PMO 2020 Qualifying Stage

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October 12, 2019

The date of the test is October 12 for all regions except Region 9, which held the test on October 19 instead. For cross-referencing purposes, this is the first year that the test numbering continues throughout the test, rather than restarting for each part. Are any explanations unclear? If so, contact me at cj@cjquines.com. More material is available on my website: https://cjquines.com.

PART I. Choose the best answer. Each correct answer is worth two points.

1. If $2^{x-1} + 2^{x-2} + 2^{x-3} = \frac{1}{16}$, find 2^x.

(a) $\frac{1}{14}$ (b) $\frac{2}{3}$ (c) $\sqrt[14]{2}$ (d) $\sqrt[3]{4}$

Answer. $\boxed{\text{(a) } \frac{1}{14}}$.

Solution. Factoring out 2^x, we get that

$$2^x\left(2^{-1} + 2^{-2} + 2^{-3}\right) = \frac{1}{16} \implies 2^x \cdot \frac{7}{8} = \frac{1}{16},$$

which means $2^x = \frac{1}{14}$.

2. If the number of sides of a regular polygon is decreased from 10 to 8, by how much does the measure of each of its interior angles decrease?

(a) $30°$ (b) $18°$ (c) $15°$ (d) $9°$

Answer. $\boxed{\text{(d) } 9°}$.

Solution 1. Each interior angle of a regular polygon with n sides is $\frac{180°(n-2)}{n}$, so the angles have measures $144°$ and $135°$, respectively. It decreases by $9°$.

Solution 2. Alternatively, the decrease in the interior angle is the same as the increase in the exterior angle. Since the exterior angles of a regular polygon are $\frac{360°}{n}$, we get $45° - 36° = 9°$.

3. Sylvester has 5 black socks, 7 white socks, 4 brown socks, where each sock can be worn on either foot. If he takes socks randomly and without replacement, how many socks would be needed to guarantee that he has at least one pair of socks of each color?

(a) 13 (b) 14 (c) 15 (d) 16

Answer. $\boxed{\text{(B) } 14}$.

Solution. In the worst case, Sylvester draws 5 black socks, 7 white socks, and 1 brown sock, without having a pair of socks in each color. When Sylvester draws the next sock, it must be brown. In total, he draws $5 + 7 + 2 = 14$ socks.

Remark. Compare PMO 2017 Qualifying I.7 "Issa has an urn containing only red and blue marbles. She selects a number of marbles from the urn at random and without replacement. She needs to draw at least N marbles in order to be sure that she has at least two red marbles. In contrast, she needs three times as much in order to be sure that she has at least two blue marbles. How many marbles are there in the urn?"

4. Three dice are simultaneously rolled. What is the probability that the resulting numbers can be arranged to form an arithmetic sequence?

(a) $\frac{1}{18}$ (b) $\frac{11}{36}$ (c) $\frac{7}{36}$ (d) $\frac{1}{6}$

Answer. $\boxed{\text{(c) } \frac{7}{36}}$.

Solution. We count how many times each possible arithmetic sequence appears in all $6^3 = 216$ possible rolls, doing casework on the common difference.

- When the difference is 0, each of $(1, 1, 1)$, $(2, 2, 2)$, and so on, can be rolled in one way. This accounts for 6 ways.
- When the difference is 1, each of $(1, 2, 3)$, $(2, 3, 4)$, up to $(4, 5, 6)$ can be rolled in $3! = 6$ possible ways. This gives for $4 \cdot 6 = 24$ ways.
- Similarly, for difference 2, we get $(1, 3, 5)$ and $(2, 4, 6)$ giving $2 \cdot 6 = 12$ ways.
- We can't have a common difference larger than 2.

In total, we get $6 + 24 + 12 = 42$ possible ways, giving a probability of $\frac{42}{216} = \frac{7}{36}$.

5. Sean and the bases of three buildings, A, B, and C are all on level ground. Sean measures the angles of elevation of the tops of buildings A and B to be $62°$ and $57°$, respectively. Meanwhile, on top of building C, CJ spots Sean and determines that the angle of depression of Sean from his location is $31°$. If the distance from Sean to the bases of all three buildings is the same, arrange buildings A, B, and C in order of increasing heights.

(a) C, B, A (b) B, C, A (c) A, C, B (d) A, B, C

Answer. $\boxed{\text{(a) } C, B, A}$.

Solution. Sean, the base of the building, and the top of the building, form the vertices of a right triangle, with the right angle at the base of the building. The angle of elevation is the angle opposite the side corresponding to the height of the building. Since the distance from Sean to the base of each building is the same, the larger the angle of elevation, the longer the length of the opposite side must be, which is the height of the building. The angle of depression from CJ to Sean is the same as his angle of elevation, so in order of increasing height, we get C, B, and A.

6. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(xy) = f(x)/y^2$ for all positive real numbers x and y. Given that

$f(25) = 48$, what is $f(100)$?

(a) 1 (b) 2 (c) 3 (d) 4

Answer. $\boxed{\text{(c) } 3}$.

Solution. Substituting $x = 25$ and $y = 4$ to the given equation, we get $f(100) = f(25)/16$, so $f(100) = 3$.

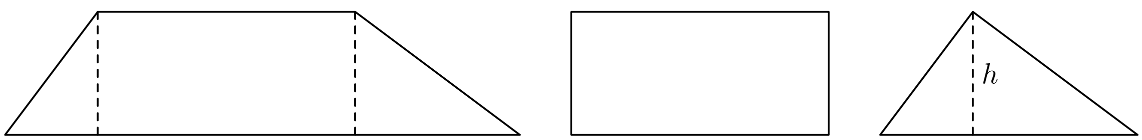
Remark. We can also solve for the function by substituing $x = 25$ and $y = x'/25$ to get $f(x') = 30000/x'^2$. Compare PMO 2017 Areas I.2 "Let f be a real-valued function such that $f(x - f(y)) = f(x) - xf(y)$ for any real numbers x and y. If $f(0) = 3$, determine $f(2016) - f(2013)$."

