13th Philippine Mathematical Olympiad

National Stage, Oral Phase

22 January 2011

Items 1-15 are worth 2 points each.

1. Seven points on a circle are numbered 1 to 7 in the clockwise direction. A grasshopper jumps in the counterclockwise direction, from one point to another on the circle. If the grasshopper is on an odd-numbered point, it moves one point, and moves two points if it is on an even-numbered point. If the grasshopper begins at the point 7, where will it be after 2011 jumps?

   (**ans**. At the point 2.

   It will only land at the points 6, 4, 2, 7, landing at 7 after every fourth jump. $2011 = 3(mod\, 4)$, so it is at 2 by the 2011th jump.)

2. What are the last two digits of 5^{2011}?

   (**ans**. 25)

3. A tournament consists of matches between exactly three players, each, respectively, garnering 2 points, 1 point, and a zero score. The ones who obtained no score are eliminated and the rest are grouped into threes to engage again in matches, with possibly one or two players having a bye. If there are 999 players in the tournament, how many matches would have been played by the time a champion, the one who earned two points in the last match, is declared?

   (**ans**. 997, which is equal to the number of players which earned a zero score in each of the 3-person matches).

4. If $\dfrac{x-a-b}{c} + \dfrac{x-b-c}{a} + \dfrac{x-c-a}{b} = 3$, where a, b, c are positive constants, find x in terms of a, b and c.

   (**ans**. $x = a + b + c$.

   Substituting this into the equation, one gets $\dfrac{c}{c} + \dfrac{a}{a} + \dfrac{b}{b} = 3$.)

5. Let $[|x|]$ be the integer part of x and $\{x\} = x - [|x|]$, the decimal part of x. Solve $2[|x|] = x + 2\{x\}$.

(**ans**. $x = 0, \frac{4}{3}, \frac{8}{3}$.

$2[|x|] = x + 2\{x\} \Rightarrow 2[|x|] = [|x|] + \{x\} + 2\{x\} \Rightarrow [|x|] = 3\{x\} < 3 \Rightarrow [|x|] = 0, 1, 3 \Rightarrow x = 0, \frac{4}{3}, \frac{8}{3}$.)

6. There is a triple k, m, n of positive integers without common factors such that $k \log_{400} 5 + m \log_{400} 2 = n$. Find the sum $k + m + n$.

(**ans**. $(k, m, n) = (2, 4, 1), k + m + n = 7$.

$\log_{400} 5^k 2^m = \log_{400} 400^n \Rightarrow 5^k 2^m = 400^n = (2^4 5^2)^n \Rightarrow m = 4n, k = 2n \Rightarrow (k, m, n) = (2, 4, 1)$.)

7. Let a, b, c be three, not necessarily distinct, numbers chosen randomly form the set $\{3, 4, 5, 6, 7, 8\}$. Find the probability that $ab + c$ is even.

(**ans**. 0.5.

Prob($ab + c$ is even)$=\frac{4(3 \cdot 3 \cdot 3)}{6 \cdot 6 \cdot 6} = \frac{4}{8}$. This is because ($ab + c$ even) iff (ab and c even OR ab and c odd) iff (a, b, c even OR a odd and b, c even OR b odd and a, c even OR a, b, c odd). These are disjoint events each with probability $\frac{3 \cdot 3 \cdot 3}{6 \cdot 6 \cdot 6}$.)

8. Evaluate $\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \frac{1}{99} + \frac{1}{143} + \frac{1}{195}$.

(**ans**. $\frac{7}{15}$.

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \frac{1}{99} + \frac{1}{143} + \frac{1}{195} =$$
$$\frac{1}{2}\left[(1 - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{7}) + \cdots + \frac{1}{13} - \frac{1}{15}\right] = \frac{1}{2}(1 - \frac{1}{15}) = \frac{7}{15}.)$$

9. How many pairs of integers solve the system $|xy| + |x - y| = 2$ if $-10 \leq x, y \leq 10$?

(**ans**. $4 : (2, 0), (0, 2), (-2, 0), (0, -2)$.

The cases are (A) $|xy| = 0$ and $|x - y| = 2 \Rightarrow (x, y) = (0, \pm 2), (\pm 2, 0)$; (B) $|xy| = 2$ and $|x - y| = 0 \Rightarrow$ no solution; (C) $|xy| = |x - y| = 1 \Rightarrow$ no solution.)

10. $x^2 + 4x + 8 = 0$ has roots a and b. Find a quadratic polynomial with integer coefficients whose roots are $\frac{1}{a}$ and $\frac{1}{b}$.

(**ans**. $8x^2 + 4x + 1$.

