VCSMS PRIME

Session 5: Algebra 2

compiled by Carl Joshua Quines

October 5, 2016

Equations

1. (13QI7) Sixty men working on a job have done 1/3 of the work in 18 days. The project is behind schedule and must be accomplished in the next twelve days. How many more workers need to be hired?
2. (14AI14) Solve the equation $(2-x^2)^{x^2-3\sqrt{2}x+4}=1$.
3. (11NE4) If $\dfrac{x-a-b}{c}+\dfrac{x-b-c}{a}+\dfrac{x-c-a}{b}=3$, where $a,b,c\in\mathbb{R}^+$, find x in terms of a,b and c.
4. (14AI19) Find the values of x in $(0,\pi)$ that satisfy the equation

$$(\sqrt{2014}-\sqrt{2013})^{\tan^2 x}+(\sqrt{2014}+\sqrt{2013})^{-\tan^2 x}=2(\sqrt{2014}-\sqrt{2013})^3$$

   .
5. (14QI7) For which m does the equation $\dfrac{x-1}{x-2}=\dfrac{x-m}{x-6}$ have no solution in x?
6. (16AI3) Determine all values of $k\in\mathbb{R}$ for which $\dfrac{4(2015^x)-2015^{-x}}{2015^x-3(2015^{-x})}=k$ admits a real solution.
7. (11ND4) Give three real roots of $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$.
8. (13NE5) Find the solution set of the equation $(\sqrt{2}-1)^x+8(\sqrt{2}+1)^x=9$.

Systems of equations

1. (8NE1) If $wxy=10, wyz=5, wxz=45, xyz=12$, what is $w+y$?
2. (16AI12) Find all real solutions to the system of equations $x(y-1)+y(x+1)=6, (x-1)(y+1)=1$.
3. (13QI14) If (a,b) is the solution of $\sqrt{x+y}+\sqrt{x-y}=4, x^2-y^2=9$, then find the value of $\dfrac{ab}{a+b}$.
4. (14NA5) Suppose that $w+4x+9y+16z=6, 4w+9x+16y+25z=7, 9w+16x+25y+36z=12$. Find $w+x+y+z$.
5. (10NA1) The nonzero numbers x,y,z satisfy $xy=2(x+y), yz=4(y+z), xz=8(x+z)$. Solve for x.

Complex numbers

1. (11AI8) Find all complex numbers x satisfying $x^3+x^2+x+1=0$.
2. (15AI3) Simplify the expression $\left(1+\dfrac{1}{i}+\dfrac{1}{i^2}+\cdots+\dfrac{1}{i^{2014}}\right)^2$.
3. (13AI1) Find all complex numbers z such that $\dfrac{z^4+1}{z^4-1}=\dfrac{i}{\sqrt{3}}$.
4. (13AI3) If $z^3-1=0$ and $z\neq 1$, find the value of $z+\dfrac{1}{z}+4$.

Polynomials

1. (13NE7) Let $P(x) = ax^7 + bx^3 + cx - 5$, where a, b, and c are constants. If $P(-7) = 7$, what is $P(7)$?
2. (13NE1) In solving a problem that leads to a quadratic equation, one student made a mistake in the constant term only, obtaining the roots 8 and 2, while another student made a mistake in the coefficient of the first degree term, obtaining the roots -9 and -1. What was the original equation?
3. (13AI5) Consider a function $f(x) = ax^2 + bx + c, a > 0$, with two distinct roots a distance p apart. By how much, in terms of a, b, c, should the function be translated downwards so that the distance between the roots becomes $2p$?
4. (15AI9) Two numbers p and q are both chosen randomly and independently of each other from the interval $[-2, 2]$. Find the probability that $4x^2 + 4px + 1 - q^2 = 0$ has imaginary roots.
5. (9NA4) What is the coefficient of x^5 in the expansion of $(2 - x + x^2)^4$?
6. (13AII2) The quartic polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Find $P(5)$.
7. (10NA8) When $(x^2 + 2x + 2)^{2009} + (x^2 - 3x - 3)^{2009}$ is expanded, what is the sum of the coefficients of the terms with odd exponents of x?
8. (16QIII5) How many terms are there when the expression of $(x+y+z)^{2015} + (x-y-z)^{2015}$ is expanded and simplified?

Polynomial factors

1. (10NE11) Find the values of a and b such that $ax^4 + bx^2 + 1$ is divisible by $x^2 - x - 2$.
2. (13NE9) If $x^2 + 2x + 5$ is a factor of $x^4 + ax^2 + b$, find the sum $a + b$.
3. (13QII9) Factorize $(r - s)^3 + (s - t)^3 + (t - r)^3$.
4. (16QII6) How many (nonconstant) polynomial factors with leading coefficient 1, with the other coefficients possibly complex, does $x^{2015} + 18$?
5. (11QIII2) Find all polynomials $p(x)$ where $xp(x - 1) = (x - 5)p(x)$ and $p(6) = 5!$.

