PMO 2021 Qualifying Stage

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February 20, 2021

Due to the pandemic, the PMO was held virtually. In lieu of a qualifying stage and an area stage, only a single qualifying stage was held, explaining the different format. This test continues last year's numbering scheme, with the numbering continuing throughout the test. Are any explanations unclear? If so, contact me at cj@cjquines.com. More material is available on my website: https://cjquines.com.

PART I. Choose the best answer. Figures are not drawn to scale. Each correct answer is worth two points.

1. In a convex polygon,the number of diagonals is 23 times the number of its sides. How many sides does it have?

   (a) 46 (b) 49 (c) 66 (d) 69

   Answer. $\boxed{\text{(b) } 49}$.

   Solution. If a polygon has n vertices, the number of diagonals it has is $\binom{n}{2} - n$. This is because a diagonal joins two vertices, but we overcounted the n sides of the polygon. From the problem, we get $\binom{n}{2} - n = 23n$, and solving yields $n = 49$.

   Remark. Another way to get the same formula. A diagonal is formed by joining each of the n vertices to one of $n - 3$ vertices: any other vertex except itself and the vertices it's next to. This counts each diagonal twice, though, so we divide by two to get $\frac{1}{2}n(n-3)$, which is the same formula.

2. What is the smallest real number a for which the function $f(x) = 4x^2 - 12x - 5 + 2a$ will always be nonnegative for all real numbers x?

   (a) 0 (b) $\dfrac{3}{2}$ (c) $\dfrac{5}{2}$ (d) 7

   Answer. $\boxed{\text{(d) } 7}$.

   Solution 1. For a quadratic to be always nonnegative, its discriminant has to be nonpositive:

$$\begin{aligned}(-12)^2 - 4(4)(-5+2a) &\le 0 \\ 144 - (-80 + 32a) &\le 0 \\ 224 &\le 32a.\end{aligned}$$

   Hence $a \ge 7$, and the minimum value for a is 7.

   Solution 2. Note that $2a - 5$ only affects the constant term of f, so we are considering translating the graph of $4x^2 - 12x$ upward or downward. It opens upward, and to make it tangent to the x-axis, it needs to be $4x^2 - 12x + 9 = (2x-3)^2$. Hence we want $2a - 5 \ge 9$, or $a \ge 7$.

3. In how many ways can the letters of the word $PANACEA$ be arranged so that the three As are not all together?

   (a) 540 (b) 576 (c) 600 (d) 720

   Answer. $\boxed{\text{(d) } 720}$.

   Solution. We do complementary counting: count the total number of arrangements, and then subtract the ones where the As are all together. The total number of arrangements is $\frac{7!}{3!}$. There are 7 letters, but we overcounted by a factor of 3!, because the As can be arranged in 3! ways without changing the arrangement.

   The number of arrangements where the As are all together is 5!. You can imagine combining the As as a single, big letter AAA. Then there would be 5 letters to arrange. The final answer is

$$\frac{7!}{3!} - 5! = \frac{7 \cdot 6 \cdot 5!}{3!} - 5! = 5!\left(\frac{7 \cdot 6}{3!} - 1\right) = 720.$$

4. How many ordered pairs of positive integers (x, y) satisfy $20x + 21y = 2021$?

   (a) 4 (b) 5 (c) 6 (d) infinitely many

   Answer. $\boxed{\text{(b) } 5}$.

   Solution. From $2021 = 2000 + 21$, we get the solution $(100, 1)$. To produce another solution, note that $20(-21) + 21(20) = 0$. We can add this to both sides to get:

$$\begin{aligned} 20(100) + 21(1) &= 2021 \\ 20(-21) + 21(20) &= 0 \\ 20(100 - 21) + 21(1 + 20) &= 2021, \end{aligned}$$

   giving us the solution $(79, 21)$. If we keep doing this, we get the solutions $(58, 41)$, $(37, 61)$, and $(16, 81)$. This gives 5 solutions.

   To prove there are no other solutions, let's say that (x, y) was another solution. We can subtract the equation $20x + 21y = 2021$ from $20(100) + 21(1) = 2021$ to get $20(100 - x) + 21(1 - y) = 0$. Note that 21 is a factor of 0, and it is also a factor of $21(1 - y)$. This means 21 must be a factor of $20(100 - x)$. Since it doesn't share factors with 20, it has to be a factor of $100 - x$. This limits the possible x, and similarly y; from here we can show that there are only 5 solutions.

   Remark. There is a general theory for solving linear Diophantine equations. Here's another way to visualize this. Consider the graph of $20x + 21y = 2021$ in the plane. It's a line, and $(100, 1)$ is one of the points on it. The slope of the line is $-\frac{20}{21}$. Interpreting this as rise over run, it means that if we go up 20 units in the y-coordinate, we go back 21 units in the x-coordinate. So another point on the line would be $(100 - 21, 1 + 20) = (79, 21)$, and if we think about drawing the line on graph paper, it wouldn't cross any other points with integer coordinates.

5. Find the sum of all k for which $x^5 + kx^4 - 6x^3 - 15x^2 - 8k^3x - 12k + 21$ leaves a remainder of 23 when divided by $x + k$.

   (a) -1 (b) $-\dfrac{3}{4}$ (c) $\dfrac{5}{8}$ (d) $\dfrac{3}{4}$

Answer. $\boxed{\text{(b) } -\dfrac{3}{4}}$.

Solution. From the remainder theorem, we know that the remainder when divided by $x+k$ is the result of substituting $-k$ for x. Setting this to 23, we get

$$\begin{aligned}(-k)^5 + k(-k)^4 - 6(-k)^3 - 15(-k)^2 - 8k^3(-k) - 12k + 21 &= 23\\ -k^5 + k^5 + 6k^3 - 15k^2 + 8k^4 - 12k - 21 &= 0\\ 8k^4 + 6k^3 - 15k^2 - 12k - 21 &= 0.\end{aligned}$$

From Vieta's formulas, we know that the sum of possible values of k is $-\frac{6}{8} = -\frac{3}{4}$.

6. In rolling three fair twelve-sided dice simultaneously, what is the probability that the resulting numbers can be arranged to form a geometric sequence?

(a) $\dfrac{1}{72}$ (b) $\dfrac{5}{288}$ (c) $\dfrac{1}{48}$ (d) $\dfrac{7}{288}$

Answer. $\boxed{\text{(d) } \dfrac{7}{288}}$.

Solution. There are 12^3 possible ordered triplets of the results. We'll count the number of triplets that can be arraged to form a geometric sequence. We'll work up from the possible common ratios, and within each one, work from the smallest term:

- When the ratio is 1, the possibilities are $(1,1,1),(2,2,2),\ldots,(12,12,12)$. These are 12 possibilities in all.
- When the ratio is $\frac{3}{2}$, the only possibility that works is $(4,6,9)$, giving $3! = 6$ permutations.
- When the ratio is 2, the possibilites are $(1,2,4),(2,4,8),(3,6,12)$. This gives $3 \cdot 6 = 18$ in total.
- When the ratio is 3, the only possibility is $(1,3,9)$, which gives 6 possible triplets.

Note that the ratio can't be any other value; the ratio of the last to first term needs to be a perfect square, and the only squares are $1,4,9$. Thus the total number of triplets is 42 and the answer is $\frac{42}{12^3} = \frac{7}{288}$.

7. How many positive integers n are there such that $\dfrac{n}{120-2n}$ is a positive integer?

(a) 2 (b) 3 (c) 4 (d) 5

Answer. $\boxed{\text{(b) } 3}$.

