VCSMS PRIME

Session 4: Combinatorics 1

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Ad hoc

1. The single-digit numbers account for $1 + 2 + \cdots + 9 = 45$ of the digits, the remaining $2015 - 45 = 1970$ digits are accounted for by two-digit numbers, which have two digits each. Thus we must have $2\left(10 + 11 + \cdots + n\right) \leq 1970$, the maximum value is $n = 44$. Thus the 1936th digit onwards is $454545\ldots$, so the 2015th is 5.

2. There are only four possible sets: $\{6,1,0\}, \{5,2,0\}, \{4,3,0\}, \{4,2,1\}$. Multiplying by 3! to account for permutations gives 24.

3. A vertically arranged block is determined by its topmost letter, which can appear anywhere in the top 10×12 part of the array, so there are 120 of them. Similarly, the horizontal blocks are determined by its leftmost letter, anywhere among the left $10 \times 12 = 120$ letters. Similarly for the diagonal blocks, but for two 10×10 blocks depending on its orientation. The total is $120 + 120 + 100 + 100 = 440$.

4. The cardinalities are $1, 3, \ldots$, so we must have $1 + 3 + \cdots + (2n - 1) \leq 2009$, which has the largest possible value of $n = 44$. Thus 2009 appears in A_{45}.

5. Each diagonal has 5 skew diagonals, and there are 12 diagonals. We divide by 2 for overcounting: $5 \times 12 \div 2 = 30$.

6. Each number appears in 2^15 subsets, depending on whether each of the other 15 numbers appear or no. Thus the sum is $2^15\left(1 + 2 + \cdots + 16\right) = 4456448$.

7. The one-digit numbers take 9 digits and the two-digit numbers take 180 digits, so we stop at $\dfrac{2016 - 189}{3} + 99 = 708$.

   From 0 to 99, in the ones place the sum is $10(0 + 1 + \cdots + 9)$ and in the tens place the sum is $10(0 + 1 + \cdots + 9)$, so the total sum is $20(45) = 900$. From 100 to 699, there are 6 0 to 99s and 100 occurences of 0 to 6 in the hundreds place, so $100(0 + 1 + \cdots + 6) + 6(900) = 7500$. Then $700, \ldots, 708$ have a sum of 99, so the total is $900 + 7500 + 99 = 8499$.

8. On the main diagonal is 0, then above and below there are two diagonals, each with $n - 1$ 1s, above and below are two diagonals, each with $n - 2$ 2s, etc. The summation is $\sum 2(n - i)i$ from $i = 1$ to n, or $2\left(n\sum i - \sum i^2\right) = \dfrac{1}{3}n^3 - \dfrac{1}{3}n$. Thus $n^3 - n = 7980$, and we observe that only $n = 20$ works.

9. The first digit has to be 1, then onwards, the digits have to be either 0 or 9. The only choices are $1999, 1099, 1009, 1000$, and the smallest is $\dfrac{1099}{19}$.

10. Burnside's, or bloody casework. We do casework on the number of white vertices. For 0 or 1 white vertices there is clearly one different way each. For 2 white vertices there are three ways: both connected by an edge, both on the same face but not adjacent, and on opposite vertices. For 3 white vertices there are four ways, all on the same face, all on opposite vertices, and two when two are connected. For 4 white vertices there are six ways: all on the same face, four where three share the same face, and one with two pairs opposite each other.

    The $5, 6, 7, 8$ white vertices are analogous to there being $3, 2, 1, 0$ black vertices, so there are the same number of ways. That makes a total of $1 + 1 + 3 + 4 + 6 + 4 + 3 + 1 + 1 = 24$.

Inclusion-Exclusion

1. Of the 100 people, 60 claim to be good, so $100 - 60 = 40$ deny to be good. Of these, 30 correctly deny, so the rest must be people who are good at math but refuse to admit it: $40 - 30 = 10$.

2. There are $\left\lfloor \frac{2015}{3} \right\rfloor = 671$ numbers less than 2015 divisible by 3. Of these, $\left\lfloor \frac{671}{5} \right\rfloor = 134$ are divisible by 5 and $\left\lfloor \frac{671}{7} \right\rfloor = 95$ are divisible by 7, with $\left\lfloor \frac{671}{35} \right\rfloor = 19$ divisible by 35. By PIE, the answer is $671 - 134 - 95 + 19 = 461$.

3. This is $\phi(10000) - 1 = 10000\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) - 1 = 3999$.

4. There are 638 numbers divisible by 3, 239 divisible by 8, 79 divisible by 24, and 319 divisible by 6 in the range $[100, 2015]$. The answer is $638 + 239 - 79 - 319 = 479$.

5. By PIE: $\left\lfloor \frac{999}{10} \right\rfloor + \left\lfloor \frac{999}{15} \right\rfloor + \left\lfloor \frac{999}{35} \right\rfloor + \left\lfloor \frac{999}{55} \right\rfloor - \left\lfloor \frac{999}{30} \right\rfloor - \left\lfloor \frac{999}{70} \right\rfloor - \left\lfloor \frac{999}{110} \right\rfloor - \left\lfloor \frac{999}{105} \right\rfloor - \left\lfloor \frac{999}{165} \right\rfloor - \left\lfloor \frac{999}{385} \right\rfloor + \left\lfloor \frac{999}{210} \right\rfloor + \left\lfloor \frac{999}{330} \right\rfloor + \left\lfloor \frac{999}{770} \right\rfloor = 146$.

