15th Philippine Mathematical Olympiad

National Stage, Written Phase

26 January 2013

Time Allotment: 4 hours *Each item is worth 8 points.*

1. Determine, with proof, the least positive integer n for which there exist n distinct positive integers $x_1, x_2, x_3, \ldots, x_n$ such that

$$\left(1-\frac{1}{x_1}\right)\left(1-\frac{1}{x_2}\right)\left(1-\frac{1}{x_3}\right)\cdots\left(1-\frac{1}{x_n}\right)=\frac{15}{2013}.$$

2. Let P be a point in the interior of $\triangle ABC$. Extend AP, BP, and CP to meet BC, AC, and AB at D, E, and F, respectively. If $\triangle APF$, $\triangle BPD$, and $\triangle CPE$ have equal areas, prove that P is the centroid of $\triangle ABC$.

3. Let n be a positive integer. The numbers $1, 2, 3, \ldots, 2n$ are randomly assigned to $2n$ distinct points on a circle. To each chord joining two of these points, a value is assigned equal to the absolute value of the difference between the assigned numbers at its endpoints.

Show that one can choose n pairwise non-intersecting chords such that the sum of the values assigned to them is n^2.

4. Let a, p, and q be positive integers with $p \leq q$. Prove that if one of the numbers a^p and a^q is divisible by p, then the other number must also be divisible by p.

5. Let r and s be positive real numbers that satisfy the equation

$$(r+s-rs)(r+s+rs)=rs.$$

Find the minimum values of $r+s-rs$ and $r+s+rs$.

Problem 1. Determine, with proof, the least positive integer n for which there exist n distinct positive integers $x_1, x_2, x_3, \ldots, x_n$ such that

$$\left(1-\frac{1}{x_1}\right)\left(1-\frac{1}{x_2}\right)\left(1-\frac{1}{x_3}\right)\cdots\left(1-\frac{1}{x_n}\right)=\frac{15}{2013}.$$

Solution. Suppose $x_1, x_2, x_3, \ldots, x_n$ are distinct positive integers that satisfy the given equation. Without loss of generality, we assume that $x_1 < x_2 < x_3 < \cdots < x_n$. Then

$$2 \le x_1 \le x_2 - 1 \le x_3 - 2 \le \cdots \le x_n - (n-1),$$

and so $x_i \ge i+1$ for $1 \le i \le n$.

$$\begin{aligned}\frac{15}{2013} &= \left(1-\frac{1}{x_1}\right)\left(1-\frac{1}{x_2}\right)\left(1-\frac{1}{x_3}\right)\cdots\left(1-\frac{1}{x_n}\right)\\ &\ge \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n+1}\right)\\ &= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n}{n+1}\\ &= \frac{1}{n+1}\end{aligned}$$

The preceding computation gives $n \ge 134$.

It remains to show that $n = 134$ can be attained. Set $x_i = i+1$ for $1 \le i \le 133$, and $x_{134} = 671$. Then

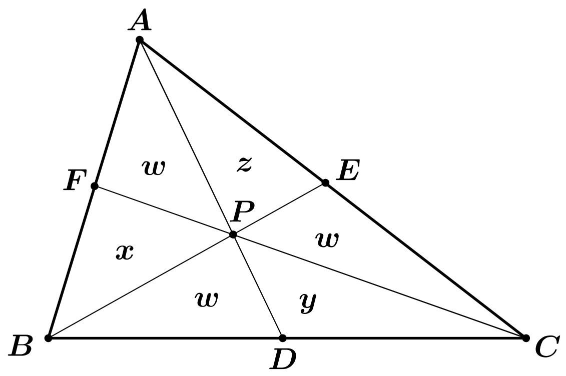
$$\left(1-\frac{1}{x_1}\right)\left(1-\frac{1}{x_2}\right)\left(1-\frac{1}{x_3}\right)\cdots\left(1-\frac{1}{x_n}\right)=\frac{1}{134}\cdot\frac{670}{671}=\frac{5}{671}=\frac{15}{2013}.$$

Therefore, the required minimum value of n is 134. Q.E.D.

Problem 2. Let P be a point in the interior of $\triangle ABC$. Extend AP, BP, and CP to meet BC, AC, and AB at D, E, and F, respectively. If $\triangle APF$, $\triangle BPD$, and $\triangle CPE$ have equal areas, prove that P is the centroid of $\triangle ABC$.

Solution. Denote by (XYZ) the area of $\triangle XYZ$. Let $w = (APF) = (BPD) = (CPE)$, $x = (BPF)$, $y = (CPD)$, and $z = (APE)$.