Solution 1. Let's say this integer is m. Then $\frac{n}{120-2n} = m$ rearranges to $2mn - 120m + n = 0$. We now complete the rectangle by using Simon's Favorite Factoring Trick. The $-120m$ suggests that it's from $(2m)(-60)$, so we want something like $(2m + \underline{\quad})(n-60)$, and we can fill it in with 1. This adds an extra term 60 to both sides:

$$\begin{aligned}2mn - 120m + n + 60 &= 60\\ (2m+1)(n-60) &= 60.\end{aligned}$$

Now, $2m+1$ is an odd factor of 60. The odd factors of 60 are $1, 3, 5, 15$. Of these, $2m+1$ can't be 1, because then m wouldn't be positive, but the other ones work. This means there are 3 solutions for m, and for each, we can find the corresponding value of n.

Solution 2. It's easier to work with a complicated numerator than a complicated denominator. Since the fraction is positive and n is positive, we know $120-2n$ must also be positive. Let's let $60-n=m$, and that way $n=60-m$. This means

$$\frac{n}{120-2n} = \frac{n}{2(60-n)} = \frac{60-m}{2m}.$$

Because the denominator is even, and this is an integer, the numerator must also be even. Hence $60-m$ is even, and thus m is even, so $m=2\ell$ for some positive integer ℓ. That makes

$$\frac{60-m}{2m} = \frac{60-2\ell}{4\ell} = \frac{30-\ell}{2\ell}.$$

Through similar reasoning, ℓ must also be even, so letting $\ell = 2k$,

$$\frac{30-\ell}{2\ell} = \frac{30-2k}{4k} = \frac{15-k}{2k}.$$

At this point, we can see that k is odd, but it's now small enough to just check the remaining choices. The choices $k=1$, 3, and 5 work, which means there are 3 solutions.

Remark. Compare PMO 2018 Areas I.7 "Determine the area of the polygon formed by the ordered pairs (x, y) where x and y are positive integers that satisfy the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{13}$." and PMO 2020 Areas I.9 "A brick with dimensions 3 by a by b units is painted blue and then cut into $3ab$ unit cubes. Exactly 1/8 of the cubes have all faces unpainted. Given a and b are positive integers, what is the volume of the brick?"

8. Three real numbers a_1, a_2, a_3 form an arithmetic sequence. After a_1 is increased by 1, the three numbers now form a geometric sequence. If a_1 is a positive integer, what is the smallest positive value of the common difference?

(a) 1 (b) $\sqrt{2}+1$ (c) 3 (d) $\sqrt{5}+2$

Answer. $\boxed{\text{(b) } \sqrt{2}+1}$.

Solution. Let's say the common difference is d, and let's write a for a_1. Then the arithmetic sequence is $a, a+d, a+2d$, and the geometric sequence is $a+1, a+d, a+2d$. Because this is a geometric sequence, the ratios of consecutive terms have to be the same:

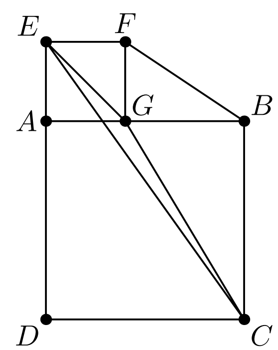
$$\begin{aligned}
\frac{a+d}{a+1} &= \frac{a+2d}{a+d} \\
(a+d)^2 &= (a+2d)(a+1) \\
a^2+2ad+d^2 &= a^2+2ad+a+2d \\
d^2-2d-a &= 0.
\end{aligned}$$

We can now work up from values of a, starting from 1, and solve for d. Alternatively, we can use the quadratic formula, or complete the square:

$$\begin{aligned}
d^2-2d+1-1-a &= 0 \\
(d-1)^2 &= a+1 \\
d &= 1 \pm \sqrt{a+1}.
\end{aligned}$$

Given that a is a positive integer, we can see the minimum positive value for d is $\sqrt{2}+1$.

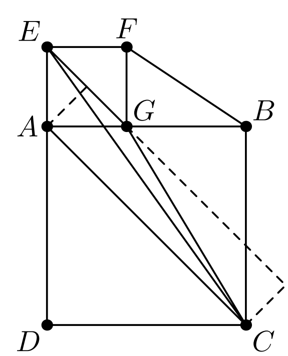
9. Point G lies on side AB of square $ABCD$ and square $AEFG$ is drawn outwards $ABCD$, as shown in the figure below. Suppose that the area of triangle EGC is $1/16$ of the area of pentagon $DEFBC$. What is the ratio of the areas of $AEFG$ and $ABCD$?



(a) $4:25$ (b) $9:49$ (c) $16:81$ (d) $25:121$

Answer. $\boxed{\text{(a) } 4:25}$.

Solution 1. The tricky part is finding the area of EGC. It's a good idea to consider the possible bases to compute the area from. Here, EG is a promising base. We now need the height from C to EG. In fact, this is equal to the height from A to EG.



To see this, draw AC. Note that AC and EG are parallel lines. The height from C to EG is thus the distance between these two parallel lines, which is equal to the height from A to EG. So triangles EGC and EGA have the same area! Now we can compute. Let the smaller square have side length x and the larger square have side length y. Then

$$\frac{[EGC]}{[DEFBC]} = \frac{[EAG]}{[AEFG]+[GFB]+[ABCD]} = \frac{\frac{x^2}{2}}{x^2+\frac{x(y-x)}{2}+y^2} = \frac{x^2}{x^2+xy+2y^2} = \frac{1}{16}.$$

Cross-multiplying this last equation and factoring, we get

$$\begin{aligned} 15x^2 - xy - 2y^2 &= 0 \\ (3x+y)(5x-2y) &= 0. \end{aligned}$$

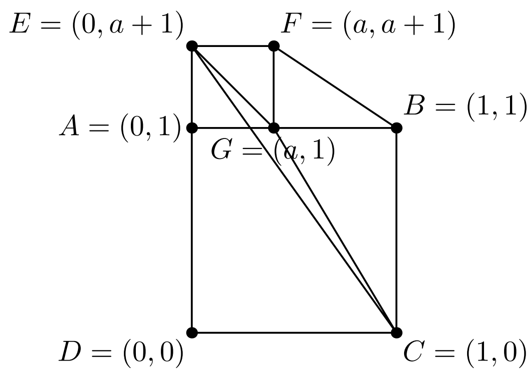
The case $3x+y=0$ isn't possible, because then one of x and y would have to be negative. So $5x-2y=0$, which means $\frac{x}{y}=\frac{2}{5}$. Squaring this gives us the ratio of the areas, $4:25$.

Solution 2. If the smaller square has side length x and the larger square have side length y,

$$[EGC]=[EAG]+[ABCD]-[GBC]-[EDC]=\frac{x^2}{2}+y^2-\frac{y(y-x)}{2}-\frac{y(x+y)}{2}=\frac{x^2}{2},$$

and the rest of the solution proceeds as in Solution 1.

