VCSMS PRIME

Session 7: Geometry 1
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Circles

1. Let the centers of the circles be A, B, one internal tangent be CD tangent to circle A at C, and to circle B at D, and let E be the intersection of the two tangents.

   Since $\angle E$ is right, and $\angle C$ is right as well, then ACE must be an isosceles right triangle. Thus AE, BE are $4\sqrt{2}$ and $2\sqrt{2}$, so AB is $6\sqrt{2}$.

2. Since $\angle QPR + \angle QSR = 180°$ then quadrilateral $PQRS$ is cyclic, so C_1 and C_2 are the same circle, and they intersect at infinitely many points.

3. Drop the perpendicular from C to AB at point D. Then $CD = 5$, and $CB = 13$, so by Pythagorean, $DB = 12$. Similarly, $AD = 12$, so the perimeter is $12 + 12 + 13 + 13 = 50$.

4. Extend CB to meet the circle again at F. By power of a point, we get $CF = 12$, and so $AF = 5$. By Stewart's on triangle OBF we find $OB = BF = 2\sqrt{6}$. Pythagorean on OCD gives $OC = 2\sqrt{15}$.

5. Let C_2 have center O, the smaller circle have center P, tangent to C_1, C_2 and AB at I, J, K, respectively. Let the smaller circle have radius r and let $OK = s$.

   Then Pythagorean on APK gives $AP^2 = AK^2 + PK^2$, or $(AI + IP)^2 = (AO + OK)^2 + PK^2$, or $(12+r)^2 = (12+s)^2+r^2$. Pythagorean on OPK gives $OP^2 = PK^2+OK^2$, or $(OJ-JP)^2 = PK^2+OK^2$, or $(12 - r)^2 = r^2 + s^2$. The r^2 term cancels in both equations, and we can equate $24r$ in both to get $144 - s^2 = s^2 + 24s$. Thus $s = 6\sqrt{3} - 6$ and $r = 3\sqrt{3}$.

6. From power of a point on E we get $AE = BE$ so ABE is equilateral. Thus $\angle ABC = 120°$. By the law of cosines $AC = 2\sqrt{7}$, and by the extended law of sines $2R = \dfrac{AC}{\sin 120°}$, so the circumradius is $\dfrac{2\sqrt{21}}{3}$.

Angles

1. Let $\angle ABD = \angle DBC = x°$. We know that $\angle ADB = \angle DBC + \angle BCD$ since it is an exterior angle, however $\angle ADB = \dfrac{180° - \angle ABD}{2}$ as triangle ABD is isosceles. Equating gives $x + 36 = \dfrac{180 - x}{2}$, or $x = 36°$. Thus since triangle ABD is isosceles, $\angle ADB = \dfrac{180° - \angle ABD}{2} = 72°$; since triangle ADE is isosceles $\angle ADE = \dfrac{180° - \angle DAB}{2} = 54°$, and so $\angle BDE = \angle ADB - \angle ADE = 17°$.

2. Let Q be the midpoint of BC. Then $\angle ABP = \angle APB = 52°$ by triangle angle sum on ABP, so $AB = BP$. Then $ABQP$ is a rhombus. Then AQ is an angle bisector since it is a diagonal, so $\angle AQP = 38°$. But $PC||AQ$ and $PQ||CD$ so $\angle PCD = \angle AQP = 38°$.

3. Note $\angle CBD = \angle ADB - \angle DCB$ upon considering exterior $\angle ADB$. But $\angle ADB = \angle ABD = \angle ABC - \angle CBD$ through isosceles triangle ABD. Substituting, $\angle CBD = (\angle ABC - \angle CBD) - \angle DCB = (\angle ABC - \angle ACB) - \angle CBD = 45° - \angle CBD$. Thus $\angle CBD = 22.5°$.