Having the same altitude, we get



$$\frac{BD}{DC} = \frac{(BAD)}{(CAD)} = \frac{2w+x}{w+y+z}$$

and

$$\frac{BD}{DC} = \frac{(BPD)}{(CPD)} = \frac{w}{y},$$

which implies

$$wy + xy = w^2 + wz. \tag{1}$$

Similarly, we also get

$$wz + yz = w^2 + wx \quad \text{and} \quad wx + xz = w^2 + wy. \tag{2}$$

Combining equations (1) and (2) gives

$$xy + yz + xz = 3w^2. \tag{3}$$

On the other hand, by Ceva's Theorem, we have

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{(APF)}{(BPF)} \cdot \frac{(BPD)}{(CPD)} \cdot \frac{(CPE)}{(APE)} = \frac{w}{x} \cdot \frac{w}{y} \cdot \frac{w}{z} = 1, \tag{4}$$

or

$$w^3 = xyz. \tag{5}$$

Applying equation (5) to equation (3) gives

$$\frac{w}{z} + \frac{w}{x} + \frac{w}{y} = 3. \tag{6}$$

Equations (4) and (6) assert that the geometric mean and the arithmetic mean of the positive numbers $\frac{w}{x}$, $\frac{w}{y}$, and $\frac{w}{z}$ are equal. By the equality condition of the AM-GM Inequality, it follows that

$$\frac{w}{x} = \frac{w}{y} = \frac{w}{z} = 1 \quad \text{or} \quad w = x = y = z.$$

Therefore, we conclude that $AF = FB$, $BD = DC$, and $CE = EA$, which means that P is the centroid of $\triangle ABC$. Q.E.D.

Problem 3. Let n be a positive integer. The numbers $1, 2, 3, \ldots, 2n$ are randomly assigned to $2n$ distinct points on a circle. To each chord joining two of these points, a value is assigned equal to the absolute value of the difference between the assigned numbers at its endpoints.

Show that one can choose n pairwise non-intersecting chords such that the sum of the values assigned to them is n^2.

Solution. First, observe that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=n+1}^{2n} i = n^2 + \frac{n(n+1)}{2},$$

which means that

$$\sum_{i=n+1}^{2n} i - \sum_{i=1}^{n} i = n^2.$$

Let $A = \{1, 2, \ldots, n\}$ and $B = \{n+1, n+2, \ldots, 2n\}$. (Here, we do not distinguish the point labeled x and the number x itself.) Because the numbers are arranged on a circle, one can find a pair $\{x_1, y_1\}$, where $x_1 \in A$ and $y_1 \in B$, such that one arc joining x_1 and y_1 contains no other labeled points. One can then remove the chord (including x_1 and y_1) joining these points. Among the remaining labeled points, one can find again a pair $\{x_2, y_2\}$, where $x_2 \in A \setminus \{x_1\}$ and $y_2 \in B \setminus \{y_1\}$, such that one arc joining x_2 and y_2 does not contain a labeled point, and then remove again the chord (including the endpoints) joining x_2 and y_2. Continuing this process, one can find pairs $\{x_3, y_3\}$, $\{x_4, y_4\}$, and so on, and then remove the chords joining the pairs.

We claim that the removed chords satisfy the required properties. Clearly, there are n such chords. Because no labeled point lies on one arc joining x_j and y_j for any $1 \le j \le n$, the removed chords are non-intersecting. Finally, the sum of the values assigned to the removed chords is

$$\sum_{j=1}^{n} (y_j - x_j) = \sum_{j=1}^{n} y_j - \sum_{j=1}^{n} x_j = \sum_{i=n+1}^{2n} i - \sum_{i=1}^{n} i = n^2.$$

This ends the proof of our claim. Q.E.D.

Problem 4. Let a, p, and q be positive integers with $p \leq q$. Prove that if one of the numbers a^p and a^q is divisible by p, then the other number must also be divisible by p.

Solution. Suppose that $p \mid a^p$. Since $p \leq q$, it follows that $a^p \mid a^q$, which implies that $p \mid a^q$.

Now, suppose that $p \mid a^q$, and, on the contrary, $p \nmid a^p$. Then there is a prime number r and a positive integer n such that $r^n \mid p$ (which implies that $r^n \leq p$) and $r^n \nmid a^p$. Since $p \mid a^q$, it follows that $r \mid a$, and so $r^n \mid a^n$. This means that $p < n$, which gives the following contradiction:

$$2^p \leq r^p < r^n \leq p.$$

Therefore, a^p must also be divisible by p. Q.E.D.

Problem 5. Let r and s be positive real numbers that satisfy the equation

$$(r+s-rs)(r+s+rs)=rs.$$

Find the minimum values of $r+s-rs$ and $r+s+rs$.

Solution. The given equation can be rewritten into

$$(r+s)^2 = rs(rs+1). \tag{1}$$

Since $(r+s)^2 \geq 4rs$ for any $r,s \in \mathbb{R}$, it follows that $rs \geq 3$ for any $r,s>0$. Using this inequality, equation (1), and the assumption that r and s are positive, we have

$$\begin{aligned} r+s-rs = \sqrt{rs(rs+1)} - rs &= \frac{1}{\sqrt{1+\frac{1}{rs}}+1} \\ &\geq \frac{1}{\sqrt{1+\frac{1}{3}}+1} = -3+2\sqrt{3}. \end{aligned}$$

Similarly, we also have

$$r+s+rs \geq 3+2\sqrt{3}.$$

We show that these lower bounds can actually be attained. Observe that if $r=s=\sqrt{3}$, then

$$r+s-rs = -3+2\sqrt{3} \quad \text{and} \quad r+s+rs = 3+2\sqrt{3}.$$

Therefore, the required minimum values of $r+s-rs$ and $r+s+rs$ are $-3+2\sqrt{3}$ and $3+2\sqrt{3}$, respectively. Q.E.D.