Solution 3. We use Cartesian coordinates. Since only the ratio of the area matters, we can say that the larger square has side length 1. Let a be the side length of the smaller square. Taking D to be the origin, we get these coordinates:



We can now use the shoelace formula to find the areas of EGC and $DEFBC$:

$$\begin{aligned}[EGC]&=\frac{1}{2}\left|0\cdot 1+a\cdot 0+1\cdot(a+1)-a\cdot(a+1)-1\cdot 1-0\cdot 0\right|\\&=\frac{1}{2}\left|-a^2\right|=\frac{1}{2}(a^2).\\ [DEFBC]&=\frac{1}{2}\left|a\cdot 1-a\cdot(a+1)-1\cdot(a+1)-1\right|\\&=\frac{1}{2}\left|-a^2-a-2\right|=\frac{1}{2}(a^2+a+2).\end{aligned}$$

Here, in the shoelace formula for $[DEFBC]$, we only write the terms that don't have a zero factor. Each of the absolute values follow from $a>0$. Now we can solve for a:

$$\frac{a^2}{a^2+a+2}=\frac{1}{16}\iff 15a^2-a-2=(3a+1)(5a-2)=0,$$

so $a=\frac{2}{5}$, and the ratio of the areas is $4:25$.

Remark. Compare with PMO 2020 Qualifying I.12 "In parallelogram $ABCD$, $CD=18$. Point F lies inside $ABCD$ and AB and DF meet at E. If $AE=12$ and the areas of FEB and FCD are 30 and 162, find the area of triangle BFC." and PMO 2017 Qualifying III.1 "A paper cut-out in the shape of an isosceles right triangle is folded in such a way that one vertex meets the edge of the opposite side, and that the constructed edges m_1 and m_2 are parallel to each other. If the length of the triangle's leg is 2 units, what is the area of the shaded region?"

10. In how many ways can 2021 be written as a sum of two or more consecutive integers?

(a) 3 (b) 5 (c) 7 (d) 9

Answer. $\boxed{\text{(c) } 7}$.

Solution. Let's say that the consecutive integers begin with a, and there are n of them. From the formula for the sum of an arithmetic series,

$$2021 = a + (a+1) + \cdots + (a + (n-1)) = \frac{n}{2}(2a+n-1) \iff 4042 = n(2a+n-1).$$

Note that if n is odd, then $n + (2a-1)$ is an odd number plus an odd number, and is even. Similarly, if n is even, the other factor is odd. Thus we need to write 4042 as a product of an even and an odd number: one of them will be n, and the other will be $2a+n-1$. As long as this is true, we can always find an integer a that works.

Note $4042 = 2 \cdot 43 \cdot 47$. Because there's only one factor of 2, if we pick any factor as n, the other will be the opposite parity. For example, if $n = 2 \cdot 43$, which is even, then $2a + n - 1 = 47$, which is odd. So any factor of 4042 corresponds to a way to write it as a sum of consecutive integers. From a well-known formula, we know 4042 has 8 factors. Subtracting the case $n = 1$, because the problem asks for "two or more consecutive integers", we get the final answer, 7.

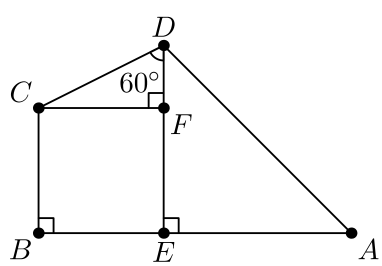
Remark. From here, it's possible to solve "Which integers can be written as the sum of two or more consecutive integers?" Because n and $2a + n - 1$ need to have opposite parity, the ones that can't are the powers of 2.

11. In quadrilateral $ABCD$, $\angle CBA = 90°$, $\angle BAD = 45°$, and $\angle ADC = 105°$. Suppose that $BC = 1 + \sqrt{2}$ and $AD = 2 + \sqrt{6}$. What is the length of AB?

(a) $2\sqrt{3}$ (b) $2 + \sqrt{3}$ (c) $3 + \sqrt{2}$ (d) $3 + \sqrt{3}$

Answer. $\boxed{\text{(c) } 3+\sqrt{2}}$.

Solution. Let E and F be the feet of the perpendiculars from D to AB, and C to DE, respectively. Because $\angle EAD = 45°$, that means $\triangle AED$ is a 45–45–90 triangle. Then $\angle EDA = 45°$, so $\angle CDF = \angle ADC - \angle EDA = 105° - 45° = 60°$, so $\triangle DFC$ is a 30–60–90 triangle.



From the fact that $\triangle AED$ is 45–45–90, we know that $AE = ED = \frac{AD}{\sqrt{2}} = \sqrt{2} + \sqrt{3}$. From rectangle $BEFC$, $BC = FE$, so we can find $DF = ED - FE = \left(\sqrt{2} + \sqrt{3}\right) - \left(1 + \sqrt{2}\right) = \sqrt{3} - 1$. Then we use the fact that $\triangle DFC$ is a 30–60–90 triangle to get $CF = 3 - \sqrt{3}$, which from rectangle $BEFC$ is also BE. Finally, $AB = AE + BE = \left(\sqrt{2} + \sqrt{3}\right) + \left(3 - \sqrt{3}\right) = 3 + \sqrt{2}$.

12. Alice tosses two biased coins, each of which has a probability p of obtaining a head, simultaneously and repeatedly until she gets two heads. Suppose that this happens on the rth toss for some integer $r \geq 1$. Given that there is a 36% chance that r is even, what is the value of p?

(a) $\dfrac{\sqrt{7}}{4}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{\sqrt{2}}{2}$ (d) $\dfrac{3}{4}$

Answer. $\boxed{\text{(a) } \dfrac{\sqrt{7}}{4}}$.

Solution 1. The probability that Alice gets two heads on the rth toss is p^2, times the probability she didn't get two heads on any of the previous $r-1$ tosses, which is $\left(1-p^2\right)^{r-1}$. Hence the probability r is even is the total probability that $r = 2, 4, 6, \ldots$, which is

$$\left(1-p^2\right)^1 p^2 + \left(1-p^2\right)^3 p^2 + \left(1-p^2\right)^5 p^2 + \cdots = \frac{(1-p^2)p^2}{1-(1-p^2)^2},$$

where we used the formula for an infinite geometric series. Setting it equal to $\frac{36}{100}$, we get

$$\begin{aligned} \frac{(1-p^2)p^2}{1-(1-p^2)^2} &= \frac{36}{100} \\ 100 - 100p^2 &= 72 - 36p^2. \end{aligned}$$

Hence $p^2 = \frac{28}{64} = \frac{7}{16}$, so $p = \frac{\sqrt{7}}{4}$.

Solution 2. Either r is even, which happens with probability 36%, or r is odd, which must happen with probability $100\% - 36\% = 64\%$. These are related—r is even is just like r being odd, if you started counting after the first flip. That is, the probability r is even is the probability that Alice doesn't get two heads in the first flip, times the probability that r is odd. This means

$$\frac{36}{100} = (1-p^2)\frac{64}{100},$$

and we can solve for $p = \frac{\sqrt{7}}{4}$.

Remark. Solution 2 uses the fact that r, being a geometric random variable, is memoryless.

13. For a real number t, $\lfloor t \rfloor$ is the greatest integer less than or equal to t and $\{t\} = t - \lfloor t \rfloor$ is the fractional part of t. How many real numbers between 1 and 23 satisfy $\lfloor x \rfloor \{x\} = 2\sqrt{x}$?

(a) 18 (b) 19 (c) 20 (d) 21

Answer. $\boxed{\text{(a) } 18}$.

Solution 1. It helps to think about what the graph of $\lfloor x \rfloor \{x\}$ looks like. Consider a given interval, say, $[2, 3)$. Here, $\lfloor x \rfloor$ is always 2, while $\{x\}$ goes $[0, 1)$. So the graph is a line from 0 to 2.

In this interval, what does $2\sqrt{x}$ look like? It goes from $2\sqrt{2} \approx 2.82$ to $2\sqrt{3} \approx 3.46$. It's also increasing. So in the interval $[2, 3)$, the value of $2\sqrt{x}$ is always at least 2.82. But the maximum value of $\lfloor x \rfloor \{x\}$ is 2. This means that there are no solutions in the interval $[2, 3)$.