4. Since in $AFGE$ we have $\angle AFG + \angle AEG = 90° + 90° = 180°$, it is a cyclic quadrilateral. Similarly, since in $BDEF$ we have $\angle BED = \angle BFD = 90°$ then it is also cyclic. Thus $\angle GAB = \angle GAF$, and $\angle GAF = \angle GEF$ by cyclic quadrilateral $AFGE$, and $\angle GEF = \angle BEF = \angle BDF$ by cyclic quadrilateral $BDEF$. However, $\angle BDF + \angle FDE = \angle CED$ since $BCDE$ is a rectangle. Thus $\angle GAB = \angle BDF = 17°$.

5. From $CA \perp CG$ and $BG \perp CG$ we have $CA||BG$. Then $\angle ABG + \angle CAB = 180°$, whence $\angle ABG = 78°$. Then $\angle ABG = \angle EBG = 2\angle EFG = 2\angle DFG$, so $\angle DFG = 39°$.

Three-dimensional

1. By Euler's formula, $V - E + F = 2$, so $V = 34$.

2. It is a regular tetrahedron of edge 1. Drop the height from the top vertex to the base, which hits its center. It forms a right triangle with one edge as the hypotenuse, the other leg is from the length from a vertex to the center. The other leg is 2/3 the median, so its length is $\frac{\sqrt{3}}{3}$. This gives its height as $\sqrt{1^2 - \left(\frac{\sqrt{3}}{3}\right)^2} = \frac{\sqrt{6}}{3}$. Its volume is one-third the area of the base times its height, or $\frac{1}{3} \cdot \frac{\sqrt{3}}{4} \cdot \frac{\sqrt{6}}{3} = \frac{\sqrt{2}}{12}$.

3. We stack the $7 \times 9 \times 11$ boxes in a $2 \times 3 \times 3$ fashion, making it take up $14 \times 27 \times 33$, which fits in the $17 \times 27 \times 37$ box. This makes the maximum number 18.

4. Let the sides of the prism be x, y, z; we have $xyz = 120$ and $(x-2)(y-2)(z-2) = 24$. WLOG z is divisible by 5. Then if $z = 5$, we see $(6, 4, 5)$ works. The surface area is then $2(6 \cdot 5 + 5 \cdot 4 + 4 \cdot 6) = 148$.

5. The centers of the spheres form a regular tetrahedron of edge 3. Through similar logic as number 2 in this section, its height is $\sqrt{3^2 - \sqrt{3}^2} = \sqrt{6}$. The overall height is the height of the tetrahedron plus two radii, so its height is $3 + \sqrt{6}$.

Areas

1. The area consists of two $150°$ sectors of a circle with radius 10, one on either side of the horse. Wrapping around the equilateral triangle gives two more $120°$ sectors, of radius $10 - 8 = 2$. The total area is thus $2 \cdot \frac{150°}{360°}\pi \cdot 10^2 + 2 \cdot \frac{120°}{360°}\pi \cdot 2^2 = 86\pi$.

2. Drop the altitude from E to AB and CD, which are parallel, so the altitude is the same line. The length of the altitude from E to AB has to be 20 for the area of AEB to be 60. Since $AB||CD$ we have $EAB\ simEDC$ and thus the length of the altitude from E to CD has to be $\frac{80}{3}$. Thus the distance between lines AB and CD is $\frac{80}{3} - 20 = \frac{20}{3}$, which is also the length of the altitude from D to AB. Thus $[BAD] = \frac{1}{2} \cdot 6 \cdot \frac{20}{3} = 20$.

3. Note that $\triangle AEB$ and $\triangle AEF$ share the same base and altitude, so they have the same area. Subtracting $[AEG]$ from both gives $[ABG] = [EFG] = 9$. Similarly, $[CDH] = [EFH] = 15$. Thus $[EGFH] = [EFG] + [EFH] = 24$.

4. (Should have E as intersection of diagonals.) Note that AEB and CED are similar with ratio $6 : 15$. Then $EB : ED = 6 : 15$ as well, as AED and AEB share the same altitude from A, their areas are in the ratios of their bases, so $[AEB] : [AED] = 6 : 15$. Thus $[AEB] = 12$.