7. A trapezoid has parallel sides of length 10 and 15; its other sides have lengths 3 and 4. Find its area.

(a) 24 (b) 30 (c) 36 (d) 42

Answer. $\boxed{\text{(b) } 30}$.

Solution. We can dissect the trapezoid into a rectangle with width 10, and two triangles. Putting the two triangles together gives a right triangle with side lengths 3, 4, and 5.



The height of the trapezoid must be the height of the triangle. Letting this height be h, we can compute the area of the triangle both as $\frac{1}{2} \cdot 3 \cdot 4$ and $\frac{1}{2} \cdot 5 \cdot h$, so $h = \frac{12}{5}$. The area of the rectangle is $\frac{12}{5} \cdot 10$, and the area of the triangle is $\frac{1}{2} \cdot 3 \cdot 4$, so the area of the whole trapezoid is 30.

Remark. Compare with PMO 2019 National Orals Average 7: "In trapezoid $ABCD$, AD is parallel to BC. If $AD = 52$, $BC = 65$, $AB = 20$, and $CD = 11$, find the area of the trapezoid."

8. Find the radius of the circle tangent to the line $3x + 2y - 4 = 0$ at $(-2, 1)$ and whose center is on the line $x - 8y + 36 = 0$.

(a) $2\sqrt{13}$ (b) $2\sqrt{10}$ (c) $3\sqrt{5}$ (d) $5\sqrt{2}$

Answer. $\boxed{\text{(a) } 2\sqrt{13}}$.

Solution. Consider the line ℓ passing through the tangent point, perpendicular to the tangent line. This line must pass through the center of the circle, so we can intersect it with the line $x - 8y + 36 = 0$ to find the center of the circle.

The slope of the tangent line is $-\frac{3}{2}$, so the slope of ℓ is $\frac{2}{3}$. It passes through the point $(-2, 1)$, so this line must be $y = \frac{2}{3}x + \frac{7}{3}$. Substituting to the second equation, we get

$$x - 8\left(\frac{2}{3}x + \frac{7}{3}\right) + 36 = 0 \implies x = 4,$$

which means $y = 5$. The distance of the tangency point $(-2, 1)$ to the center $(4, 5)$ is $2\sqrt{13}$, which must be the radius.

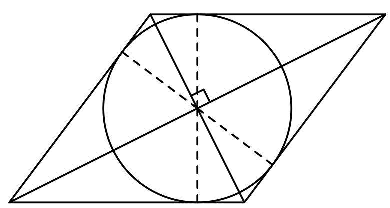
Remark. Compare with PMO 2019 Qualifying I.9 "A circle is tangent to the line $2x - y + 1 = 0$ at the point $(2, 5)$ and the center is on the line $x + y - 9 = 0$. Find the radius of the circle."

9. A circle is inscribed in a rhombus which has a diagonal of length 90 and area 5400. What is the circumference of the circle?

(a) 36π (b) 48π (c) 72π (d) 90π

Answer. $\boxed{\text{(c) } 72\pi}$.

Solution. The area of the rhombus is half the product of its diagonals, meaning the other diagonal has length 120. The intersecting diagonals form four triangles. As the diagonals are the perpendicular bisectors of each other, they form a right triangle with legs 45 and 60. By the Pythagorean theorem, the rhombus has side length 75.



Consider each of these four triangles. Dropping the perpendicular from the intersection of the diagonals to the sides, they must be the same length h. So the inscribed circle must be centered at the intersection, and have radius of length h.

We can compute the area of each triangle as $\frac{1}{2} \cdot 75 \cdot h$, so the area of the whole rhombus is $4 \cdot \frac{1}{2} \cdot 75 \cdot h$. But this is 5400, so $h = 36$. The circumference of the circle must then be $2\pi h = 72\pi$.

Remark. We computed the area of the rhombus as half the product of the perimeter and the radius of the inscribed circle. This is similar to the formula sr for the area of a triangle, where s is the semiperimeter and r is the inradius.

10. Suppose that n identical promo coupons are to be distributed to a group of people, with no assurance that everyone will get a coupon. If there are 165 more ways to distribute these to four people than there are ways to distribute these to three people, what is n?

(a) 12 (b) 11 (c) 10 (d) 9

Answer. $\boxed{\text{(d) } 9}$.

Solution. The number of ways to distribute n identical objects to four distinct people is, by balls and urns, $\binom{n+3}{3}$. Similarly, the number of ways to distribute to three people is $\binom{n+2}{2}$. This means that $\binom{n+3}{3} - \binom{n+2}{2} = 165$. Trying each choice, we see that $n = 9$ works.

Remark. Compare to PMO 2016 Qualifying III.4: "Let $N = \{0, 1, 2 \ldots\}$. Find the cardinality of the set $\{(a, b, c, d, e) \in N^5 : 0 \le a + b \le 2, 0 \le a + b + c + d + e \le 4\}$", or PMO 2016 Areas I.9: "How many ways can you place 10 identical balls in 3 baskets of different colors if it is possible for a basket to be empty?", or PMO 2016 Nationals Easy 11: "How many solutions does $x + y + z = 2016$ have, where x, y, and z are integers with $x > 1000$, $y > 600$, and $z > 400$?", or PMO 2017 Qualifying II.9: "How many ordered triples of positive integers (x, y, z) are there such that $x + y + z = 20$ and two of x, y, z are odd?", or PMO 2019 Qualifying I.5: "Juan has 4 distinct jars and a certain number of identical balls. The number of ways that he can distribute the balls into the jars where each jar has at least one ball is 56. How many balls does he have?", or PMO 2019 Areas I.16 "Compute the number of ordered 6-tuples (a, b, c, d, e, f) of positive integers such that $a + b + c + 2(d + e + f) = 15$."

11. Let x and y be positive real numbers such that

$$\log_x 64 + \log_{y^2} 16 = \frac{5}{3} \qquad \text{and} \qquad \log_y 64 + \log_{x^2} 16 = 1.$$

What is the value of $\log_2(xy)$?

(a) 16 (b) 3 (c) $\frac{1}{3}$ (d) $\frac{1}{48}$

Answer. $\boxed{\text{(a) } 16}$.