Permutations

1. Casework: only 100 has a sum of 1 and 999 has a sum of 27. By balls and urns, there are $\binom{10}{8} = 45$ that sum to 8, except 9 of these start with 0. The sum is $1 + 1 + 45 - 9 = 38$.

2. There are $\frac{10!}{2!}$ permutations, since I is repeated. Except we overcount: we want to consider only AIIU out of its $\frac{4!}{2!} = 12$ possible permutations. So we divide by 12. This gives $\frac{10!}{12 \cdot 2!} = 151200$.

3. In a line, this is $2 \times 6! \times 6!$ – pick either girl or boy to go first and alternate, then multiply by the number of ways to permute per gender. Divide by 12 to account for rotation in a circle: $2 \times 6! \times 6! \div 12 = 86400$.

4. There are $\frac{7!}{2!} = 2520$ permutations since A is repeated. However, we want to consider only AEA out of its 3 permutations, so divide by 3 to get 840.

5. Since $33750 = 2 \cdot 3^3 \cdot 5^4$, we split to four cases: all prime numbers, which is $\frac{8!}{4!3!} = 280$, one of them is 6, which is $\frac{7!}{4!2!} = 105$, one of them is 9, which is $\frac{7!}{4!} = 210$, and when both 6 and 9 are present, $\frac{6!}{4!} = 30$. The sum is $280 + 105 + 210 + 30 = 625$.

6. There are 4! starting with A, 4! with M, 4! with R, a total of 72. Then 3! start with SA, a total of 78. The first starting with SM is SMART, so that must be the 79th.

7. There are $\frac{7!}{2!2!2!} = 630$ ways without restriction. There are $\frac{5!}{2!} = 60$ ways where PHI appears and also $\frac{5!}{2!} = 60$ ways where ILL appears. For strings with both PHI and ILL, it can be either as PHILL, I, P which is $3! = 6$ ways, or as PHI, ILL and P for another $3! = 6$ ways. By PIE, there are $630 - 60 - 60 + 6 + 6 = 522$ ways.

8. We use PIE. There are $\frac{6!}{2!2!2!} = 90$ ways to arrange MURMUR. By symmetry, when two Ms, Us, or Rs are together, there are $\frac{5!}{2!2!} = 30$ ways. Similarly, if two pairs of letters are together, there are $\frac{4!}{2!} = 12$ ways. Finally there are 3! ways when each pair is together. By PIE, there are $90-30-30-30+12+12+12-6 = 30$ ways.

Combinations

1. There are 2^n subsets, 1 with no elements and n with one element. Thus $2^n - n - 1 = 57$, which only has the solution $n = 6$.

2. One of the numbers chosen has to be 7, another has to be either $3, 6$ or 9, the last number can be anything. This gives $1 \cdot 3 \cdot 7 = 21$, but we overcounted when the last number is also divisible by 3, which happens 6 ways, when we only want to count it 3 times. So $21 - 6 + 3 = 18$.

3. Order the boys arbitrarily, then there are $6! = 720$ ways to arrange the girls to form pairs.

Balls and urns

1. Let Amy, Bob and Charlie receive a, b, c cookies respectively; we have $a+b+c = 15$, or $(a-4)+b+c = 11$. Now each of $a-4, b, c$ are positive integers, so this is balls and urns. There are 10 slots in between and we pick 2 of them, so $\binom{10}{2} = 45$ ways.

2. Balls and urns directly: $\binom{12}{2} = 66$.

3. Consider $(x - 1000) + (y - 600) + (z - 400) = 16$, where each variable is now a positive integer, so by balls and urns there are $\binom{15}{2} = 105$ ways.

4. Consider the 6 integers that are left. You are placing four integers such that no two are adjacent, so you have to place them in-between, in front, or behind the 6 integers, giving 7 slots. Each slot can only go to one integer so no two are adjacent, giving $\binom{7}{4} = 35$ ways.

5. Consider the 7 books that are left. You are placing 5 books so that no two of them are adjacent, so you have to place them in-between, in front, or behind the 7 books, giving 8 slots. Each slot can only go to one book so no two are adjacent, so that's $\binom{8}{5} = 56$ ways.

6. Similar as above, there are $\binom{8}{5} = 56$ ways in a row. However, we overcount: we don't want to count the $\binom{6}{3} = 20$ ways when a book is placed in front and behind. This is $56 - 20 = 36$ ways.

7. Bloody casework on $a + b$. When $a + b = 0$, there is 1 solution for (a, b). Then either $c + d + e = 0$, with $\binom{2}{2}$ solutions, $c + d + e = 1$ with $\binom{3}{2}$ solutions, etc., up to $c + d + e = 4$ with $\binom{6}{2}$ solutions, all by balls and urns. This is thus $\binom{2}{2} + \binom{3}{2} + \cdots + \binom{6}{2} = \binom{7}{3} = 35$.

   Similarly, $a + b = 1$ has $\binom{2}{1} = 2$ solutions, and the hockeystick sum is $\binom{2}{2} + \cdots + \binom{5}{2} = \binom{6}{3} = 20$, so there are $2 \times 20 = 40$ ways. When $a + b = 2$ then the sum is $\binom{3}{1} \times \left(\binom{2}{2} + \cdots + \binom{4}{2}\right) = 30$. The sum of all cases is 105.