Remainder theorem

1. (14AI11) Let r be some real constant, and $P(x)$ a polynomial which has remainder 2 when divided by $x - r$, and remainder $-2x^2 - 3x + 4$ when divided by $(2x^2 + 7x - 4)(x - r)$. Find all values of r.
2. (11AI5) Let $f(x)$ be a cubic polynomial. If $f(x)$ is divided by $2x + 3$, the remainder is 4, while if it is divided by $3x + 4$, the remainder is 5. What will be the remainder when $f(x)$ is divided by $6x^2 + 17x + 22$?
3. (9NA7) Let $P(x)$ be a polynomial, that, when divided by $x-19$, has the remainder 99, and when divided by $x - 99$, has the remainder 19. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?

Root-finding

1. (16QII2) What is the difference between the largest and smallest real zeros of the function $f(x) = 2x^4 - 7x^3 + 2x^2 + 7x + 2$?
2. (13NA8) There are values of m for which $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$ has equal roots. What are these equal roots?
3. (13AI11) Let f be a polynomial function that satisfies $f(x - 5) = -3x^2 + 45x - 108$. Find the roots of $f(x)$.

4. (14AII3) If p is a real constant such that the roots of the equation $x^3 - 6px^2 + 5px + 88 = 0$ form an arithmetic sequence, find p.
5. (11NE11) $x^3 + kx - 128 = 0$ has a root of multiplicity 2. Find k.
6. (11AI17) Find all real numbers a such that $x^3 + ax^2 - 3x - 2$ has exactly two distinct real zeros.

Vieta's

1. (16NE4) There are two distinct real numbers which are larger than their reciprocals by 2. Find the product of these numbers.
2. (16NA4) Let $f(x)$ be a polynomial function of degree 2016 whose 2016 zeros have a sum of S. Find the sum of the 2016 zeros of $f(2x - 3)$ in terms of S.
3. (9QII1) The roots of the quadratic equation $x^2 - 51x + k = 0$ differ by 75, where k is a real number. Determine the sums of the squares of the roots.
4. (11NE10) Find a quadratic polynomial with integer coefficients whose roots are the reciprocals of $x^2 + 4x + 8 = 0$.
5. (9NE13) Find the sum of the reciprocals of the roots of $4x^4 - 3x^3 - x^2 + 2x - 6 = 0$.
6. (13QII5) The equation $x^2 - bx + c$ has two roots p and q. If the product pq is to be maximum, what value of b will make $b + c$ minimum?
7. (14NE11) Suppose a, b, c are the roots of $x^3 - 4x + 1 = 0$. Find the value of $\dfrac{a^2bc}{a^3+1} + \dfrac{ab^2c}{b^3+1} + \dfrac{abc^2}{c^3+1}$.

Coordinate plane

1. (11QI4) For what values of a does the system $x^2 - y^2 = 0, (x - a)^2 + y^2 = 0$ have a unique solution?
2. (10NE6) If the parabola $y + 1 = x^2$ is rotated clockwise by 90° about its focus, what will be the new coordinates of its vertex?
3. (14AI8) For what real values of p will the graph of the parabola $y = x^2 - 2px + p + 1$ be on or above that of the line $y = -12x + 5$?
4. (14QII4) Let (a, b) and (c, d) be the two distinct points of intersection of circles C_1 and C_2. The circle C_1 is centered at the origin and passes through $P(16, 16)$, while the circle C_2 is centered at P and passes through the origin. Find $a + b + c + d$.
5. (13NA9) Let $A(-3, 0), B(3, 0), C(0, 5)$ and $D(0, -5)$. How many points $P(x, y)$ on the plane satisfy both $PA + PB = 10$ and $|PC - PD| = 6$?
6. (11QII7) A line with y-intercept 5 and positive slope is drawn such that the line intersects $x^2 + y^2 = 9$. What is the least slope of such a line?
7. (16AI17) Find the area of the region bounded by the graph of $|x| + |y| = \dfrac{1}{4}|x + 15|$.
8. (13AI4) Find the equation of the line that contains the point $(1, 0)$, that is of least positive slope, and that does not intersect the curve $4x^2 - y^2 - 8x = 12$.
9. (13AI6) Find the equation of the circle, in the form $(x - h)^2 + (y - k)^2 = r^2$, inscribed in a triangle whose vertices are located at the points $(-2, 1), (2, 5), (5, 2)$.
10. (13NE12) What is the length of the shortest path that begins at the point $(-3, 7)$, touches the x-axis, and then ends at a point on the circle $(x - 5)^2 + (y - 8)^2 = 25$?