Solution. We can use the change of base formula to write all of the logarithms in base 2. This gives the two equations as

$$\frac{\log_2 64}{\log_2 x} + \frac{\log_2 16}{\log_2 y^2} = \frac{5}{3} \qquad \text{and} \qquad \frac{\log_2 64}{\log_2 y} + \frac{\log_2 16}{\log_2 x^2} = 1.$$

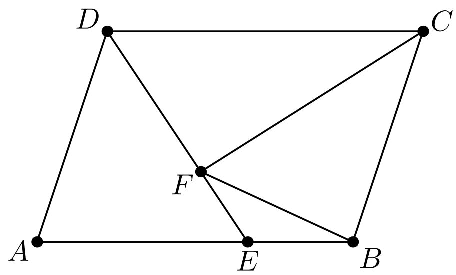
Writing $\log_2 x^2 = 2\log_2 x$ and $\log_2 y^2 = 2\log_2 y$, we can rewrite the equations entirely using $u = \dfrac{1}{\log_2 x}$ and $v = \dfrac{1}{\log_2 y}$:

$$6u + 2v = \frac{5}{3} \qquad \text{and} \qquad 6v + 2u = 1.$$

We add the two equations to get $8u + 8v = \frac{8}{3}$ and divide by 4 to get $2u + 2v = \frac{2}{3}$. Subtracting from the first equation, we get $u = \frac{1}{4}$, and so $v = \frac{1}{12}$. Finally,

$$\log_2(xy) = \log_2 x + \log_2 y = \frac{1}{u} + \frac{1}{v} = 16.$$

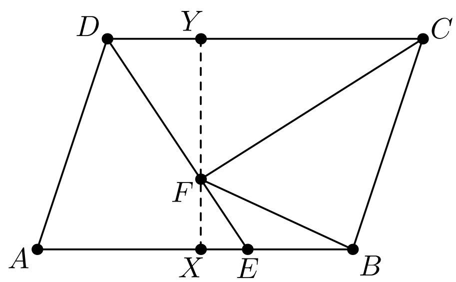
12. The figure below shows a parallelogram $ABCD$ with $CD = 18$. Point F lies inside $ABCD$ and lines AB and DF meet at E. If $AE = 12$ and the areas of triangles FEB and FCD are 30 and 162, respectively, find the area of triangle BFC.



(a) 162 (b) 156 (c) 150 (d) 144

Answer. $\boxed{\text{(d) } 144}$.

Solution. Let the feet of the perpendiculars from F to EB and CD be X and Y, respectively.



As the area of triangle FEB is 30, we get $\frac{1}{2} \cdot FX \cdot EB = \frac{1}{2} \cdot FX \cdot 6 = 30$, which means $FX = 10$. Similarly, we find $FY = 18$. So the height of the parallelogram is $10 + 18 = 28$. The area of trapezoid $EBCD$ is $\frac{1}{2}(18 + 6)(28) = 336$, so the area of triangle BFC is $336 - 30 - 168 = 144$.

13. A *semiprime* is a natural number that is the product of two primes, not necessarily distinct. How many subsets of the set $\{2, 4, 6, \ldots, 18, 20\}$ contain at least one semiprime?

(a) 768 (b) 896 (c) 960 (d) 992

Answer. $\boxed{\text{(c) } 960}$.

Solution. We use complimentary counting. The total number of subsets is $2^{10} = 1024$. We now count the number of subsets which don't contain any semiprime. The semiprimes are $4, 6, 10$, and 14, so there are 6 remaining elements, which form $2^6 = 64$ subsets. Subtracting these gives $1024 - 64 = 960$.

14. The number whose base-b representation is 91_b is divisible by the number whose base-b representation is 19_b. How many possible values of b are there?

(a) 2 (b) 3 (c) 4 (d) 5

Answer. $\boxed{\text{(b) } 3}$.

Solution. In base-10, 91_b is $9b + 1$ and 19_b is $b + 9$. So the ratio $\dfrac{9b+1}{b+9} = 9 - \dfrac{80}{b+9}$ must be an integer. For $\dfrac{80}{b+9}$ to be an integer, we must have $b = 1, 7, 11, 31, 71$. But for 91_b to be a number in base-b, we must have $b > 9$. So there are 3 possible values of b.

Remark. Several problems boil down to determining when a quotient like this is an integer, and the strategy of doing division works. Compare PMO 2009 Qualifying II.7 "How many values of n are there for which n and $\frac{n+3}{n-1}$ are both integers?" or PMO 2015 Areas I.11 "Find all integer values of n that will make $\frac{6n^3-n^2+2n+32}{3n+1}$ an integer", or Problem 21 later on.

15. The number of ordered pairs (a, b) of relatively prime positive integers such that $ab = 36!$ is

(a) 128 (b) 1024 (c) 2048 (d) 4096

Answer. $\boxed{\text{(c) } 2048}$.

Solution. Suppose the prime factorization of 36! is $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. For a and b to be relatively prime, they must share no prime divisors. That means that all of $p_1^{e_1}$ must go into either a or b; we can't split it between the two of them. So each ordered pair (a, b) corresponds to a different way to partition the prime factors of 36!.

There are 11 primes less than 36: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$, and 31. So there are $2^{11} = 2048$ different ways to partition these primes, and thus 2048 different ordered pairs.

PART II. Choose the best answer. Each correct answer is worth three points.

16. Which of the following **cannot** be the difference between a positive integer and the sum of its digits?

(a) 603 (b) 684 (c) 765 (d) 846

Answer. $\boxed{\text{(b) } 684}$.

Solution 1. Each of the choices, except 684 can be written as the difference between a positive integer and the sum of its digits. For example, $610 - 7 = 603$, $780 - 15 = 765$, and $860 - 14 = 846$. However, we get that $690 - 15 = 675$, but $700 - 7 = 693$. This skips over 684, so it cannot be written as the difference between a positive integer and the sum of its digits.