Let's look at a different example, like $[5, 6)$. Again, the $\lfloor x \rfloor \{x\}$ part would go from 0 to 5. The $2\sqrt{x}$ part would go from $2\sqrt{5} \approx 4.47$ to $2\sqrt{6} = 4.90$. That means that their graphs would intersect at some point in the interval. Since both graphs are increasing, that means they also intersect at only one point.

We can do similar reasoning for the rest of the intervals. Each of $[1, 2), \ldots, [4, 5)$ have no solutions, while each of $[5, 6), \ldots, [22, 23)$ have one solution, giving 18 such real numbers.

Solution 2. Let $n = \lfloor x \rfloor$ and $d = \{x\}$. Then

$$\begin{aligned} \lfloor x \rfloor \{x\} &= 2\sqrt{x} \\ nd &= 2\sqrt{n+d} \\ n^2 d^2 - 4n - 4d &= 0 \\ d &= \frac{4 \pm \sqrt{16 - 4(n^2)(-4n)}}{2n^2} \\ d &= \frac{2 + 2\sqrt{1+n^3}}{n^2}. \end{aligned}$$

Here, we use the quadratic formula to solve for d, and take the positive solution because $d \geq 0$. The value of n determines the value of d, and thus the value of $x = n + d$. Thus, we only need to count the number of n that make $d < 1$:

$$\begin{aligned} \frac{2 + 2\sqrt{1+n^3}}{n^2} &< 1 \\ 2\sqrt{1+n^3} &< n^2 - 2 \\ 4 + 4n^3 &< n^4 - 4n^2 + 4 \\ n^2\left(n^2 - 4n - 4\right) &> 0. \end{aligned}$$

This becomes $(n-2)^2 > 8$, which is satisfied by each of $n = 5, 6, \ldots, 22$, giving 18 solutions.

14. Find the remainder when $\displaystyle\sum_{n=2}^{2021} n^n$ is divided by 5.

(a) 1 (b) 2 (c) 3 (d) 4

Answer. $\boxed{\text{(d) } 4}$.

Solution. To find $n^n \bmod 5$, we'll simplify both the base and the exponent. The base can just be taken mod 5. For the exponent, we know by Fermat's little theorem that $n^4 \equiv 1 \pmod 5$, as long as n isn't 0. This means we only need to take the exponent mod 4, because if the exponent is, say, $4k+2$, then $n^{4k+2} \equiv \left(n^4\right)^k n^2 \equiv 1 \cdot n^2 \equiv n^2 \pmod 5$.

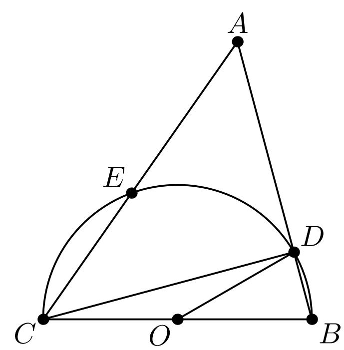
Because we're taking the base mod 5 and the exponent mod 4, this means that $n \bmod 20$ completely determines the value of n^n. So we only need to find the value of the first 20 numbers, and then multiply by the number of times they appear in the sum.

$$\begin{matrix} 1^1 & 2^2 & 3^3 & 4^4 & 5^5 \\ 6^6 & 7^7 & 8^8 & 9^9 & 10^{10} \\ 11^{11} & 12^{12} & 13^{13} & 14^{14} & 15^{15} \\ 16^{16} & 17^{17} & 18^{18} & 19^{19} & 20^{20} \end{matrix} \equiv \begin{matrix} 1^1 & 2^2 & 3^3 & 4^4 & 0 \\ 1^2 & 2^3 & 3^0 & 4^1 & 0 \\ 1^3 & 2^0 & 3^1 & 4^2 & 0 \\ 1^0 & 2^1 & 3^2 & 4^3 & 0 \end{matrix} \equiv \begin{matrix} 1 & 4 & 2 & 1 & 0 \\ 1 & 3 & 1 & 4 & 0 \\ 1 & 1 & 3 & 1 & 0 \\ 1 & 2 & 4 & 4 & 0 \end{matrix} \pmod 5$$

The sum of n^n from 1 to 20 is thus 4. By our previous argument, this is also the sum of n^n from 21 to 40, and from 41 to 60, and so on. Going from 1 to 2020, this repeats 101 times, so the sum would be $404 \equiv 4 \pmod 5$. Finally, note that the sum starts from 2 and ends at 2021, so we have to subtract 1^1 and add 2021^{2021}. This means the final answer is $4 - 1 + 1 \equiv 4 \pmod 5$.

Remark. An interesting observation is the columns of the previous table sum to either 4 or 0 mod 5. This follows from the fact that, for a prime p, $a^0 + a^1 + \cdots + a^{p-1} \equiv 0 \pmod p$ when $a \not\equiv 1$, which can be proven using the geometric series formula.

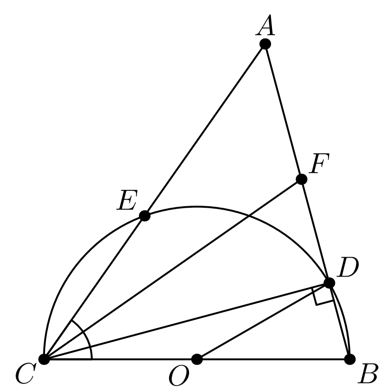
15. In the figure below, BC is the diameter of a semicircle centered at O, which intersects AB and AC at D and E respectively. Suppose that $AD = 9$, $DB = 4$, and $\angle ACD = \angle DOB$. Find the length of AE.



(a) $\dfrac{117}{16}$ (b) $\dfrac{39}{5}$ (c) $2\sqrt{13}$ (d) $3\sqrt{13}$

Answer. $\boxed{\text{(b) } \dfrac{39}{5}}$.

Solution 1. Our strategy is to use power of a point on A, and to do that, we want to find AC. The key observation is that, because $\angle DCB$ is an inscribed angle, its measure is half of $\angle DOB$, and thus, half of $\angle ACD$. This encourages us to draw the angle bisector of $\angle ACD$, so let F be on segment AD such that CF bisects $\angle ACD$.



Now if we let $\angle ACD = \angle DOB = 2\theta$ for some θ, then $\angle DCB = \frac{1}{2}\angle DOB = \theta$, and $\angle ACF = \angle FCD = \frac{1}{2}\angle ACD = \theta$. Because BC is a diameter of a semicircle, it follows $\angle CDB = \angle CDF = 90^\circ$, and hence $\triangle FCD \cong \triangle BCD$ by ASA. Thus $DF = DB = 4$ and $AF = AD - DF = 5$.

We now apply the angle bisector theorem on $\triangle ACD$ with angle bisector CF. This tells us that $\frac{AC}{CD} = \frac{AF}{FD} = \frac{5}{4}$. Hence, let $AC = 5x$, $CD = 4x$ for some x. Using the Pythagorean theorem on right $\triangle ADC$, we get $CD^2 + AD^2 = AC^2$, or $(4x)^2 + 9^2 = (5x)^2$. It follows that $x = 3$ and $AC = 15$. Finally, applying power of a point on A, we get that $AD \cdot AB = AE \cdot AC$, or $9 \cdot 13 = AE \cdot 15$, and hence $AE = \frac{39}{15}$.