$ab = 8, a + b = -4 \Rightarrow \frac{1}{a}\frac{1}{b} = \frac{1}{8}, \frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b} = -\frac{1}{2}$. This means that the reciprocals are roots of the polynomial $x^2 + \frac{1}{2}x + \frac{1}{8}$ and hence of $8x^2 + 4x + 1$.)

11. $x^3 + kx - 128 = 0$ has a root of multiplicity 2. Find k.

(**ans**. -48

$x^3 + kx - 128 = (x - r)^2(x - s) = x^3 + (s - 2r)x^2 + (r^2 + 2rs)x - r^2s = 0 \Rightarrow r^s = 128, s + 2r = 0 \Rightarrow s = -2r, -2r^3 = 128 \Rightarrow r = -4, s = 8$. Thus, $k = r^2 + 2rs = 16 - 64 = -48$.)

12. Let $f(m) = \left.2^{2^{2^{\cdot^{\cdot^{2}}}}}\right\}m$. Find the least m so that $\log_{10} f(m)$ exceeds 6.

(**ans**. 5.

$f(4) = 2^{16} = 66048 < 10^6 < 2^{66048} = f(5)$.)

13. $x + \frac{1}{x}$ has a maximum in $x < 0$ and a minimum in $x > 0$. Find the area of the rectangle whose sides are parallel to the axes and two of whose vertices are the maximum and minimum values of $x + \frac{1}{x}$.

(**ans**. area$= 4 \cdot 2 = 8$.

Vertex at maximum is $(-1, -2)$, vertex at minimum is $(1, 2)$. Thus, width is 2 and height is 4. Vertices are obtained from the the inequalities $\frac{(x+1)^2}{x} \geq 0$ for $x > 0$ and $\frac{(x-1)^2}{x} \leq 0$ for $x < 0$.)

14. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $xf(y) = yf(x)$ for all $x, y \in \mathbb{R}$. Find the intersection of the graphs of $y = f(x)$ and $y = x^2 + 1$ if $f(1) = -1$.

(**ans** ϕ =Null set.

We have that $\frac{f(x)}{x} = \frac{f(y)}{y} = c$, a constant$\Rightarrow f(x) = cx \Rightarrow f(x) = -x$ from the given condition. $y = f(x) = -x$ does not intersect the parabola $y = x^2 + 1$ because $x^2 - x + 1 = 0$ has no real solutions.)

15. From what country is the only the second South East Asian mathematician who recently received what is considered the highest recognition in mathematics, the Fields Medal.

(**ans**. Vietnam (Ngo Bao Chau))

Items 16-25 are worth 3 points each.

16. How many times does the graph of $y = \pm\sqrt{\dfrac{x-1}{x+1}}$ cross the x-axis, the y-axis or the line $y = x$?

(**ans**. Twice, at $(1, 0)$ and in the third quadrant.

First $x \geq 1$ or $x < -1$. Solving x in terms of y, we get $x = \dfrac{1+y^2}{1-y^2}$. The graph of this equation has asymptotes $y = 1$ and $y = -1$. For $x \geq 1$ in this graph, $-1 < y < 1$ and this portion of the graph does not intersect by $y = x$. For $x < -1$, we have $y > 1$ or $y < -1$ with $x = -1$ as asymptote. $y = x$ intersects this portion of the graph once in the third quadrant.)

17. How many roots has the equation $\sin x - \log_{10} x = 0$?

(**ans**. 3.

$\sin x = \log x \Rightarrow x \leq 10$. $10 < 2 \cdot 2\pi$ means that in $[0, 2\pi]$ there is a complete period of sin and part of a second period. There is an intersection in the first period, and after the first period, near $\dfrac{5\pi}{2}$ to the left and to the right, sin has value < 1 (and > 0, of course). Since $2\pi + \dfrac{\pi}{2} = \dfrac{5\pi}{2} < 10$, $\sin x$ intersect $\log x$ at two points to the left and right side of $\dfrac{5\pi}{2}$.)

18. The system $x^2 - y^2 = 0$, $(x-a)^2 + y^2 = 1$ has generally at most four solutions. Find the values of a so that the system has two or three solutions.

(**ans**. $a = \pm 1$ for two solutions, $a = \pm\sqrt{2}$ for three solutions.

The solutions are given by $x = \dfrac{a \pm \sqrt{2-a^2}}{2}, y = \pm x$. There are two solutions if the quadratic equation involving x has a single solution$\Rightarrow$ $2 - a^2 = 0 \Rightarrow a = \pm\sqrt{2}$. If one value of x is 0, then there will at most be 3 solutions. Solving $x = 0$ in a yields $a = \pm 1$ and this gives exactly 3 solutions).