Solution 2. A systematic way to find this would be, if the positive integer was $\overline{abc}$, we're solving the equation

$$100a + 10b + c - (a + b + c) = 9(11a + b) = 603, 684, 765, 846.$$

So for 603, for example, we get $11a + b = 67$, and we have $(a, b) = (6, 1)$ as a solution. Similarly, for 765 we get $11a + b = 85$ and thus $(a, b) = (7, 8)$ is a solution, and for 846, we get $11a + b = 94$, so $(a, b) = (8, 6)$ is a solution. But for 684, we get $11a + b = 76$, and there's no solution for this where a and b are digits.

Remark. These numbers are OEIS A282473. They have the interesting property that the difference between consecutive terms seems to almost always be 9. Sharvil informs me this is similar to a problem from the Australian Mathematical Olympiad 2016.

17. Evaluate the sum

$$\sum_{n=0}^{2019} \cos\left(\frac{n^2\pi}{3}\right).$$

(a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

Answer. $\boxed{\text{(a) } 0}$.

Solution. Writing out the values of $\cos\left(\frac{n^2\pi}{3}\right)$ for $n = 0, 1, 2, \ldots$, it takes the values

$$1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, \ldots$$

So we would guess that it repeats every 6 terms. The sum of every 6 terms is 0, meaning that the only remaining terms are the $n = 2016$ through $n = 2019$ terms, which must be the same as the $n = 0$ through $n = 3$ terms. The sum of these terms is also 0, so the whole sum is 0.

Remark. Compare PMO 2017 Qualifying I.9 "Evaluate the following sum: $1 + \cos\frac{\pi}{3} + \cos\frac{2\pi}{3} + \cos\frac{3\pi}{3} + \cdots + \cos\frac{2016\pi}{3}$", or PMO 2018 Qualifying I.7 "Evaluate the sum $\sum_{n=3}^{2017} \sin\left(\frac{(n!)\pi}{36}\right)$."

18. There is an unlimited supply of red 4×1 tiles and blue 7×1 tiles. In how many can an 80×1 path be covered using nonoverlapping tiles from this supply?

(a) 2381 (b) 3382 (c) 5384 (d) 6765

Answer. $\boxed{\text{(c) } 5384}$.

Solution. Suppose we use r red tiles and b blue tiles. As the total number of tiles is 80, they must satisfy $4r + 7b = 80$, which has solutions (r, b) as $(20, 0)$, $(13, 4)$, and $(6, 8)$. There's one way to arrange 20 tiles in a row. There are $\binom{17}{4}$ ways to arrange 13 red tiles and 4 blue tiles in a row: we choose which of the $13 + 4 = 17$ tiles are blue, and the rest of the tiles are red. Similarly, there are $\binom{14}{8}$ ways for the other case. This gives a total of $1 + 2380 + 3003 = 5384$ ways.

19. For a real number t, $\lfloor t \rfloor$ is the greatest integer less than or equal to t. How many natural numbers n are there such that $\left\lfloor \frac{n^3}{9} \right\rfloor$ is prime?

(a) 3 (b) 9 (c) 27 (d) infinitely many

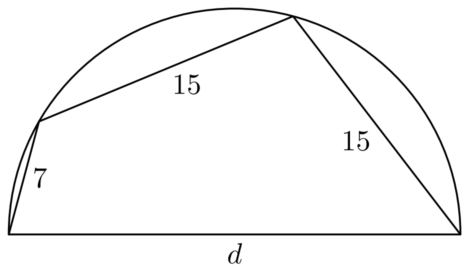
Answer. $\boxed{\text{(a) } 3}$.

Solution. Considering that we're dividing by 9, we do casework on the value of n modulo 3. Letting $n = 3k + r$, for some $r = 0, 1, 2$, we get that

$$\frac{(3k + r)^3}{9} = \frac{27k^3 + 27k^2r + 9kr^2 + r^3}{9} = 3k^3 + 3k^2r + kr^2 + \frac{r^3}{9}.$$

For $r = 0, 1, 2$, the last term is always less than 1, so taking the floor leaves only $3k^3 + 3k^2r + kr^2$. This factors as $k(3k^2 + 3kr + r^2)$. For this to be prime, one of these factors must be 1, which must be k. When $k = 1$, we can check $n = 3, 4, 5$ to get $3, 7, 13$, which all work. So there are only 3 possible values of n.

20. A quadrilateral with sides of length $7, 15, 15$, and d is inscribed in a semicircle with diameter d, as shown in the figure below.

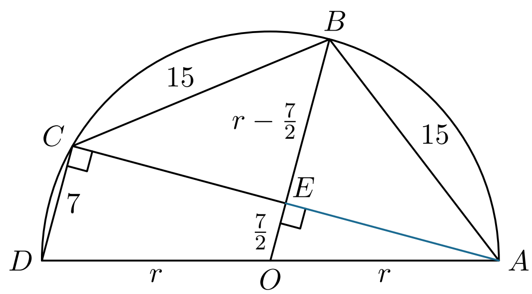


Find the value of d.

(a) 18 (b) 22 (c) 24 (d) 25

Answer. $\boxed{\text{(d) } 25}$.

Solution 1. Let r be the radius, and label the points as in the figure below.

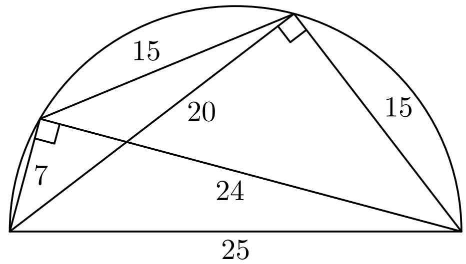


Draw the radius OB. This radius bisects the chord AC, and therefore must be perpendicular to it. This produces two similar right triangles, $\triangle DCA \sim \triangle OFA$. The hypotenuse of $\triangle DCA$ is $2r$ while the hypotenuse of $\triangle OFA$ is r, so because $DC = 7$, we get $OF = \frac{7}{2}$. The length of AE can then be computed in two different ways using the Pythagorean theorem:

$$r^2 - \left(\frac{7}{2}\right)^2 = 15^2 - \left(r - \frac{7}{2}\right)^2 \implies 2r^2 - 7r - 225 = 0,$$

which factors as $(2r - 25)(r + 9) = 0$. Discarding the negative solution, $d = 2r = 25$.