Solution 2. We pick up from Solution 1, after deducing $\angle ACD = 2\theta$, $\angle DCB = \theta$ and $\angle CDB = \angle CDA = 90°$. Then using right $\triangle ADC$ and $\triangle BDC$, we get that $CD = \frac{9}{\tan 2\theta} = \frac{4}{\tan\theta}$. From the tangent double angle formula,

$$\begin{aligned} \frac{9}{\frac{2\tan\theta}{1-\tan^2\theta}} &= \frac{4}{\tan\theta} \\ 1 - \tan^2\theta &= \frac{4 \cdot 2\tan\theta}{9\tan\theta}, \end{aligned}$$

and hence $\tan\theta = \frac{1}{3}$. (We discard $\tan\theta = 0$ and $\tan\theta = -\frac{1}{3}$ because θ is acute.) Hence $CD = 12$, and from the Pythagorean theorem, $AC = 15$, and the rest proceeds as in Solution 1.

Solution 3. There's a solution that involves no geometric insight, although it is a lot of algebra. We pick up from Solution 1, after noticing $\angle CDB = \angle CDA = 90°$. Somehow we have to use the fact that $\angle ACD = \angle DOB$. But $\triangle ADC$ is right, so we can find $\sin\angle ACD$ using AC. Then we can use the cosine law on $\angle DOB$ to get $\cos\angle DOB$, and then try to use $\sin^2\angle ACD + \cos^2\angle DOB = 1$.

Let $AC = x$. Let $AC = x$. From right $\triangle ADC$, $\sin\angle ACD = \frac{AD}{AC} = \frac{9}{x}$. Now let $OB = OC = OD = r$, the radius of the semicircle. If we use the cosine law on $\triangle DOB$, we can find

$$\begin{aligned} DB^2 &= DO^2 + OB^2 - 2 \cdot DO \cdot OB \cdot \cos\angle DOB \\ \cos\angle DOB &= \frac{r^2 + r^2 - 4^2}{2 \cdot r \cdot r} \\ \cos\angle DOB &= 1 - \frac{8}{r^2}. \end{aligned}$$

So $\sin^2\angle ACD + \cos^2\angle DOB = 1$ relates r and x, but to solve for them, we need one more way to relate r and x. Well, we can use the Pythagorean theorem on right $\triangle ADC$ and $\triangle BDC$. Note that $CD^2 = BC^2 - BD^2 = AC^2 - AD^2$, so $(2r)^2 - 4^2 = x^2 - 9^2$, giving us $r^2 = \frac{x^2-65}{4}$. Finally,

$$\begin{aligned} \sin^2\angle ACD + \cos^2\angle DOB &= 1 \\ \left(\frac{9}{x}\right)^2 + \left(1 - \frac{8}{r^2}\right)^2 &= 1 \\ \frac{81}{x^2} + \left(1 - \frac{32}{x^2-65}\right)^2 &= 1 \\ \frac{81}{x^2} + 1 - \frac{64}{x^2-65} + \frac{1024}{x^4 - 130x^2 + 4225} &= 1 \\ \frac{17x^4 - 5346x^2 + 342225}{x^6 - 130x^4 + 4225x^2} &= 0 \\ \left(17x^2 - 39^2\right)\left(x^2 - 15^2\right) &= 0. \end{aligned}$$

The positive possibilities are $x = \frac{39}{\sqrt{17}}$ and $x = 15$. Of these, the former can be ruled out as being too small, or because it doesn't lead to an answer in the choices. The latter gives us $AC = 15$, and the rest proceeds as in Solution 1.

Remark. In Solution 1, from $AC : AD = 5 : 4$, we can deduce that $\triangle ACD$ is 3–4–5, and then get $AC = 15$.

PART II. All answers are positive integers. Do not use commas if there are more than 3 digits, e.g. type 1234 instead of $1,234$. A positive fraction a/b is in lowest terms if a and b are both positive integers whose greatest common factor is 1. Each correct answer is worth five points.

16. Consider all real numbers c such that $|x-8| + \left|4 - x^2\right| = c$ has exactly three real solutions. The sum of all such c can be expressed as a fraction a/b in lowest terms. What is $a+b$?

Answer. $\boxed{93}$.

Solution. It helps to think about the graph of $|x-8| + \left|4 - x^2\right| = y$ to get a sense of what the y would be. When does it increase and decrease? To analyze it, we can split it up based on the value of x. We'll split on $x = -2, 2, 8$, because these are where the absolute values would change.

- When $x \le -2$, it's $(8-x) + \left(x^2 - 4\right) = y$, or $y = x^2 - x + 4$. This is a parabola, pointing up, whose vertex is at $x = \frac{1}{2}$, so it just decreases in this interval.
- When $-2 \le x \le 2$, it's $(8-x) + \left(4 - x^2\right) = y$, or $y = -x^2 - x + 12$. This is a parabola, pointing down, whose vertex is at $x = -\frac{1}{2}$. In this interval, it increases, then decreases.
- When $2 \le x \le 8$, it's $(8-x) + \left(x^2 - 4\right) = y$, or $y = x^2 - x + 4$. This is the parabola we saw earlier, which means that it just increases in this interval.
- Finally, when $x \ge 8$, it's $(x-8) + \left(x^2 - 4\right) = y$, or $y = x^2 + x - 12$. Again, this is a parabola pointing up with vertex at $-\frac{1}{2}$, so it continues increasing in this interval.

The graph changes direction thrice, at $x = -2, -\frac{1}{2}$, and 2. We can compute the y values at these points as $10, \frac{49}{4}, 6$. Using this information, we can sketch what the graph would look like, and determine that the y that produce three solutions are 10 and $\frac{49}{4}$. Their sum is $\frac{89}{4}$, so the answer is $89 + 4 = 93$.

17. Find the smallest positive integer n for which there are exactly 2323 positive integers less than or equal to n that are divisible by 2 or 23, but not both.

Answer. $\boxed{4644}$.

Solution. The number of integers at most n that are divisible by 2 is $\left\lfloor \frac{n}{2} \right\rfloor$, and similarly, the number divisible by 23 is $\left\lfloor \frac{n}{23} \right\rfloor$. But this double-counts the numbers divisible by both. To not count those numbers, we can subtract $2\left\lfloor \frac{n}{46} \right\rfloor$. So we're looking for the smallest n such that

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{23} \right\rfloor - 2\left\lfloor \frac{n}{46} \right\rfloor = 2323.$$

Let's over-estimate n and then go down bit-by-bit until we find the right one. $\lfloor x \rfloor$ is always at most x, so we want $\frac{n}{2} + \frac{n}{23} - \frac{2n}{46} \le 2323$, which solves to $n \le 4646$. Plugging into the original equation, we see that 4646 works, but is it the smallest? We can check that 4645 and 4644 both work, but 4643 gives 2322, which is too small. So the answer must be 4644.

18. Let $P(x)$ be a polynomial with integer coefficients such that $P(-4) = 5$ and $P(5) = -4$. What is the maximum possible remainder when $P(0)$ is divided by 60?

Answer. $\boxed{41}$.

