VCSMS PRIME

Session 1: Algebra 1
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Domain and range

1. Notice that $x^2 - 4x + 1 = (x-2)^2 - 3$. The minimum is thus 2^{-3} and it is unbounded, the range is thus $[1/8, +\infty)$.

2. For the domain, $x^2 - 10x + 29 = (x-5)^2 + 4 \geq 4$, thus there is no restriction for the square root. The denominator cannot be 0, thus the radical cannot be 2/5, but this is impossible. The domain is $(-\infty, +\infty)$.

   From above, the radical can be anything in $[2, +\infty)$. The maximum is when the radical is 2, giving 3/4. As the radical grows larger, it approaches 0. The range is $(0, 3/4]$.

3. We have $25 - x^2 - y^2 \geq 0, |x| - y \geq 0$. The first is a circle with radius 5, the second is an absolute value function. The intersection is a sector with angle 270°, which has area $75\pi/4$.

4. $\lfloor x^2 - x - 2 \rfloor$ will be 0 if $0 \leq x^2 - x - 2 < 1$. Solving yields $(\dfrac{1-\sqrt{13}}{2}, -1] \cup [2, \dfrac{1+\sqrt{13}}{2})$.

5. For f, as x approaches $-\infty$, 3^{-x} approaches $+\infty$ and the fraction approaches 2. As x approaches $+\infty, 3^{-x}$ approaches 0 and the fraction approaches 1/2. The range of f is thus $(-\infty, 1/2) \cup (2, \infty)$. Similarly the range of g is $(-3, 4)$.

6. Solving for y yields $y = \dfrac{12e^x + 3}{3e^x + 1}$. By a similar argument as number $5, m = 3$.

7. We have $f^4(x) > 0, f^3(x) > 1, f^2(x) > e, f(x) > e^e, x > e^{e^e}$. The domain is $(e^{e^e}, +\infty)$.

8. When $x = a, b, c$, f is 1. Since the degree of f is at most 2, and we have three distinct values of f, by interpolating, $f(x) = 1$. The range is $\{1\}$.

Logarithms

1. The sum is $1 \times 3 + \cdots + 20 \times 22$. This is equal to $(2^2 - 1) + \cdots + (21^2 - 1)$, which we can evaluate using the sum of squares formula as 3290.

2. Raising both sides to the base, we have $4 = (x^2 - 3x)^2$. Thus $x^2 - 3x = +2, -2$. We see that the negative case is impossible after substituting in the original equation. Thus $x^2 - 3x = 2$, which has two real roots.

3. We have $\left|\log_{\frac{1}{2}} |x|\right| - 1 = 0$. Thus $\log_{\frac{1}{2}} |x| = \pm 1$, or $|x| = \frac{1}{2}, 2$. This has four real solutions, thus the graph crosses the x-axis four times.

4. After noting that $x > 0$ from the $\log_{2014} x$ in the exponent, taking the base-x logarithm of both sides yields $\log_x \sqrt{2014} + \log_{2014} x = 2014$. Substituting $u = \log_{2014} x$ and using the fact that $\log_x \sqrt{2014} = \dfrac{1}{2u}$, we see that $2u^2 - 4028u + 1 = 0$. Suppose that the roots of this are $u_1 = \log_{2014} x_1, u_2 = \log_{2014} x_2$ and thus by Vieta's and the product rule for logarithms we have $u_1 + u_2 = 2014 = \log_{2014}(x_1 x_2)$. The product of the roots x_1 and x_2 to the original equation is thus 2014^{2014} which has units digit 6.

5. Multiplying the three given equations yields $(xyz)^2 = 10^{a+b+c}$, taking the logarithms of both sides yields $\log x + \log y + \log z = \dfrac{a+b+c}{2}$.

6. Note that $a = \log_{14} 16 = 4 \log_{14} 2$. Thus $\log_{14} 2 = a/4$. Thus $\log_8 14 = \dfrac{1}{\log_{14} 8} = \dfrac{1}{3 \log_{14} 2} = \dfrac{4}{3a}$.

Exponents

1. a) Note that $4^3 = 2^6$. Equating exponents, $2^x = 6$, and thus $x = \log_2 6$.

   b) We see that $x = 1$ is a solution. Equating exponents yields $x = 2$. Thus $x = 1, 2$.

   c) Equating exponents, $x^x = x^2$. From b, we have $x = 1, 2$. Thus $x = 1, 2$.

   d) Again, we see that $x = 1$ is a solution. Equating exponents yields $x = \pm \sqrt[2010]{2010}$. Thus $x = 1, \pm \sqrt[2010]{2010}$.

2. Taking hundredth roots yields $n^3 > 3^5 = 243$. The smallest integral n that satisfies this is 7.

3. First, compare 11^{16} and $25^{12} = 5^{24}$ by taking the eighth root, reducing the comparison to 11^2 and 5^3. It is clear that the former is lesser. Compare $25^{12} = 5^{24}$ and $16^{14} = 2^{56}$ by taking the eighth root, reducing the comparison to 5^3 and 2^7. It is clear that the former is lesser. From least to greatest, we have $11^{16}, 25^{12}, 16^{14}$.

4. We factor the LHS as $(9^{2x-1})(9 - 1) = 8\sqrt{3}$, by equating exponents, we have $2x - 1 = \frac{1}{2}$. Thus $(2x - 1)^{2x} = \sqrt{2}/8$.