Solution 2. The lengths 7, 15, and the choices, suggest both the 7–24–25 right triangle, and the 15–20–25 right triangle. This would make sense given the two right triangles formed by the diagonals. We make a guess that the lengths in the diagram are like so:



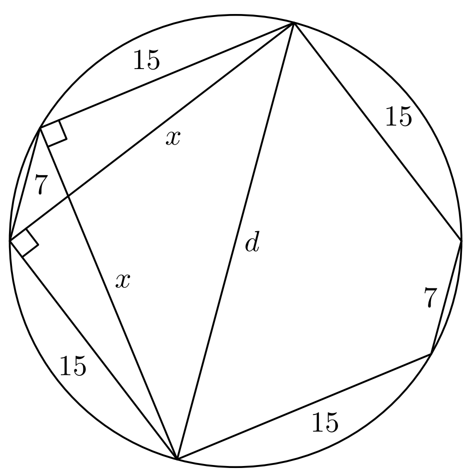
We can verify that the top length is 15 through Ptolemy's, as $20 \cdot 24 = 15 \cdot 25 + 7 \cdot 15$. So the answer is 25.

Solution 3. There's a solution that does not involve guessing that uses the same idea. By the Pythagorean theorem, the length of the two diagonals are $\sqrt{d^2 - 49}$ and $\sqrt{d^2 - 225}$. Using Ptolemy's, we find that

$$\begin{aligned}\sqrt{d^2 - 49} \cdot \sqrt{d^2 - 225} &= 15d + 7 \cdot 15 \\ (d^2 - 49)(d^2 - 225) &= (15d + 105)^2 \\ d^4 - 499d^2 - 3150d &= 0.\end{aligned}$$

We can then factor the last equation as $d(d+7)(d+18)(d-25) = 0$ to get $d = 25$, or we can plug in each of the choices to see that $d = 25$ is a root. So the answer must be 25.

Solution 4. Rotate the semicircle about its center by $180°$ to complete the hexagon. All diameters are the same length, so we can find the length of any of these diameters instead. In particular, we focus on an isosceles trapezoid with bases 7 and d, and legs 15.



Both of its diagonals have the same length; call it x. By Ptolemy's, $x^2 = 7d + 225$. As a diagonal forms a right triangle with the diameter as the hypotenuse, by the Pythagorean theorem, we get $x^2 + 225 = d^2$. Combining these, we get $d^2 - 7d - 450 = 0$, which factors as $(d+18)(d-25) = 0$, so $d = 25$.

Remark. Dr. Eden shared Solution 1 to me. Siva and Andrew shared Solution 4 to me. In a contest, I would do Solution 2. It falls under the class of techniques I describe as *engineering*. I think engineering is helpful for short-answer competition math problems, and it's a skill that people don't exercise as often. I'm working on an article about this, so stay tuned. Compare to AIME 2013 II Problem 8.

21. Find the sum of all real numbers b for which all the roots of the equation $x^2 + bx - 3b = 0$ are integers.

(a) 4 (b) -8 (c) -12 (d) -24

Answer. $\boxed{\text{(d) } -24}$.

Solution 1. Let the roots be r and s. By Vieta's, $r + s = -b$ and $rs = -3b$, so $rs = 3r + 3s$. We can solve for r as

$$r = \frac{3s}{s-3} = 3 + \frac{9}{s-3},$$

and so $s - 3$ has to be a factor of 9, so it can be either -9, -3, -1, 1, 3, or 9. This gives

$$(r, s) = (2, -6), (0, 0), (-6, 2), (12, 4), (6, 6), (4, 12).$$

As $b = -(r + s)$, its possible values are $4, 0, -16, -12$, which has sum -24.

Solution 2. For the roots of the equation to be integers, the discriminant $b^2 + 12b$ must be a perfect square a^2. We complete the square to get

$$\begin{aligned} a^2 &= b^2 + 12b \\ a^2 + 36 &= (b+6)^2 \\ 36 &= (b + 6 - a)(b + 6 + a), \end{aligned}$$

by using the difference of two squares. If a is an integer, then b must be an integer as well, so we're looking at factorizations of 36 into two integers.

To solve the system of equations, for both a and b to be integers, we must have $b + 6 - a$ and $b + 6 + a$ as both odd or both even. Also, $b + 6 - a$ is less than $b + 6 + a$. So the possible factors of 36 that correspond to $(b + 6 - a, b + 6 + a)$ are $(-18, -2), (-6, -6), (2, 18)$, and $(6, 6)$. The bs that correspond to each of these are $-16, -12, 4, 0$. This gives a sum of -24.

Solution 3. From the fact that the discriminant is $b^2 + 12b = (b + 6)^2 - 36$, we can see that if b is a solution, then $-12 - b$ has to be a solution as well, because

$$(b+6)^2 - 36 = ((-12 - b) + 6)^2 + 36.$$

By pairing up b and $-12 - b$, we see the answer has to be some multiple of -12. Looking at the choices, we see 4, which happens to be a solution. By inspection, 0 has to be another solution. So we have at least two pairs of solutions adding up to -12, which means the answer is at most -24. But this is the smallest choice, so it must be correct.

Remark. Completing the square is a neat trick here, and is a good idea whenever we have a quadratic Diophantine equation. Compare AMC 2015 12A Problem 18 / 10A Problem 23, "The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of all possible values of a?"

22. A number x is selected randomly from the set of all real numbers such that a triangle with side lengths $5, 8$, and x may be formed. What is the probability that the area of this triangle is greater than 12?

(a) $\dfrac{3\sqrt{15}-5}{10}$ (b) $\dfrac{3\sqrt{15}-\sqrt{41}}{10}$ (c) $\dfrac{3\sqrt{17}-5}{10}$ (d) $\dfrac{3\sqrt{17}-\sqrt{41}}{10}$

Answer. $\boxed{\text{(c) } \dfrac{3\sqrt{17}-5}{10}}$.