Solution 1. By the remainder theorem, we know that $P(x) = (x + 4)Q(x) + 5$, where $Q(x)$ is some other polynomial with integer coefficients. (To see this, consider substituting $x = -4$.) We want $P(5) = -4$, so substituting $x = 5$ gives

$$\begin{aligned} P(5) &= (5 + 4)Q(5) + 5 \\ Q(5) &= -1 \\ Q(x) &= (x - 5)R(x) - 1, \end{aligned}$$

where we again use the remainder theorem. Plugging it back into the first equation and substituting $x = 0$,

$$\begin{aligned} P(x) &= (x + 4)Q(x) + 5 \\ P(x) &= (x + 4)\left((x - 5)R(x) - 1\right) + 5 \\ P(0) &= 4\left(-5R(0) - 1\right) + 5 \\ P(0) &= -20R(0) + 1. \end{aligned}$$

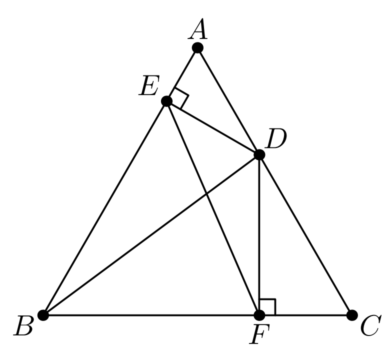
Now $R(0)$ is some constant. Modulo 60, the value of $-20R(0)+1$ is either 1, $-20+1$, or $-40+1$. These are 1, 41, and 21 modulo 60, so the largest possible remainder is 41.

Solution 2. We use the fact that $a-b \mid P(a)-P(b)$ to get $4 \mid P(0)-P(-4)$ and $-5 \mid P(0)-P(5)$. This means $P(0) \equiv P(-4) \equiv 1 \pmod 4$ and $P(0) \equiv P(5) \equiv 1 \pmod 5$. This means $P(0) \equiv 1 \pmod{20}$, and the possible values of $P(0)$ mod 60 would be 1, 21, 41, the largest of which is 41.

19. Let $\triangle ABC$ be an equilateral triangle with side length 16. Points D, E, F are on CA, AB, and BC, respectively, such that $DE \perp AE$, $DF \perp CF$, and $BD = 14$. The perimeter of $\triangle BEF$ can be written in the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are integers. Find $a + b + c + d$.

Answer. $\boxed{31}$.

Solution 1. Let $AE = x$ and $CF = y$. Then note that $\triangle DAE$ and $\triangle DCF$ are 30–60–90 triangles, so $AD = 2x$, $DE = x\sqrt{3}$, $DC = 2y$, and $DF = y\sqrt{3}$.



From $AD + DC = AC$ we get $2x + 2y = 16$, or $x + y = 8$. We also have $BE = 16 - x$ and $BF = 16 - y$. So $BE + BF = 32 - (x + y) = 24$, and the only thing we need to find is EF.

Applying Stewart's theorem on cevian BD, we get

$$\begin{aligned} AC \cdot AD \cdot DC + BD^2 \cdot AC &= BC^2 \cdot AD + AB^2 \cdot DC \\ 16 \cdot 2x \cdot 2y + 14^2 \cdot 16 &= 16^2 \cdot 2x + 16^2 \cdot 2y \\ 4xy + 196 &= 32(x+y) \\ xy &= 15. \end{aligned}$$

Because $\angle BED = \angle BFD = 90^\circ$, it follows $\angle BED + \angle BFD = 180^\circ$ and $BEDF$ is cyclic. We can now apply Ptolemy's theorem to find EF:

$$\begin{aligned} BD \cdot EF &= BF \cdot DE + BE \cdot DF \\ 14 \cdot EF &= (16-y)\left(x\sqrt{3}\right) + (16-x)\left(y\sqrt{3}\right) \\ 14 \cdot EF &= 16\sqrt{3}(x+y) - 2xy\sqrt{3} \\ EF &= 7\sqrt{3}. \end{aligned}$$

Hence the perimeter is $BE + BF + EF = 24 + 7\sqrt{3}$ and the answer is $24 + 0 + 7 + 0 = 31$.

Solution 2. We proceed from the first paragraph of Solution 1. Using the Pythagorean theorem on right $\triangle AED$, we get that $ED^2 + EB^2 = BD^2$, or

$$\begin{aligned} \left(x\sqrt{3}\right)^2 + (16-x)^2 &= 14^2 \\ 3x^2 + x^2 - 32x + 256 &= 196. \end{aligned}$$

This is $(x-5)(x-3) = 0$. Hence $x = 3, 5$, and as $x + y = 8$, we get $(x, y) = (3, 5)$ or $(5, 3)$. These are symmetric about swapping A and C, so we know that both choices will give the same perimeter. From here, we can proceed using Ptolemy's, as in Solution 1, to find EF.

Solution 3. Alternatively, we could also use the cosine law on, say, $\triangle DEF$. From quadrilateral $BEDF$ we get $\angle EDF = 120^\circ$, hence

$$\begin{aligned} DE^2 + DF^2 - 2 \cdot DE \cdot DF \cdot \cos \angle EDF &= EF^2 \\ \left(x\sqrt{3}\right)^2 + \left(y\sqrt{3}\right)^2 - 2\left(x\sqrt{3}\right)\left(y\sqrt{3}\right)\cos 120^\circ &= EF^2 \\ 3x^2 + 3y^2 - 6xy\left(-\frac{1}{2}\right) &= EF^2 \\ 3\left((x+y)^2 - xy\right) &= EF^2. \end{aligned}$$

If we knew $x + y$ and xy from Solution 1, or if we knew the values of x and y from Solution 2, we can now find EF. It's also possible to use the cosine law on $\triangle BEF$ itself.

Solution 4. Let G be the foot of the perpendicular from B to AC. Then $BG = 8\sqrt{3}$, because it's the height of an isosceles triangle with side length 16. Using the Pythagorean theorem on right $\triangle BGD$, we get $DG^2 = BD^2 - BG^2 = 196 - 192 = 4$, so $DG = 2$. From here we can get $AD = 6$ and $DC = 10$, and we can proceed as in Solution 1.

20. How many subsets of the set $\{1, 2, 3, \ldots, 9\}$ do not contain consecutive odd integers?

Answer. $\boxed{208}$.

Solution. Such a set can contain any subset of $\{2, 4, 6, 8\}$, and then a subset of $\{1, 3, 5, 7, 9\}$ with no two consecutive odd integers. There are 2^4 ways to pick a subset of the even numbers. Then we want to pick a subset of 5 things, no two of which are consecutive. It's well-known that this is F_7, the seventh Fibonacci number, which is 13. So the answer is $2^4 \cdot 13 = 208$.

Why is F_{n+2} the number of ways to choose a subset of $\{1, 2, \ldots, n\}$ containing no consecutive integers? Say there are a_n such subsets. We count based on whether they contain n or not:

- If it doesn't contain n, then it can be a subset of $\{1, 2, \ldots, n-1\}$ with no consecutive integers. So there are a_{n-1} subsets that don't contain n.
- If it does contain n, then it can't contain $n-1$. So it's a subset of $\{1, 2, \ldots, n-2\}$, with no consecutive integers, with n added in. There are a_{n-2} of these, so there are a_{n-2} subsets that do contain n.

This means $a_n = a_{n-1} + a_{n-2}$. Now, $a_0 = 1$, because there's only one way, the empty set. And $a_1 = 2$, because it's either the whole set or the empty set. This means $a_0 = F_2$ and $a_1 = F_3$, and from the recursion, we can prove $a_n = F_{n+2}$.

Remark. A common interpretation of the Fibonacci numbers is the number of ways to tile a $2 \times n$ rectangle with 2×1 dominoes. It's possible to construct a bijection from this to the number of subsets of $\{1, 2, \ldots, n-1\}$ with no consecutive integers. See A000045 for more interpretations.

Remark. It's also possible to construct the recursion directly. Let a_n be the number of subsets of $\{1, 2, \ldots, n\}$ with no consecutive odd integers. Then we can show, by considering whether or not 1 is part of the subset, that $a_n = 2a_{n-2} + 4a_{n-4}$.