19. Find the domain of the function $f(x) = \dfrac{1}{[|x^2 - x - 2|]}$.

(**ans**. $x \in (-\infty, \frac{1-\sqrt{13}}{2}] \cup (-1, 2) \cup [\frac{1+\sqrt{13}}{2}, +\infty)$.

x must not satisfy $0 \leq x^2 - x - 2 < 1 \Rightarrow x$ must not satisfy $x^2 - x - 2 = (x-2)(x+1) \geq 0 \Leftrightarrow x \in (-\infty, -1] \cup [2, +\infty)$ AND must not satisfy $x^2 - x - 2 \leq 1 \Leftrightarrow x^2 - x - 3 < 0 \Leftrightarrow x \in (\frac{1-\sqrt{13}}{2}, \frac{1+\sqrt{13}}{2})$. Thus, $x \in (-\infty, \frac{1-\sqrt{13}}{2}] \cup (-1, 2) \cup [\frac{1+\sqrt{13}}{2}, +\infty)$.

20. Find all nonnegative integer solutions of the system

$$5x + 7y + 5z = 37$$

$$6x - y - 10z = 3.$$

(**ans**. $(x, y, z) = (4, 1, 2)$.

Eliminating z by multiplying the first equation by 2 and taking the sums, we obtain $16x+13y = 77$. This is equivalent to $16(x-4)+13(y-1) = 0$, hence $(x, y, z) = (4, 1, 2)$ is a solution. All other solutions are given by $x = 4 + 16t, y = 1 - 13t, t$ an integer. $x \geq 0 \Rightarrow t \geq 0$. This implies $y \leq 0 \Rightarrow t \leq 0$, hence $t = 0$.)

21. Find the sum $\displaystyle\sum_{k=1}^{19} k \binom{19}{k}$.

(**ans**. $19 \cdot 2^{18}$.

$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \Rightarrow k\binom{n}{k} = n\binom{n-1}{k-1}$. Then, $n = 19 \Rightarrow$ $\displaystyle\sum_{k=1}^{19} k\binom{19}{k} = \sum_{k=0}^{18} 19 \binom{18}{k-1} = 19 \cdot 2^{18}$.)

22. Find the square root of $25 \cdot 26 \cdot 27 \cdot 28 + 1$.

(**ans** 701.

$a(a+1)(a+2)(a+3)+1 = a(a+3)(a+1)(a+2)+1 = [a^2+3a][a^2+3a+2]+1 = [(a^2+3a+1)-1][(a^2+3a+1)+1]+1 = (a^2+3a+1)^2$. If $a = 25$, the square root is $25^2 + 3 \cdot 25 + 1 = 701$.)

23. Let $a = \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ and $b = \dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$. Find the value of $a^4+b^4+(a+b)^4$.

(**ans**. $64^2 + 62^2 - 2 (=7938)$.

$ab = 1, a^2b^2 = 1$. $(a+b) = \dfrac{10+6}{5-3} = 8 \Rightarrow (a+b)^2 = 64 \Rightarrow (a+b)^4 = 64^2$ and $a^2+b^2 = 62$. $a^4+b^4 = (a^2+b^2)^2 - 2a^2b^2 = 62^2 - 2$. Thus $a^4+b^4+(a+b)^4 = 64^2+62^2-2 = 7938$.)

24. Find the sum of all even factors of 1152.

(**ans.**3302.

$1152 = 2^7 3^2$. The even factors of 1152 are each of the form $2^k 3^l$, $1 \le k \le 7$, $0 \le l \le 2$. Hence the sum can be written as $(2+2^2+\cdots+2^7)(1+3+3^2) = 2\dfrac{(1-2^7)}{1-2} \cdot 13 = 3302$)

25. Find the polynomial expression in $Z = x - \dfrac{1}{x}$ of $x^5 - \dfrac{1}{x^5}$.

(**ans**. $Z^5 + 5Z^3 + 5Z$.

$x^5 - \dfrac{1}{x^5} = (x - \dfrac{1}{x})(x^4 + x^2 + 1 + \dfrac{1}{x^2} + \dfrac{1}{x^4}) = (x - \dfrac{1}{x})(x^2 + \dfrac{1}{x^2} + x^4 + \dfrac{1}{x^4} + 1)$. Now $x^2 + \dfrac{1}{x^2} = (x - \dfrac{1}{x})^2 + 2 = Z^2 + 2$, $x^4 + \dfrac{1}{x^4} = (x^2 + \dfrac{1}{x^2})^2 - 2 = (Z^2+2)^2 - 2 = Z^4 + 4Z^2 + 2 \Rightarrow x^5 - \dfrac{1}{x^5} = Z(Z^2 + 2 + z^4 + 4Z^2 + 2 + 1) = Z^5 + 5Z^3 + 5Z$.