Solution 1. Consider how the area of the triangle varies as x varies. When x is close to 3, the area is close to zero. As x increases, the area gradually increases until it reaches a maximum. Then as x continues increasing, the area decreases, until when x is close to 13, and it becomes close to zero again. So there's some minimum value of x and some maximum value of x, where in between these values the area of the triangle is at least 12. At these values, the area of the triangle must be 12.

We use Heron's formula to determine when the area is 12. The semiperimeter is $s = \dfrac{x+13}{2}$, and we get

$$\begin{aligned} 12 &= \sqrt{s(s-a)(s-b)(s-c)} \\ 144 &= \frac{x+13}{2} \cdot \frac{x+3}{2} \cdot \frac{x-3}{2} \cdot \frac{13-x}{2} \\ 2304 &= (169-x^2)(x^2-9), \end{aligned}$$

which conveniently factors into $(x^2-25)(x^2-153)=0$. Discarding the negative solutions, the area of the triangle is 12 when $x=5$ or $x=3\sqrt{17}$, and is greater than 12 for any value in between. Since x can be any value between 3 and 13, the probability is $\dfrac{3\sqrt{17}-5}{10}$.

Solution 2. Here's another way to find for which x the area would be 12. Let θ be the angle opposite x, and use the formula $\frac{1}{2}bc\sin\theta$ for the area. We find that

$$\frac{1}{2} \cdot 5 \cdot 8 \cdot \sin\theta = 12 \implies \sin\theta = \frac{3}{5}.$$

From $\sin^2\theta + \cos^2\theta = 1$, we find that $\cos\theta = \pm\frac{4}{5}$. We can then use the law of cosines to find the possible x:

$$x = \sqrt{5^2+8^2-2\cdot 5\cdot 8\cdot\cos\theta} = \sqrt{89 \pm 64} = 5, 3\sqrt{17}.$$

23. Two numbers a and b are chosen randomly from the set $\{1, 2, \ldots, 10\}$ in order, and with replacement. What is the probability that the point (a, b) lies above the graph of $y = ax^3 - bx^2$?

(a) $\dfrac{4}{15}$ (b) $\dfrac{9}{50}$ (c) $\dfrac{19}{100}$ (d) $\dfrac{1}{5}$

Answer. $\boxed{\text{(c) } \dfrac{19}{100}}$.

Solution. For (a, b) to lie above the graph, it must be the case that $y > ax^3 - bx^2$, or $b > a^4 - a^2b$. This rearranges to $b > \dfrac{a^4}{a^2+1} = a^2 + \dfrac{1}{a^2+1}$. As b has to be an integer, we get $b \geq a^2$. We now do casework on the value of a:

- If $a = 1$, then any value of b works, so there are 10 possible pairs (a, b) that work.

- If $a = 2$, then $b \geq 4$ work, giving 7 possible pairs.
- If $a = 3$, then $b \geq 9$ work, giving 2 pairs.
- For $a \geq 4$, we find that $a^2 > 10$, so there's no value of b that works.

This gives us $10 + 7 + 2 = 19$ possible pairs that work out of 100, so the probability is $\dfrac{19}{100}$.

24. For a real number t, $\lfloor t \rfloor$ is the greatest integer less than or equal to t. How many integers n are there with $4 \leq n \leq 2019$ such that $\lfloor \sqrt{n} \rfloor$ divides n and $\lfloor \sqrt{n+1} \rfloor$ divides $n+1$?

(a) 44 (b) 42 (c) 40 (d) 38

Answer. $\boxed{\text{(b) } 42}$.

Solution. We determine which integers n satisfy $\lfloor \sqrt{n} \rfloor$ divides n. Let $k = \lfloor \sqrt{n} \rfloor$, and let r be such that $n = k^2 + r$. Note that $0 \leq r < 2k + 1$, because otherwise $k + 1$ would be the largest integer less than or equal to $\sqrt{n}$. If k divides n, it then means r must be either 0, k, or $2k$.

Now if $n = k^2$, then $n+1$ would be k^2+1, which cannot satisfy the conditions for $\lfloor \sqrt{n} \rfloor$ to divide n. Similarly, if $n = k^2 + k$, then $k^2 + k + 1$ also can't satisfy the conditions for $\lfloor \sqrt{n} \rfloor$ to divide n.

So it must be the case that $n = k^2 + 2k$, and $n + 1 = k^2 + 2k + 1$, which is $(k + 1)^2$. This indeed satisfies the conditions for $\lfloor \sqrt{n+1} \rfloor$ to divide $n + 1$. So any positive integer n equal to $k^2 + 2k$ for some k works.

For $4 \leq k^2 + 2k \leq 2019$, we must have $k = 2, 3, \ldots, 43$. So there are 42 choices of k that works.

Remark. The numbers n such that $\lfloor \sqrt{n} \rfloor$ divides n are OEIS A006446.

25. The number $20^5 + 21$ has two prime factors which are three-digit numbers. Find the sum of these numbers.

(a) 1112 (b) 1092 (c) 1062 (d) 922

Answer. $\boxed{\text{(a) } 1112}$.

Solution 1. Letting $x = 20$, we're motivated to factor $x^5 + x + 1$. Let ω be a primitive cube root of unity; that is, ω satisfies $\omega^3 = 1$ but $\omega \neq 1$. This is

$$\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0$$

and as $\omega \neq 1$, we get $\omega^2 + \omega + 1 = 0$. Also, from $\omega^3 = 1$, we can multiply both sides by ω^2 to get $\omega^5 = \omega^2$. So

$$\omega^5 + \omega + 1 = \omega^2 + \omega + 1 = 0,$$

and ω is a root of $x^5 + x + 1$, so $x^2 + x + 1$ must be a factor of $x^5 + x + 1$. Indeed, $x^5 + x + 1 = (x^2 + x + 1)(x^3 - x^2 + 1)$, and so we get $20^5 + 21 = 421 \cdot 7601$. We're given that the number has two prime factors which are three-digit numbers, so 7601 has to be divisible by some small prime. Checking small primes shows that it's divisible by 11, as $7601 = 11 \cdot 691$. So the two primes are 421 and 691, and their sum is 1112.