21. For a positive integer n, define $s(n)$ as the smallest positive integer t such that n is a factor of $t!$. Compute the number of positive integers n for which $s(n) = 13$.

Answer. $\boxed{792}$.

Solution. An important fact about the factorials is that $t! = t \cdot (t-1)!$. So any factor of $(t-1)!$ is also a factor of $t!$. By induction, if $s < t$, then any factor of $s!$ is also a factor of $t!$. This means that if $s(n) = 13$, then n is a factor of $13!$, and it isn't a factor of $12!$. If $\tau(n)$ is the number of factors of n, the answer must be $\tau(13!) - \tau(12!)$. We can compute that $12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1 \cdot 11^1$, and from a well-known formula, we get that $\tau(12!) = (1+10)(1+5)(1+2)(1+1)(1+1) = 792$. Similarly $\tau(13!) = 1584$, so the answer is $1584 - 792 = 792$.

Remark. From multiplicativity, $\tau(13!) - \tau(12!) = \tau(13)\tau(12!) - \tau(12!) = \tau(12!)$. So we don't need to find the factors of $13!$, although it's not hard to do so if we already have the factors of $12!$.

22. Alice and Bob are playing a game with dice. They each roll a die six times, and they take the sums of the outcomes of their own rolls. The player with the higher sum wins. If both players have the same sum, then nobody wins. Alice's first three rolls are 6, 5, and 6, while Bob's first three rolls are 2, 1, and 3. The probability that Bob wins can be written as a fraction a/b in lowest terms. What is $a + b$?

Answer. $\boxed{3895}$.

Solution. Let's say that Alice's next three rolls are a, b, c and Bob's next three rolls are d, e, f. We count the number of possibilities for a, b, c, d, e, f such that Bob wins. We want $6+5+6+a+b+c < 2+1+3+d+e+f$, or

$$a + b + c + (7 - d) + (7 - e) + (7 - f) < 10.$$

Here, we write $7 - d$ so that it becomes a positive integer; because d is between 1 and 6, $7 - d$ would also be between 1 and 6. We can check that a, b, c are also forced to be between 1 and 6, otherwise the sum would be more than 10.

So we have six positive integers that sum to some integer less than 10. We can count the number of possibilities with balls and urns. When the sum is n, the number of solutions is $\binom{n-1}{5}$. Since the sum can be either 6, 7, 8, or 9, the total number of solutions is

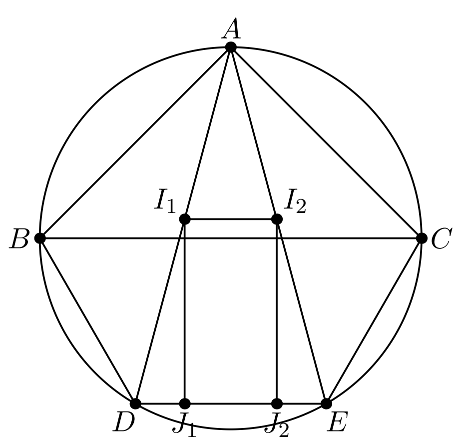
$$\binom{6-1}{5} + \binom{7-1}{5} + \binom{8-1}{5} + \binom{9-1}{5} = \binom{9}{6} = 84,$$

where we use the hockeystick identity. The number of possible rolls is 6^6, so the probability is $\frac{84}{6^6} = \frac{7}{3888}$ and the answer is $7 + 3888 = 3895$.

23. Let ABC be an isosceles triangle with a right angle at A, and suppose that the diameter of its circumcircle Ω is 40. Let D and E be points on the arc BC not containing A such that D lies between B and E, and AD and AE trisect $\angle BAC$. Let I_1 and I_2 be the incenters of $\triangle ABE$ and $\triangle ACD$ respectively. The length of I_1I_2 can be expressed in the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are integers. Find $a + b + c + d$.

Answer. $\boxed{20}$.

Solution 1. We have an incenter and a circumcircle, so the key idea is to use the incenter–excenter lemma. Because AD and AE trisect $\angle BAC$, it follows AD is the angle bisector of $\angle BAE$. Hence, by the lemma, $DI_1 = DB = DE$. Similarly, $EI_2 = EC = DE$.



Now we claim that $DI_1 = BD = DE = EC = EI_2 = 20$, half the radius of the circumcircle. One way to see this is to imagine rotating quadrilateral $BDEC$ 180° about the center of the circle, which produces a regular hexagon, which is well-known to have side length equal to the radius of the circle. Another way is to note that $\angle DAE$ is one-third of $\angle BAC$, so it's 30°, and by the extended law of sines $DE = 2 \cdot 20 \cdot \sin 30° = 20$.

Let J_1 and J_2 lie on DE such that I_1J_1 and I_2J_2 are both perpendicular to DE. Finally,

$$I_1I_2 = J_1J_2 = DE - DJ_1 - EJ_2,$$

so using right triangles DJ_1I_1 and EJ_2I_2,

$$DE - DI_1 \cos 75^\circ - EI_2 \cos 75^\circ = 20 - 20\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) - 20\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) = 20 - 10\sqrt{6} + 10\sqrt{2},$$

so the final answer is $20 + 10 + 0 - 10 = 20$.

Solution 2. We use Cartesian coordinates. Center Ω on the origin O and choose $A = (0, 20)$, $B = (-20, 0)$, and $C = (20, 0)$. By symmetry, we know that I_1I_2 is parallel to BC and is bisected by OA, so it's enough to just find the distance of, say, I_2 to OA, and then double it. But this is just the x-coordinate of I_2!

To find the x-coordinate of I_2, we can use the formula for the incenter of a triangle in Cartesian coordinates: it's the weighted average of the triangle's vertices, where each vertex is weighted by the length of the opposite side. This means we need to coordinates of D. Because $\angle BAD = \frac{1}{3}\angle BAC = 30^\circ$, we know $\angle BOD = 2\angle BAD = 60^\circ$, and using some trigonometry we get $D = \left(-10, -10\sqrt{3}\right)$.

Now we need the side lengths of $\triangle ACD$. Using the distance formula, we get $AC = 20\sqrt{2}$, $CD = 20\sqrt{3}$ and $DA = 10\sqrt{2} + 10\sqrt{6}$. Then the x-coordinate of I_2 would be

$$\frac{0\left(20\sqrt{3}\right) + 20\left(10\sqrt{2}+10\sqrt{6}\right) - 10\left(20\sqrt{2}\right)}{20\sqrt{3}+10\sqrt{2}+10\sqrt{6}+20\sqrt{2}} = \frac{200\sqrt{2}+200\sqrt{6}-200\sqrt{2}}{30\sqrt{2}+20\sqrt{3}+10\sqrt{6}} = \frac{20\sqrt{6}}{3\sqrt{2}+2\sqrt{3}+\sqrt{6}}.$$

We want to write this in the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$. The simplest way is to do this is equate it with $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, and cross-multiply to get

$$20\sqrt{6} = (6b + 6c + 6d) + (3a + 3c + 6d)\sqrt{2} + (2a + 2b + 6d)\sqrt{3} + (a + 2b + 3c)\sqrt{6}.$$

Solving the system of equations gives $(a, b, c, d) = (10, 5, 0, -5)$. Hence the x-coordinate of I_2 is $10 + 5\sqrt{2} - 5\sqrt{6}$, which means the length of I_1I_2 is double that, $20 + 10\sqrt{2} - 10\sqrt{6}$, and the answer is $20 + 10 + 0 - 10 = 20$.

