VCSMS PRIME

Session 2: Trigonometry
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Circular functions

1. As $\cos x = -\cos(180^\circ - x)$, the sum is 0.
2. Rearranging, $x/y = 5/3 = \tan\theta$. Thus $\sin\theta = 5/\sqrt{34}$.
3. The line is the terminal side of an angle θ. Note that $\tan\theta = \tan 75^\circ$, so the angle is 75°. The tangent line to the unit circle makes an angle of 165° with the origin, so its slope is $\tan 165^\circ = -2 + \sqrt{3}$.
4. We let $x = 1$ to get the sum of the coefficients as $\cos(2\cos^{-1}(0)) = -1$.

Identities

1. The half-angle identity gives $\cos 15^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.
2. We wish to evaluate $\log_2 \sin(\pi/8)\cos(15\pi/8)$. By the product-to-sum identity, this is $\log_2(1/2)(\sin(2\pi) + \sin(7\pi/4)) = -3/2$.
3. We use the fact that $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ to get $\tan x \tan y = \dfrac{1}{2}$. Then $\cot^2 x + \cot^2 y = \dfrac{(\tan x + \tan y)^2 - 2\tan x \tan y}{\tan^2 x \tan^2 y} = 96$.
4. Note that $\cot(37^\circ + 8^\circ) = \dfrac{\cot 37^\circ \cot 8^\circ - 1}{\cot 37^\circ + \cot 8^\circ} = 1$, so $\cot 37^\circ \cot 8^\circ - 1 = \cot 37^\circ + \cot 8^\circ$. This rearranges to $(1 - \cot 37^\circ)(1 - \cot 8^\circ) = 2$.
5. We see $\cot(\cot^{-1} 2 + \cot^{-1} 3) = \dfrac{2 \cdot 3 - 1}{2 + 3} = 1$. Similarly, $\cot(\cot^{-1} 4 + \cot^{-1} 5) = 19/9$. Finally, $\cot(\cot^{-1} 1 + \cot^{-1} 19/9) = 5/14$.
6. Note that $\tan\theta^\circ \cos 1^\circ + \sin 1^\circ = \dfrac{\sin\theta^\circ \cos 1^\circ + \sin 1^\circ \cos\theta^\circ}{\cos\theta^\circ} = \dfrac{\sin(\theta^\circ + 1^\circ)}{\cos\theta^\circ}$. The product telescopes using cofunctions and the result is $\dfrac{1}{\sin 1^\circ} = \csc 1^\circ$.
7. Interpret this with the unit circle: there is a right triangle with legs of length $\sec\alpha$ and $\csc\alpha$, and its hypotenuse is $\tan\alpha + \cot\alpha$. The area of the triangle is equal to half the product of its legs, or $\frac{1}{2}\sec\alpha\csc\alpha$. It is also equal to half the product of the hypotenuse and the altitude to the hypotenuse, or $\frac{1}{2}(\tan\alpha + \cot\alpha)$. The answer is $\sqrt{14}$.

Equations

1. (The equation holds for all x.) By phase shift, $2\sin 3x = 2\cos\left(3x - \dfrac{\pi}{2} + 2k\pi\right) = -2\cos\left(3x + \dfrac{\pi}{2} + 2k\pi\right)$ for some $k \in \mathbb{Z}$. The product ac in both cases is $(4k-1)\pi$.
2. Square both sides to yield $1 - 2\sin 2\theta \cos 2\theta = 1 - \sin 4\theta = 3/2$, giving $\sin 4\theta = -1/2$. Since $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, it follows $4\theta \in (-2\pi, 2\pi)$. In this interval, $\sin 4\theta$ becomes $-1/2$ four times, so the equation has four solutions.
3. Square both sides and substitute $\cos^2\theta = 1 - \sin^2\theta$ to yield $5\sin^2\theta + 2\sin\theta - 3 = (5\sin\theta - 3)(\sin\theta + 1) = 0$. Either $\sin\theta = 3/5$ or $\sin\theta = -1$, but we can eliminate the latter as $0 < \theta < \pi/2$. Thus $\sin\theta = 3/5$.

4. Substituting $\sec^2 x = \tan^2 x + 1$ and simplifying gives the quadratic equation $\tan^2 x + 6\tan x - 16 = (\tan x + 8)(\tan x - 2) = 0$, thus $x \in \{\tan^{-1} 2 \pm k\pi, \tan^{-1}(-8) \pm k\pi | k \in \mathbb{Z}\}$.

5. Transpose $\frac{1}{\cos x}$ and square both sides. Substitute $\sin^2 x = 1 - \cos^2 x$ and then $\cos x = u$ to get the equation $\frac{3}{1-u^2} = 16 + \frac{1}{u^2} - \frac{8}{u}$. Clear the denominators to get $16u^4 - 8u^3 - 12u^2 + 8u - 1 = 0$.

   By inspection, $u = \frac{1}{2}$ works; dividing through gives $8u^3 - 6u + 1 = 0$. This reminds one of the triple angle formula $\cos 3x = 4\cos^3 x - 3\cos x$. We rewrite the equation as $4u^3 - 3u = -\frac{1}{2} = \cos 3x$. Keeping in mind $x \in (-\pi/2, 0)$, we let $3x = -\frac{4\pi}{3}$ and get $x = -\frac{4\pi}{9}$.