5. Let the triangle be ABC intersecting the circle with center O at B' and C' lying on AB and AC, respectively. The required region is quadrilateral $AB'OC'$ minus the sector with arc $B'C'$. This is twice the area of a unit equilateral triangle minus the unit sector of $60°$, or $2 \cdot \frac{\sqrt{3}}{4} - \frac{1}{6}\pi = \frac{3\sqrt{3} - \pi}{6}$.

6. In rectangle $ABMN$ with area 2, triangles APM and BPN form half the area, so the sum of their areas is 1. P is vertically halfway between AM and BN, so its distance to DC is $\frac{3}{2}$. The area of DPC is thus $\frac{1}{2} \cdot 2 \cdot \frac{3}{2} = \frac{3}{2}$. Then triangles PQR and DCP are similar, but the height from P to QR is the

distance from P to AB, which is $\frac{1}{2}$. Thus the ratio of similarity is $1:3$, so the ratio of their areas is $1:9$, thus the area of PQR is $\frac{1}{6}$. The sum is $\frac{8}{3}$.

7. It is simplest to Cartesian bash. Set $M(0,0)$, $B(0,18)$, $I(16,0)$. Thus $H(0,8)$ and $A(6,0)$. Line BA is $\frac{x}{6}+\frac{y}{18}=1$ in intercept form, also line IH is $\frac{x}{16}+\frac{y}{8}=1$. Equating gives $\frac{x}{6}-\frac{x}{16}=\frac{y}{8}-\frac{y}{18}$ or $\frac{10x}{6\cdot 16}=\frac{10y}{8\cdot 18}$, cancelling gives $3x=2y$. Substituting back to either equation gives $T(4,6)$. Using the shoelace formula on $MATH$ gives its area as 34.

8. It is also simple to Cartesian bash: set $C(0,0)$, $B(0,16)$, $A(13,16)$ and $D(11,0)$. Then E is a midpoint so $E(12,8)$. The slope of AD is 8 so the slope of EF is $-\frac{1}{8}$. Point F lies on BC so its x-coordinate is zero; it lies on EF so $F(0,9.5)$. Using the shoelace formula gives 91.

9. Suppose that point C is C' after folding, and DC' and EC' intersect AB at A' and B' respectively. Drop altitudes H from C to DE and M from C to AB. Clearly C,H,M,C' are collinear. The ratio $[A'B'C']:[ABC]=16:100$ is given, thus the ratio $C'M:CM=4:10$ due to similarity. Also, $CH=C'H$ since they are the same altitude after folding. Since $CH+C'H=CM+C'M$ due to collinearity, $2CH=CM+\frac{2}{5}CM$ from earlier. By similarity, $CH:CM=7:10=DE:AB$, so $DE=\frac{56}{5}$.

10. Let x be the side of the square. The Pythagorean theorem on right CEH gives $\left(r-\frac{x}{2}\right)^2+x^2=r^2$, so $x=\frac{4}{5}r$. Thus $\angle HCE=\tan^{-1}\frac{4}{3}$. The required area is equal to $[CHGF]$ minus the sector with arc HM; the former is $\frac{1}{2}\left(r+\frac{x}{2}+x\right)x$ while the latter is $\frac{1}{2}r^2\tan^{-1}\frac{4}{3}$. Simplifying yields $r^2\left(\frac{22}{25}-\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$.

11. Official solution uses algebra and whatever. We use Cartesian. Take an affine transformation to $A(0,1)$, $B(1,0)$, $C(0,0)$ which preserves the problem, and let $P(a,b)$. It is easy to bash $D\left(-\frac{a}{b-1},0\right)$, $E\left(0,-\frac{b}{a-1}\right)$, $F\left(\frac{a}{a+b},\frac{b}{a+b}\right)$. Then $[DBP]=[ECP]=[FAP]$ and bashing gives $a=b=\frac{1}{3}$, which is as required.