VCSMS PRIME

Session 6: Combinatorics 2
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Random variable

1. (14QI3) What is the probability of getting a sum of 10 when rolling three fair six-sided dice?
2. (13QII7) A fair die is thrown three times. What is the probability that the largest outcome of the three throws is a 3?
3. (11AII1) Sherlock and Mycroft play a game which involves flipping a single fair coin. The coin is flipped repeatedly until one person wins. Sherlock wins if the sequence TTT shows up first while Mycroft wins if the sequence HTT shows up first. Who among the two has a higher probability of winning?
4. (16ND5) The faces of a 12-sided die are numbered $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$, and 12 such that the sum of the numbers on opposite faces is 13. The die is meticulously carved so that it is biased: the probability of obtaining a particular face F is greater than $1/12$, the probability of obtaining the face opposite F is less than $1/12$, while the probability of obtaining any one of the other ten faces, is $1/12$. When two such dice are rolled, the probability of obtaining a sum of 13 is $29/384$. What is the probability of obtaining face F?

Random selection

1. (11QII2) Mica has six different colored crayons. She can use one ore more colors in her painting. What is the likelihood that she will use only her favorite color?
2. (16QI15) The exterior of a cube of side length 3 units is painted red, and is then divided into 27 unit cubes. If one of these cubes is randomly selected and then rolled, what is the probability that a red face comes up?
3. (11NE7) Let a, b, c be three, not necessarily distinct, numbers chosen randomly from the set $\{3, 4, 5, 6, 7, 8\}$. Find the probability that $ab + c$ is even.
4. (16NE6) An urn contains five red chips numbered 1 to 5, five blue chips numbered 1 to 5, and five white chips numbered 1 to 5. Two chips are drawn from this urn without replacement. What is the probability that they have either the same color or the same number?
5. (16NA8) In a certain school, there are 5000 students. Each student is assigned an ID number from 0001 to 5000. No two students can have the same ID number. If a student is selected uniformly at random, what is the probability that the ID number of the student does not contain any 2s among its digits?
6. (16NA6) Suppose that Ethan has four red chips and two white chips. He selects three chips at random and places them in urn 1, while the remaining chips are placed in urn 2. He lets his brother Josh draw one chip from each urn at random. What is the probability that the chips drawn by Josh are both red?
7. (14NA6) A bag contains 3 red and 3 green balls. Three balls are drawn at random from the bag and replaced with 3 white balls. Afterwards, another 3 balls are drawn at random from the bag. Find the probability that the color of the 3 balls in the second draw are all different.
8. (13NA7) Twenty-five people sit around a circular table. Three of them are chosen randomly. What is the probability that two of the three are sitting next to each other?
9. (12N1) A computer generates even integers half of the time and another computer generates even integers a third of the time. If a_i and b_i are the integers generated by the computers, respectively, at time i, show that the probability $a_1b_1 + \cdots + a_kb_k$ is even is $\dfrac{1}{2} + \dfrac{1}{2 \cdot 3^k}$.

Geometric probability

1. (14NE8) Given two distinct points A and B, a point P is randomly and uniformly chosen in the interior of segment AB. Let $r > 0$. Find the probability, in terms of r, that $\dfrac{AP}{BP} < r$.

2. (16QII1) A married couple has PhP $50,000$ in their joint account. In anticipation of their upcoming anniversary, they agreed to split-up evenly and buy each one a gift. However, they agreed that they must have enough money left such that when combined, a minimum bank balance of PhP $5,000$ is maintained. If the couple individually bought gifts without regard to how much the other's gift will cost, and each gift is randomly price between PhP 0 to PhP $25,000$, what is the probability that they will be able to maintain the minimum required balance after buying the gifts?

3. (11AI7) Find the probability of obtaining two numbers x and y in the interval $[0, 1]$ such that $x^2 - 3xy + 2y^2 > 0$.

4. (16AI19) The amount 4.5 is split into two nonnegative real numbers uniformly at random. Then each number is rounded to its nearest integer. For instance, if 4.5 is split into $\sqrt{2}$ and $4.5 - \sqrt{2}$, then the resulting integers are 1 and 3, respectively. What is the probability that the two integers sum to 5?

Existence combinatorics

1. (10QII9) Seven distinct integers are chosen randomly from the set $\{1, 2, \ldots, 2009\}$. What is the probability that two of these integers have a difference that is a multiple of 6?

2. (13AI13) From the xy-plane, select five distinct points that have integer coordinates. Find the probability that there is a pair of points among the five whose midpoint has integer coordinates.

3. (9N3) Each point of a circle is colored either red or blue. Prove that there always exists an isosceles triangle inscribed in this circle such that all its vertices are colored the same. Does there always exist an equilateral triangle inscribed in this circle such that all its vertices are colored the same?

4. (10N4) There are 2008 blue, 2009 red, and 2010 yellow chips on a table. At each step, one chooses two chips of different colors, and recolor both of them using the third color. Can all chips be of the same color after some steps?

5. (10N5) Determine the smallest positive integer n such that for every choice of n integers, there exists at least two whose sum or difference is divisible by 2009.

6. (11N3) The 2011th prime number is 17483, and the next prime is 17489. Does there exist a sequence of 2011^{2011} consecutive positive integers that contains exactly 2011 prime numbers? Prove your answer.

7. (13N3) Let n be a positive integer. The numbers $1, 2, 3, \ldots, 2n$ are randomly assigned to $2n$ distinct points on a circle. To each chord joining two of these points, a value is assigned equal to the absolute value of the difference between the assigned numbers at its endpoints. Show that one can choose n pairwise non-intersecting chords such that the sum of the values assigned to them is n^2.

8. (12N5) There are exactly 120 Twitter subscribers from National Science High School. Statistics shows that each of 10 given celebrities have at least 85 followers from National Science High School. Prove that there must be two students such that each of the 10 celebrities is being followed in Twitter by at least one of these students.