Remark. In Solution 2, the fact that we could rewrite $\frac{20\sqrt{6}}{3\sqrt{2}+2\sqrt{3}+\sqrt{6}}$ in the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ is a fancy consequence of the fact that $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$ is a field extension of $\mathbb{Q}$. This is why the $\sqrt{6}$ is necessary, for example, $\frac{4}{1+\sqrt{2}+\sqrt{3}} = 2 + \sqrt{2} - \sqrt{6}$. Sorry, it's been a while since I've done number theory, and I need to cite little facts like these to prove to myself that I can still do it.

24. Find the number of functions f from the set $S = \{0, 1, 2, \ldots, 2020\}$ to itself such that, for all $a, b, c \in S$, all three of the following conditions are satisfied:

(i) If $f(a) = a$, then $a = 0$;
(ii) If $f(a) = f(b)$, then $a = b$; and
(iii) If $c \equiv a + b \pmod{2021}$, then $f(c) \equiv f(a) + f(b) \pmod{2021}$.

Answer. $\boxed{1845}$.

Solution. The important condition is (iii). We can show through induction that, because $f(a) \equiv f(a-1) + f(1) \mod 2021$, then $f(a) \equiv af(1) \pmod{2021}$. If we determine the value of $f(1)$, we know what the entire function is. Now we consider the other two conditions and what restrictions they give for $f(1)$.

Condition (i) means that for each $a \neq 0$, we want $af(1) \neq a$. Because $f(a) \equiv af(1) \pmod{2021}$, this means $af(1) \not\equiv a \pmod{2021}$, or $a\left(f(1)-1\right) \not\equiv 0 \pmod{2021}$. Now, note that this has to be true for *every* $a \neq 0$, so we can choose the a we want. Picking $a = 43$, we get that $43\left(f(1)-1\right) \not\equiv 0 \pmod{2021}$. Now we can divide by 43 to get $f(1) - 1 \not\equiv 0 \pmod{47}$. This means $f(1) \not\equiv 1 \pmod{47}$. Similarly, by picking $a = 47$, we can prove that $f(1) \not\equiv 1 \pmod{43}$.

Let's look at condition (ii) with $b = 0$. Note that $f(0) = 0f(1) = 0$, regardless of the choice of $f(1)$. This means that, for each $a \neq 0$, we want $f(a) \neq 0$, or $af(1) \not\equiv 0 \pmod{2021}$. Again, we choose $a = 43$ to show $f(1) \not\equiv 0 \pmod{47}$, and $a = 47$ to show $f(1) \not\equiv 0 \pmod{43}$.

So, we have that $f(1) \not\equiv 0, 1 \pmod{43}$ and $f(1) \not\equiv 0, 1 \pmod{47}$. The values of $f(1)$ mod 43 and 47 completely determine it mod 2021. Since $f(1) \equiv 2, \ldots, 42 \pmod{43}$, it has 41 possibilities mod 43, and similarly 45 possibilities mod 47, it has $41 \cdot 45 = 1845$ possibilities mod 2021.

Remark. To give a full proof, we have to show that if $f(1) \not\equiv 0, 1 \pmod{43, 47}$, then the function satisfies all three conditions. This isn't necessary for the contest and follows a similar idea to the solution above, but it does need to be shown when the proof is required.

25. A sequence $\{a_n\}$ of positive real numbers is defined by $a_1 = 1$ and for all integers $n \geq 1$,

$$a_{n+1} = \frac{a_n\sqrt{n^2+n}}{\sqrt{n^2+n+2a_n^2}}.$$

Compute the sum of all positive integers $n < 1000$ for which a_n is a rational number.

Answer. $\boxed{131}$.

Solution. The first few terms are $1, \frac{\sqrt{2}}{2}, \frac{\sqrt{21}}{7}, \frac{\sqrt{10}}{5}, \frac{\sqrt{65}}{13}, \ldots$. There's a lot of square roots here, so let's try squaring all the terms. This gives us the numbers $1, \frac{1}{2}, \frac{3}{7}, \frac{2}{5}, \frac{5}{13}, \ldots$. Now it looks like the numerators are $1, 2, 3, \ldots$. In fact, we can rewrite the squares of the terms as $\frac{1}{1}, \frac{2}{4}, \frac{3}{7}, \frac{4}{10}, \frac{5}{13}, \ldots$, so we can guess that $a_n = \sqrt{\frac{n}{3n-2}}$. In fact, we can prove this with induction. The base case is clear, and for the inductive step,

$$\frac{a_n\sqrt{n^2+n}}{\sqrt{n^2+n+2a_n^2}} = \sqrt{\frac{n}{3n-2}} \cdot \frac{\sqrt{n^2+n}}{\sqrt{n^2+n+2\cdot\frac{n}{3n-2}}} = \frac{\sqrt{n^2(n+1)}}{\sqrt{n^2(3n+1)}} = \sqrt{\frac{n+1}{3\left(n+1\right)-2}},$$

as desired. Now we need to find all positive integers $n < 1000$ for which $\sqrt{\frac{n}{3n-2}}$ is a rational number. For this to happen, the fraction $\frac{n}{3n-2}$, when put in simplest terms, must have a perfect square in the numerator and a perfect square in the denominator. To put it in simplest forms, we divide n and $3n-2$ by their GCD. By a property of GCD, we get that $(n, 3n-2) = (n, 3n-2-3n) = (n, -2)$, so the GCD is either 2 or 1.

If the GCD is 2, that means $n = 2n'$ for some n', and $\frac{n}{3n-2} = \frac{2n'}{6n'-2} = \frac{n'}{3n'-1}$, which is now in simplest terms. Now the numerator must be a perfect square, so let's say $n' = x^2$ for some integer x. That means $3n' - 1$, which is $3x^2 - 1$, must also be a perfect square. But modulo 3, this is -1, and -1 is not a perfect square modulo 3. So $3n' - 1$ can't be a perfect square, and there are no solutions in this case.

The remaining case is when the GCD is 1. Then both n and $3n - 2$ are perfect squares. Letting $n = x^2$ for some integer x, we get that $3x^2 - 2$ is also a perfect square. At this point we can just try some values of x until we get one that works: $x = 1$ works, so does $x = 3$, and $x = 11$. Since $n = x^2$, we only have to check up to 31, because $32^2 > 1000$. We find that the only ones that work are $1, 3, 11$, which means $n = 1, 9, 121$ are the solutions, and they have sum 131.

Remark. The equation $3x^2 - 2 = y^2$ is a generalized Pell equation; it can be written in the more familiar form $x^2 - 3y^2 = -2$. The morally correct method is to use the solution $(x, y) = (1, 1)$, and then generate solutions through multiplying by the solutions of the regular Pell equation $u^2 - 3v^2 = 1$. For example, the Pell equation has solution $(u, v) = (2, 1)$, and note that

$$\left(x + y\sqrt{3}\right)\left(u + v\sqrt{3}\right) = (1 + \sqrt{3})(2 + \sqrt{3}) = 5 + 3\sqrt{3},$$

and indeed, $(x, y) = (5, 3)$ is a solution to $x^2 - 3y^2 = -2$. We can get the solutions of the regular Pell equation $u^2 - 3v^2 = 1$ through computing the powers of the fundamental solution $(u, v) = (2, 1)$. For example, $(2 + \sqrt{3})^3 = 26 + 15\sqrt{3}$, and $(u, v) = (26, 15)$ is a solution to $u^2 - 3v^2 = 1$.

With thanks to Nathanael Joshua Balete and Richard Eden for comments.