Items 26-30 are worth 6 points each.

26. Let x and y be the integral and fractional parts of $\sqrt{37 - 20\sqrt{3}}$. Find the value of $x + y + \dfrac{4}{y}$.

(**ans**. 9.

$\sqrt{37 - 2(10)\sqrt{3}} = \sqrt{(5 - 2\sqrt{3})^2} = 5 - 2\sqrt{3} = 1 + (4 - 2\sqrt{3}) = 1 + 2(2 - \sqrt{3}) \Rightarrow x = 1, y = 4 - 2\sqrt{3}$. Thus, $x + y + \dfrac{4}{y} = 1 + 4 - 2\sqrt{3} + \dfrac{4}{4 - 2\sqrt{3}} = 5 - 2\sqrt{3} + (4 + 2\sqrt{3}) = 9$.)

27. Find the positive integers n so that $2^8 + 2^{11} + 2^n$ is a perfect square.

(**ans**. 12.

$m^2 = 2^8 + 2^{11} + 2^n \Rightarrow 2^n = m^2 - 2^8 - 2^{11} = m^2 - 2^8(1 + 2^3) = m^2 - (3 \cdot 2^4)^2 = (m - 3 \cdot 2^4)(m + 3 \cdot 2^4) = (m - 48)(m + 48) \Rightarrow m - 48 = 2^k, m + 48 = 2^l, k + l = n \Rightarrow 2^l - 2^k = 96 \Rightarrow 2^k(2^{l-k} - 1) = 2^5 \cdot 3 \Rightarrow$

$2^{l-k}-1=3$, $2^k=2^5$ by unique factorization of integers$\Rightarrow l-k=2$ and $l=7 \Rightarrow n=12$. OR complete the square: $(2^s)^2+2(2^6)(2^4)+(2^4)^2=(2^s+2^4)^2 \Rightarrow s=6 \Rightarrow n=2s=12$.)

28. How many positive-integer pairs (x, y) are solutions to the equation $\dfrac{xy}{x+y}=1000$.

(**ans**. 49

$(2a_1+1)(2a_2+1)\cdots(2a_k+1)$ where $1000=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}=2^3 5^3$; so 49. Let $\dfrac{xy}{x+y}=n \Rightarrow xy-nx-ny=0 \Rightarrow (x-n)(y-n)=n^2 \Rightarrow x>n, y>n$. In the factorization $n^2=p_1^{2a_1}p_2^{2a_2}\cdots p_k^{2a_k}$ each divisor of n^2 determines a solution, hence the answer.)

29. Give three real roots of $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$.

(**ans**. Any three numbers in $[5,10]$.

$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$

can be written

$\sqrt{(x-1)-4\sqrt{x-1}+4}+\sqrt{(x-1)-6\sqrt{x-1}+9}=\sqrt{(\sqrt{x-1}-2)^2}+\sqrt{(\sqrt{x-1}-3)^2}=1$ hence $|\sqrt{x-1}-2|+|\sqrt{x-1}-3|=1$. There are four cases. (I) $\sqrt{x-1}-2\geq 0$ and $\sqrt{x-1}-3\geq 0$ which implies $x\geq 10 \Rightarrow x=10$. (II) $\sqrt{x-1}-2\leq 0$ and $\sqrt{x-1}-3\leq 0$ which implies $x\leq 5 \Rightarrow x=5$. (III) $\sqrt{x-1}-2\geq 0$ and $\sqrt{x-1}-3\leq 0$ implies $5\leq x\leq 10$ and (IV) $\sqrt{x-1}-2\leq 0$ and $\sqrt{x-1}-3\geq 0$, which is impossible. Thus, any $x\in[5,10]$ solves the equation.)

30. Let $x+\dfrac{1}{x}=\sqrt{2}$. Find the value of $x^8+\dfrac{1}{x^8}$.

(**ans**. 2.

Let $x+\dfrac{1}{x}=2\cos\theta \Rightarrow x^2-2\cos\theta\cdot x+1=0 \Rightarrow x=\cos\theta\pm i\sin\theta \Rightarrow x^n=\cos n\theta\pm i\sin n\theta$ and $x^{-n}=\cos n\theta\mp i\sin n\theta$. Thus, $x^n+\dfrac{1}{x^n}=2\cos n\theta$ Let $\sqrt{2}=2\cos(\pi/4)$ and $n=8$ so that $x^8+\dfrac{1}{x^8}=2\cos 8(\pi/4)=2$.)