Solution 2. There are several other ways to find the factorization. We can quickly check that x^5+x+1 doesn't have $x+1$ or $x-1$ as a factor, so if it is factorable, it must be a cubic times a quadratic:

$$x^5+x+1 = (x^3+ax^2+bx+1)(x^2+cx+1).$$

Expanding the right-hand side, we get

$$x^5+(a+c)x^4+(ac+b+1)x^3+(a+bc+1)x^2+(b+c)x+1=0.$$

By comparing coefficients, we find the solution $(a,b,c)=(-1,0,1)$, which gives us the factorization.

Remark. I've used the same problem in one of the tests I've written. See PRIME 2017 Final III.3, "What is the sum of the prime factors of 3 200 021?"

PART III. All answers should be in simplest form. Each correct answer is worth six points.

26. Find the number of ordered triples of integers (m,n,k) with $0<k<100$ satisfying

$$\frac{1}{2^m}-\frac{1}{2^n}=\frac{3}{k}.$$

Answer. $\boxed{13}$.

Solution. Rearranging the equation, we get

$$\frac{2^{n-m}-1}{2^n}=\frac{3}{k} \implies k(2^{n-m}-1)=3\cdot 2^n.$$

For the left-hand side to be positive, we must have $n>m$. So the factor $2^{n-m}-1$ has to be odd. We get that 2^n has to divide k, and we can write k as $2^n\ell$ for some integer ℓ, to get

$$\ell(2^{n-m}-1)=3.$$

We have two cases:

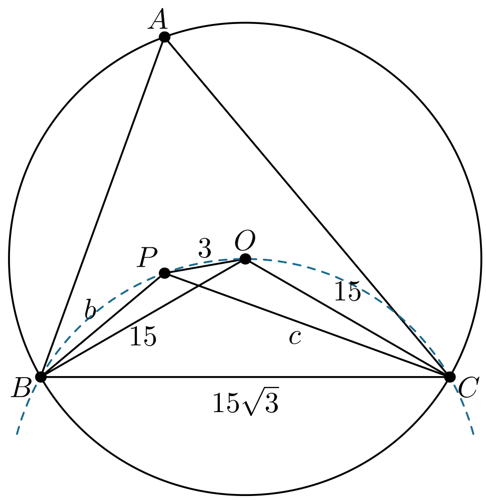
- $\ell=3$ and $2^{n-m}-1=1$. In this case, $n-m=1$, and $k=3\cdot 2^n$. For $0<3\cdot 2^n<100$, we get that $n=0,1,\ldots,5$, giving 6 possible triples.
- $\ell=1$ and $2^{n-m}-1=3$. In this case, $n-m=2$, and $k=2^n$. For $0<2^n<100$, we must have $n=0,1,\ldots,6$, giving 7 possible triples.

In total, this gives us 13 possible triples.

27. Triangle ABC has $\angle BAC=60^\circ$ and circumradius 15. Let O be the circumcenter of ABC and let P be a point inside ABC such that $OP=3$ and $\angle BPC=120^\circ$. Determine the area of triangle BPC.

Answer. $\boxed{54\sqrt{3}}$.

Solution. As $\angle BAC=60^\circ$ and O is the circumcenter, we get that $\angle BOC=120^\circ$. As this is equal to $\angle BPC$, quadrilateral $BPOC$ is cyclic. This motivates us to draw the radii BO and CO, both of which have length 15. Using the law of cosines on $\triangle BOC$, we find that $BC=15\sqrt{3}$.



Let $b = BP$ and $c = CP$. Using the law of cosines again on $\triangle BPC$, we find that $b^2+bc+c^2 = 675$. Applying Ptolemy's theorem on quadrilateral $BPOC$ gives us

$$15c = 45\sqrt{3} + 15b \implies c - b = 3\sqrt{3}.$$

We can square this equation to get $b^2 - 2bc + c^2 = 27$. Combining with the previous equation, we find $bc = 216$. The area of $\triangle BPC$ is then

$$\frac{1}{2}bc\sin 120^\circ = \frac{1}{2} \cdot 216 \cdot \frac{\sqrt{3}}{2} = 54\sqrt{3}.$$

28. A string of 6 digits, each taken from the set $\{0, 1, 2\}$, is to be formed. The string should **not** contain any of the substrings 012, 120, and 201. How many such 6-digit strings can be formed?

Answer. $\boxed{492}$.

Solution 1. We use PIE and complementary counting. In particular, we need to know the number of 6-digit strings that **do** contain some of these substrings. By using symmetry, we only have to consider three cases:

- The string contains 012. We have to choose the other 3 digits in 3^3 ways, and then arrange 012 with the other 3 digits in 4 ways. But this counts the string 012012 twice, so the actual count is $3^3 \cdot 4 - 1 = 107$.
- The string contains both 012 and 120. This can appear as 0120, which by similar logic, happens in $3 \cdot 3^2 = 27$ possible ways, or as 12012, which happens in $2 \cdot 3 = 6$ possible ways, or as 012120, or 120012, which are 2 ways. But the strings 012012 and 120120 are counted twice, so the actual count is $27 + 6 + 2 - 2 = 33$.
- The string contains all of 012, 120, and 201. There are, again, 6 possible ways for it to contain each of 01201, 12012, or 20120. But the strings 012012, 120120, and 201201 are each counted twice. So the actual count is $6 + 6 + 6 - 3 = 15$.

As there are 729 possible strings in total, the answer is $729 - 3 \cdot 107 + 3 \cdot 33 - 15 = 492$.

Solution 2. Let the string be $s_1 s_2 \cdots s_6$, and consider the transformed string $t_1 t_2 \cdots t_6$, where $t_i = s_i - i$ modulo 3. Then instead of avoiding the substrings 012, 120, and 201, we want to avoid the substrings 000, 111, and 222 instead. Call a string *valid* if it satisfies this.

Let a_n denote the number of such strings of length n. Given a valid string $t_1 t_2 \cdots t_n$, there are two cases. The first case is that $t_n = t_{n-1}$, which means $t_{n-1} \neq t_{n-2}$ and removing the last two characters gives a valid string of length $n-2$. The second case is that $t_n \neq t_{n-1}$, and removing the last character gives a valid string of length $n-1$.