6. Transpose the first term of the left hand side, use the double angle formulae, and then use cofunctions to get $\cos(2x + b) = \sin(2ax - \pi) = \cos(3\pi/2 - 2ax)$. We can see that there are two cases: when $a = 1$ and $b = \pi/2 + 2k\pi, k \in \mathbb{Z}$, or when $a = -1$ and $b = 3\pi/2 + 2k\pi, k \in \mathbb{Z}$.

7. Substitute $\cot\alpha = \frac{1}{\tan\alpha}$ and simplify to get $\tan\beta = \frac{1-\tan\alpha}{1+\tan\alpha}$. Cross-multiply and rearrange the terms to get $\tan\alpha + \tan\beta = 1 - \tan\alpha\tan\beta$, which is $\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \tan(\alpha+\beta) = 1$, so $\alpha + \beta = \pi/4$.

8. Note $\cos 8\theta = 2\cos^2 4\theta - 1$, so $\frac{1}{2} + \frac{1}{2}\cos 8\theta = \cos^2 4\theta$. Taking the positive root and repeating gives $\cos\theta$. Thus $\cos 4\theta, \cos 2\theta$ and $\cos\theta$ must all be at least 0. This is when $\theta \in \left[0, \frac{\pi}{8}\right] \cup \left[\frac{15\pi}{8}, 2\pi\right]$.

Triangle laws

1. This is a $45^\circ - 45^\circ - 90^\circ$ triangle, thus $\angle ACD = 60^\circ$ and $\angle CDA = 75^\circ$. By the sine law, $\frac{CD}{\sin 45^\circ} = \frac{AC}{\sin 75^\circ}$, so $CD = \sqrt{3} - 1$. The altitude of ADC with respect to the base AC has length $CD \sin 60^\circ = \frac{1}{2}(3 - \sqrt{3})$, thus the area is $\frac{1}{4}(3 - \sqrt{3})$.

2. There is a solution with the sine law, but the synthetic solution involves letting D be the foot of the altitude from C to AB, making ADC a $30^\circ - 60^\circ - 90^\circ$ triangle and BCD a $45^\circ - 45^\circ - 90^\circ$ triangle. AD has length $\frac{\sqrt{2}}{2}$ and CD and BD both have length $\frac{\sqrt{6}}{2}$. The area is then $\frac{3+\sqrt{3}}{2}$.

3. Let $BM = MC = x$. By Apollonius', $AC = \sqrt{2x^2 - 14}$. We use the cosine law to get $\cos\angle BAC = \frac{4^2 - \left(\sqrt{2x^2-14}\right)^2 - (2x)^2}{2 \cdot 4\sqrt{2x^2-14}} = \frac{1-x^2}{4\sqrt{2x^2-14}}$. We want to maximize this, and upon seeing the numerator being negative, we are inspired to take the negative and minimize using AM-GM. Then $\cos\angle BAC = -\frac{1}{4\sqrt{2}}\left(\frac{x^2-1}{\sqrt{x^2-7}}\right) = -\frac{1}{4\sqrt{2}}\left(\frac{x^2-7}{\sqrt{x^2-7}} + \frac{6}{\sqrt{x^2-7}}\right) = -\frac{1}{4\sqrt{2}}\left(\sqrt{x^2-7} + \frac{6}{\sqrt{x^2-7}}\right) \le -\frac{1}{4\sqrt{2}} \cdot 2\sqrt{6} = -\frac{\sqrt{3}}{2}$ by AM-GM. Thus $\angle BAC \ge 150^\circ$.

4. By the cosine law, $\frac{a^2+b^2-c^2}{ab} = 2\cos\gamma$. Since $2\cos\gamma = 2\cos(\pi - \alpha - \beta) = -2\cos(\alpha+\beta)$, we can use the sum formula for cosine to get the answer as $\frac{32}{65}$.

5. There is a straightforward solution with the sine law, but we will proceed synthetically. Let A' be the point on the line AB that is not N such that $A'A = 6$. Then $AA' = AC = AN = 6$, thus A is the center of a circle with diameter $A'N$ containing point C, and $\angle A'CN = 90^\circ$. Draw a line through N parallel

to CA' and let it intersect lines CM and CB at P and Q respectively. Since $\triangle A'MC \sim \triangle PMN$ and $\triangle A'BC \sim \triangle NBQ$, we have $PN = \frac{MN}{MA'} \cdot CA'$ and $QN = \frac{BN}{BA'} \cdot CA'$, and substituting the given shows that $PN = QN$, which implies $\triangle CNP \cong \triangle CNQ$, which implies $\angle MCN = \angle NCB$.

6. By the cosine law, $a^2 = b^2 + c^2 - bc$. Factoring, $b^3 + c^3 = (b+c)(b^2 + c^2 - bc) = (b+c)a^2$. Add a^3 to both sides and rearrange to get the desired equality.