VCSMS PRIME

Session 3: Number theory
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September 28, 2016

Ad hoc

1. (14QII8) How many trailing zeroes does 126! have when written in decimal notation?
2. (14AI7) What is the largest positive integer k such that 27! is divisible by 2^k?
3. (14NE2) What is the smallest number of integers that need to be selected from $\{1, 2, \ldots, 50\}$ to guarantee that two of the selected numbers are relatively prime?
4. (16QII4) How many positive integers n make the expression $7^n + 7^3 + 2 \cdot 7^2$ a perfect square?
5. (11ND2) Find all positive integers n make the expression $2^8 + 2^{11} + 2^n$ a perfect square.
6. (14NA7) What is the largest positive integer $abcdef$ that can be formed from the digits $1, 2, 3, 4, 5, 6$, each used exactly once, if $abcdef$ is divisible by 6, $abcde$ is divisible by 5, $abcd$ is divisible by 4, abc is divisible by 3, and ab is divisible by 2?
7. (14NA8) Let N be the smallest integer such that the quotient $\dfrac{10!}{N}$ is odd. If x and y are nonnegative numbers such that $2x + y = N$, what is the maximum value of x^2y^2?
8. (16AI14) Let P be the product of all prime numbers less than 90. Find the largest integer N so that for each $n \in \{2, 3, 4, \ldots, N\}$, the number $P + n$ has a prime factor less than 90.

Factors

1. (16QI7) What is the fifth largest divisor of the number $2,015,000,000$?
2. (11NA9) What is the sum of all the even positive divisors of 1152?
3. (15AI1) What is the fourth smallest positive integer having exactly 4 positive integer divisors, including 1 and itself?
4. (13NE4) If $25 \cdot 9^{2x} = 15^y$ is an equality of integers, what is the value of x?
5. (14QI14) How many factors of 7^{9999} are greater than $1,000,000$?
6. (13QI9) Determine the number of factors of $5^x + 2 \cdot 5^{x+1}$.
7. (14AI10) Let p and q be positive integers such that $pq = 2^3 \cdot 5^5 \cdot 7^2 \cdot 11$ and $\dfrac{p}{q} = 2 \cdot 5 \cdot 7^2 \cdot 11$. Find the number of positive integer divisors of p.
8. (11QIII3) Let $n = 2^{31}3^{19}$. How many positive divisors of n^2 are less than n but do not divide n?
9. (16AI7) Find the sum of all the prime factors of $27,000,001$.
10. (9AI16) Give the prime factorization of $3^{20} + 3^{19} - 12$.
11. (16NA10) Let m be the product of all positive integral divisors of $360,000$. Suppose the prime factors of m are $p_1, p_2, \ldots, p_k$ for some positive integer k, and $m = p_1^{e_1} p_2^{e_2} \cdot \cdots \cdot p_k^{e_k}$, for some positive integers $e_1, e_2, \ldots, e_k$. Find $e_1 + e_2 + \cdots + e_k$.
12. (12N2) Let f be a polynomial function with integer coefficients and p be a prime number. Suppose that there are at least four distinct integers satisfying $f(x) = p$. Show that f does not have integer zeros.

Divisibility

1. (9QII7) How many values of n for which n and $\dfrac{n+3}{n-1}$ are both integers?
2. (15AI11) Find all integer values of n that will make $\dfrac{6n^3-n^2+2n+32}{3n+1}$ an integer.
3. (15AII2) What is the greatest common factor of all integers of the form p^4-1, where p is a prime number greater than 5?
4. (14AII2) Let a, b, and c be positive integers such that $\dfrac{a\sqrt{2013}+b}{b\sqrt{2013}+c}$ is a rational number. Show that $\dfrac{a^2+b^2+c^2}{a+b+c}$ and $\dfrac{a^3-2b^3+c^3}{a+b+c}$ are both integers.
5. (11ND3) How many positive integer pairs (x, y) are solutions to the equation $\dfrac{xy}{x+y}=1000$?
6. (16N2) Prove that the arithmetic sequence $5, 11, 17, 23, 29, \ldots$ contains infinitely many primes.

Diophantine equations

1. (9QI5) How many ordered pairs (x, y) of positive integers satisfy $2x+5y=100$?
2. (16NE12) Find all values of integers x and y satisfying $2^{3x}+5^{3y}=189$.
3. (13NE3) Find the values of a such that the system $x+2y=a+6, 2x-y=25-2a$ has a positive integer pair solution (x, y).
4. (14QII3) If $2xy+y=43+2x$ for positive integers x, y, find the largest value of $x+y$.
5. (14NE4) Find positive integers a, b, c such that $a+b+ab=15, b+c+bc=99$ and $c+a+ca=399$.
6. (11NA5) Find all nonnegative integer solutions of the system $5x+7y+5z=37, 6x-y-10z=3$.
7. (13NE8) If $7x+4y=5$ and x and y are integers, find the value of $\lfloor y/x \rfloor$.
8. (14NE9) How many triples (x, y, z) of positive integers satisfy the equation $x^{y^z}y^{z^x}z^{x^y}=3xyz$?
9. (13N1) Determine, with proof, the least positive integer n for which there exists n distinct positive integers $x_1, x_2, x_3, \ldots, x_n$ such that
$$\left(1-\frac{1}{x_1}\right)\left(1-\frac{1}{x_2}\right)\left(1-\frac{1}{x_3}\right)\cdots\left(1-\frac{1}{x_n}\right)=\frac{15}{2013}.$$

Modulo

1. (11AI4) Find the last 2 nonzero digits of 16!.
2. (16QI12) When $n+5$ is divided by 4, the remainder is 3. When $n+4$ is divided by 5, the remainder is 5. What is the remainder when $n+6$ is divided by 20?
3. (16NE7) Let n be a positive integer greater than 1. If $2n$ is divided by 3, the remainder 2. If $3n$ is divided by 4, the remainder is 3. If $4n$ is divided by 5, the remainder is 4. If $5n$ is divided by 6, the remainder is 5. What is the least possible value of n?
4. (14NA2) Find the remainder when $3!^{5!^{7!^{\cdots^{2013!}}}}$ is divided by 11.
5. (15AI17) What is the remainder when $16^{15}-8^{15}-4^{15}-2^{15}-1^{15}$ is divided by 96?
6. (16AII1) The 6-digit number $739ABC$ is divisible by $7, 8$, and 9. What values can A, B, and C take?
7. (13N4) Let a, p, and q be positive integers with $p \leq q$. Prove that if one of the numbers a^p and a^q is divisible by p, then the other number must also be divisible by p.