001 days from now fall?

(a) Saturday (b) Monday (c) Thursday (d) Friday

13. The lines $2x + ay + 2b = 0$ and $ax - y - b = 1$ intersect at the point $(-1, 3)$. What is $2a + b$?

(a) -6 (b) -4 (c) 4 (d) 6

14. Let x be a real number that satisfies the equation

$$16\left(\log_9 x\right)^4 = \left(\log_3 x^3\right)^2 + 10.$$

Determine $\left(\log_9 x\right)^2$.

(a) 10 (b) $\sqrt{10}$ (c) $\frac{5}{2}$ (d) $\frac{\sqrt{5}}{2}$

15. Let r and s be the roots of the equation $x^2 - 2mx - 3 = 0$. If $r + s^{-1}$ and $s + r^{-1}$ are the roots of the equation $x^2 + px - 2q = 0$, what is q?

(a) 1 (b) $\frac{2}{3}$ (c) -3 (d) $-\frac{4}{3}$

Part II. Each correct answer is worth three points.

16. On the blackboard, 1 is initially written. Then each of ten students, one after another, erases the number he finds on the board, and write its double plus one. What number is erased by the tenth student?

(a) $2^{11} - 1$ (b) $2^{11} + 1$ (c) $2^{10} - 1$ (d) $2^{10} + 1$

17. For how many real numbers x is $\sqrt{2009 - \sqrt{x}}$ an integer?

(a) 0 (b) 45 (c) 90 (d) 2009

18. How many distinct natural numbers less than 1000 are multiples of 10, 15, 35, or 55?

(a) 145 (b) 146 (c) 147 (d) 148

19. Let x and y be nonnegative real numbers such that $2^{x+2y} = 8\sqrt{2}$. What is the maximum possible value of xy?

(a) $8\sqrt{2}$ (b) 49/4 (c) 49/32 (d) 1

20. In how many ways can ten people be divided into two groups?

(a) 45 (b) 511 (c) 637 (d) 1022

21. Let P be the point inside the square $ABCD$ such that $\triangle PCD$ is equilateral. If $AP = 1$ cm, what is the area of the square?

(a) $3+\sqrt{3}$ cm^2 (b) $2+\sqrt{3}$ cm^2 (c) $\frac{9}{4}$ cm^2 (d) 2 cm^2

22. Let x and y be real numbers such that $2^{2x} + 2^{x-y} - 2^{x+y} = 1$. Which of the following equations is always true?

(a) $x + y = 0$ (b) $x = 2y$ (c) $x + 2y = 0$ (d) $x = y$

23. In $\triangle ABC$, M is the midpoint of BC, and N is the point on the bisector of $\angle BAC$ such that $AN \perp NB$. If $AB = 14$ and $AC = 19$, find MN.

(a) 1 (b) 1.5 (c) 2 (d) 2.5

24. Seven distinct integers are randomly chosen from the set $\{1, 2, \ldots, 2009\}$. What is the probability that two of these integers have a difference that is a multiple of 6?

(a) $\frac{7}{2009}$ (b) $\frac{2}{7}$ (c) $\frac{1}{2}$ (d) 1

25. A student on vacation for d days observed that (1) it rained seven times, either in the morning or in the afternoon, (2) there were five clear afternoons, and (3) there were six clear mornings. Determine d.

(a) 7 (b) 8 (c) 9 (d) 10

Part III. Each correct answer is worth six points.

26. How many sequences containing two or more consecutive positive integers have a sum of 2009?

(a) 3 (b) 4 (c) 5 (d) 6

27. In $\triangle ABC$, let D, E, and F be points on the sides BC, AC, and AB, respectively, such that $BC = 4CD$, $AC = 5AE$, and $AB = 6BF$. If the area of $\triangle ABC$ is 120 cm^2, what is the area of $\triangle DEF$?

(a) 60 cm^2 (b) 61 cm^2 (c) 62 cm^2 (d) 63 cm^2

28. A function f is defined on the set of positive integers by $f(1) = 1$, $f(3) = 3$, $f(2n) = n$, $f(4n + 1) = 2f(2n + 1) - f(n)$, and $f(4n + 3) = 3f(2n + 1) - 2f(n)$ for all positive integers n. Determine

$$\sum_{n=1}^{10} [f(4n + 1) + f(2n + 1) - f(4n + 3)].$$

(a) 55 (b) 50 (c) 45 (d) 40

29. A sequence of consecutive positive integers beginning with 1 is written on the blackboard. A student came along and erased one number. The average of the remaining numbers is $35\frac{7}{17}$. What number was erased?

(a) 7 (b) 8 (c) 9 (d) 10

30. Let M be the midpoint of the side BC of $\triangle ABC$. Suppose that $AB = 4$ and $AM = 1$. Determine the smallest possible measure of $\angle BAC$.

(a) 60° (b) 90° (c) 120° (d) 150°

Answers

1. a	**6.** d	**11.** b	**16.** c	**21.** b	**26.** c
2. d	**7.** a	**12.** c	**17.** b	**22.** d	**27.** b
3. a	**8.** a	**13.** d	**18.** c	**23.** d	**28.** d
4. d	**9.** c	**14.** c	**19.** c	**24.** d	**29.** a
5. d	**10.** c	**15.** b	**20.** b	**25.** c	**30.** d