More logarithms

1. We see that $2^3 < 3^2$, thus $2 < 3^{2/3}, \log_3 2 < 2/3$. Since $625^2 < 75^3, 625^{2/3} < 75, 2/3 < \log_{625} 75$. Finally, we see that $\log_{625} 75 = \dfrac{\log_5 75}{4} < \log_5 3$. Thus from least to greatest, we have $\log_3 2, 2/3$, $\log_{625} 75, \log_5 3$.

2. After solving, we see $x = 1/2$. The infinite geometric series evaluates to 2.

3. Simplifying, we see that this is equivalent to $1 - \log_a b + 1 - \log_b a$. The minimum value of $\log_a b + \log_b a$ is 2 by AM-GM, thus the maximum value of the expression is 0.

4. Simplifying, we see $5^k 2^m = 400^n = (5^2 2^4)^n$. We have $k = 2n, m = 4n$. Since the greatest common divisor must be 1, we have $n = 1, k = 2, m = 4, k + m + n = 7$.

5. After trial and error, we find $m = 5$ works.

6. Let $u = 5^{\frac{1}{2x}}$. Simplifying, we have $u^2 + 125 < 30u$ which factors into $(u-5)(u-25) < 0$, thus $u \in (5, 25)$ and $x \in (1/4, 1/2)$.

7. We have $x \geq 2(x - 1)$, thus $x \leq 2$. But from the argument of $\log(x - 1)$ we have $x > 1$. Combining, we see all $x \in (1, 2]$ work.

Floor, ceiling, fractional

1. The equation is $2\lfloor x \rfloor = \lfloor x \rfloor + \{x\} + 2\{x\}$, which is $\lfloor x \rfloor = 3\{x\}$. As $\{x\} \in [0, 1)$, the only values for which $3\{x\}$ is an integer is $\{x\} \in \{0, 1/3, 2/3\}$. These give solutions $x = 0, 4/3, 8/3$.

2. Note that x must be nonnegative. We do casework on $\lfloor x \rfloor$. When $\lfloor x \rfloor = 0$, clearly $x = 0$. When $\lfloor x \rfloor = 1$ then $2x(x - 1) = 1$, which has solution $\dfrac{1 + \sqrt{3}}{2}$. When $\lfloor x \rfloor = 2$, then $2x(x - 2) = 4$, which has solution $1 + \sqrt{3}$. If $\lfloor x \rfloor \geq 3$, then examining the discriminant reveals there is no solution. Thus $x = 0, \dfrac{1 + \sqrt{3}}{2}, 1 + \sqrt{3}$.

3. In the interval $(1/4^2, 1/4]$, y is 1, its length is $1/4 - 1/4^2$. In the interval $(1/4^4, 1/4^3]$, y is 3, its length is $1/4^3 - 1/4^4$. Continuing the pattern, the desired sum is $1/4 - 1/4^2 + 1/4^3 - 1/4^4 + \cdots$, an infinite geometric series with sum $1/5$.

Value-finding

1. Letting $x = 0$, we see $f(0) = 2$. Similarly, we see $f(7) = 383$. The difference is 381.
2. We set $f(a) = 1$ and subtract $f(1)$ on both sides. We see that $f(b)^2 = 1$ for all b. Thus $f(1) - f(-1)$ can be anything in $\{-2, 0, 2\}$.
3. We substitute $x = 0$ and $x = 3$ to get the system of equations $2f(0)-2f(3) = -18, -f(3)-2f(0) = -30$. Solving, we get $f(0) = 7$.

Cauchy functional equation

Note: if we have $f(x + y) = f(x) + f(y)$, the solution from $\mathbb{Q} \to \mathbb{R}$ is $f(x) = kx$. Similarly, the solution to $f(x + y) = f(x)f(y)$ is $f(x) = k^x$ and the solution to $f(xy) = f(x) + f(y)$ is $f(x) = \log_k x$.

1. Letting $y = 0$ in the second equation and cancelling $f(0)$ on both sides yields $f(x) = 0$ for all x. Thus $f(\pi^{2013}) = 0$.
2. As per the note, the solution is $f(x) = kx$. We see that $k = 3/2$ and thus $f(2009) = 3013.5$.
3. As per the note, the solution is $f(x) = k^x$. We see that $k = 5$ and $3f(-2) = 3/25$.

Other functional equations

1. Letting $x = y = 0$ gives $f(0) = 1/2009$. Letting $x = y$ gives $f(x) = \pm 1/2009$. The negative case fails, thus $f(\sqrt{2009}) = 1/2009$.
2. Let $x = 0$ to get $f(-1) = f(y) - 2y - 2$. Let $y = 0$ to get $f(-1) = -1$. Equating gives us $f(y) = 2y + 1$ for all y.
3. Let $y = 0$ to get $f(0) = 0$. Let $x = 0$ to get f is odd. Switch x and y and equate to the original, use $f(y - x) = -f(x - y)$; rearrange to get

   $$f(x + y)/(x + y) = f(x - y)/(x - y).$$

   Thus $f(a)/a$ is a constant k for all a, and $f(a) = ka$. We have $k = 3/5$ and thus $f(2015) = 1209$.
4. Let $g(x) = (x + 2009)/(x - 1)$. The given is $x + f(x) + 2f(g(x)) = 2010$. Replace x with $g(x)$ to get $g(x) + f(g(x)) + 2f(x) = 2010$. Solving, $f(x) = \dfrac{x^2 + 2007x - 6028}{3x - 3}$.
5. Let $f(0) = a$, set $x = 0$ to get $f(a) = 1$. Set $x = a$ to get $f(1) = 1 - a$, set $x = 1$ to get $f(1 - a) = a$. Set $x = 1 - a$ to get $f(a) = 1 - a + a^2$. We get either $a = 0, 1$, either of which make a contradiction. Thus no f exists.