So given a valid string of length $n-2$, we can append two of 00, 11, or 22 to make a valid string of length n. Similarly, given a valid string of length $n-1$, we can append two of 0, 1, or 2 to make a valid string of length n. We thus get the recursion $a_n = 2a_{n-1} + 2a_{n-2}$. The base cases are $a_1 = 3$ and $a_2 = 9$. We continue to find 24, 66, 180, then 492, which is the answer.

Solution 3. We proceed from the previous solution, and count the number of strings that avoid 000, 111, and 222. Let b_n be the number of valid strings $t_1 \cdots t_n$ such that $t_{n-1} = t_n$, and let c_n be the number of valid strings such that $t_{n-1} \neq t_n$. We can then write the recursions

$$\begin{aligned} b_n &= c_{n-1} \\ c_n &= 2c_{n-1} + 2b_{n-1}, \end{aligned}$$

with the base cases $b_1 = 0$, $b_2 = 3$, $c_1 = 3$, and $c_2 = 6$. It can be checked that this gives the same answer of 492.

Remark. The sequence a_n is OEIS A121907. Compare to PMO 2019 Areas I.11 "A *Vitas word* is a string of letters that satisfies the following conditions: it consists of only the letters B, L, R; it begins with a B and ends in an L; no two consecutive letters are the same. How many Vitas words are there with 11 letters?"

29. Suppose a, b, and c are positive integers less than 11 such that

$$\begin{aligned} 3a + b + c &\equiv abc \pmod{11} \\ a + 3b + c &\equiv 2abc \pmod{11} \\ a + b + 3c &\equiv 4abc \pmod{11}. \end{aligned}$$

What is the sum of all possible values of abc?

Answer. $\boxed{198}$.

Solution. The key idea is that, modulo 11, we can divide both sides of congruences by numbers that aren't multiples of 11. For example, to divide by 5, we can multiply both sides by 9, because $5 \cdot 9 \equiv 1 \pmod{11}$. For example, if we add the three congruences, we can multiply both sides by 9 to get

$$\begin{aligned} 5a + 5b + 5c &\equiv 7abc \pmod{11} \\ a + b + c &\equiv 8abc \pmod{11}. \end{aligned}$$

Subtracting this from the original congruences, and then multiplying both sides by 6 to get rid of the 2, we find

$$a \equiv 2abc, \qquad b \equiv 8abc, \qquad c \equiv 9abc \pmod{11}.$$

Multiplying all three, we get $abc \equiv (abc)^3$. As $abc \not\equiv 0$, we find that $abc \equiv 1$ or $abc \equiv 10$. Substituting these to the previous equivalences, we get that $(a, b, c) = (2, 8, 9)$ or $(a, b, c) = (9, 3, 2)$. The sum of all possible abc is then $2 \cdot 8 \cdot 9 + 9 \cdot 3 \cdot 2 = 198$.

Remark. Working in congruences modulo primes are nice, because you can divide by nonzero numbers. In other words, the integers modulo a prime form a field, where we can add, subtract, multiply, and divide. For small primes, we can figure out how to divide by trial and error. Compare PMO 2017 Qualifying I.11, "When $2a$ is divided by 7, the remainder is 5. When $3b$ is divided by 7, the remainder is also 5. What is the remainder when $a + b$ is divided by 7?"

30. Find the minimum value of $\dfrac{7x^2 - 2xy + 3y^2}{x^2 - y^2}$ if x and y are positive real numbers such that $x > y$.

Answer. $\boxed{2\sqrt{6} + 2}$.

Solution 1. The fact that we're minimizing something involving positive real numbers, and that we have $x > y$, motivates us to do something using AM–GM. We want to rewrite the expression as a sum of terms so that they cancel. We would want to find a, b, and c such that

$$\frac{7x^2 - 2xy + 3y^2}{x^2 - y^2} = a \cdot \frac{x + y}{x - y} + b \cdot \frac{x - y}{x + y} + c,$$

for all x and y. By expanding the right-hand side and comparing coefficients, we get the system of equations

$$\begin{aligned} a + b + c &= 7 \\ 2a - 2b &= -2 \\ a + b - c &= 3. \end{aligned}$$

Adding all three equations gives $4a = 8$, so $a = 2$. Substituting to the second equation gives $b = 3$, and substituting to the first equation gives $c = 2$. Using AM–GM on the first two terms, we find

$$\frac{7x^2 - 2xy + 3y^2}{x^2 - y^2} = 2 \cdot \frac{x + y}{x - y} + 3 \cdot \frac{x - y}{x + y} + 2 \geq 2\sqrt{6} + 2,$$

and equality can be achieved by setting $2 \cdot \frac{x+y}{x-y}$ and $3 \cdot \frac{x-y}{x+y}$ equal to each other.

Solution 2. Let the minimum value be k. Because it is the minimum, then for all positive x and y such that $x > y$, we get

$$\begin{aligned} \frac{7x^2 - 2xy + 3y^2}{x^2 - y^2} &\geq k \\ 7x^2 - 2xy + 3y^2 - k(x^2 - y^2) &\geq 0 \\ (7 - k)\left(\frac{x}{y}\right)^2 - 2 \cdot \frac{x}{y} + (3 + k) &\geq 0. \end{aligned}$$

This is a quadratic in terms of $\frac{x}{y}$ that is always nonnegative. That means that it has exactly one real root, which means its discriminant must be zero. This means that

$$4 - 4(7 - k)(3 + k) = 0 \implies k = 2 \pm 2\sqrt{6},$$

but as k can't be negative, it must be $2 + 2\sqrt{6}$.

Remark. Compare PMO 2017 Qualifying II.4, "If $b > 1$, find the minimum value of $\frac{9b^2-18b+13}{b-1}$", or PMO 2018 Qualifying III.3, "Find the minimum value of $\frac{18}{a+b} + \frac{12}{ab} + 8a + 5b$, where a and b are positive real numbers".

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