VCSMS PRIME

Session 8: Algebra 3

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Manipulation

1. (13NE10) If $\dfrac{x}{y} + \dfrac{y}{x} = 4$ and $xy = 3$, find the value of $xy(x+y)^2 - 2x^2y^2$.
2. (14NE1) If $x = \sqrt{2013 - yz}, y = \sqrt{2014 - zx}$ and $z = \sqrt{2015 - xy}$, find $(x+y)^2 + (y+z)^2 + (z+x)^2$.
3. (14QIII4) If $m^3 - 12mn^2 = 40$ and $4n^3 - 3m^2n = 10$, find $m^2 + 4n^2$.
4. (13AII1) If $x + y + xy = 1$, where x, y are nonzero real numbers, find the value of $xy + \dfrac{1}{xy} - \dfrac{y}{x} - \dfrac{x}{y}$.
5. (14QIII3) If $\dfrac{a}{a^2+1} = \dfrac{1}{3}$, determine $\dfrac{a^3}{a^6 + a^5 + a^4 + a^3 + a^2 + a + 1}$.
6. (14QII9) It is known that $1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \cdots = \dfrac{\pi^2}{6}$. Find the sum $1 + \dfrac{1}{9} + \dfrac{1}{25} + \dfrac{1}{49} + \cdots$.
7. (14AI18) Let x be a real number so that $x + \dfrac{1}{x} = 3$. Find the last two digits of $x^{2^{2013}} + \dfrac{1}{x^{2^{2013}}}$.
8. (13ND5) Let $x = cy + bz, y = az + cx, z = bx + ay$. Find $\dfrac{(x-y)(y-z)(z-x)}{xyz}$ in terms of a, b, and c.

Surds

1. (15AI6) Rationalize the denominator of $\dfrac{6}{\sqrt[3]{4} + \sqrt[3]{16} + \sqrt[3]{64}}$ and simplify.
2. (13QII10) If $36 - 4\sqrt{2} - 6\sqrt{3} + 12\sqrt{6} = (a\sqrt{2} + b\sqrt{3} + c)^2$, find the value of $a^2 + b^2 + c^2$.
3. (14ND2) Solve for x: $\sqrt{x + \sqrt{3x+6}} + \sqrt{x - \sqrt{3x+6}} = 6$.
4. (13NA6) If $\sqrt[3]{a + \sqrt{b}} = 12 + \sqrt{5}$, find the value of $\sqrt[3]{a - \sqrt{b}}$.
5. (11NA8) Let $a = \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ and $b = \dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$. Find the value of $a^4 + b^4 + (a+b)^4$.
6. (11AII3) Show that $\sqrt[n]{2} - 1 \le \sqrt{\dfrac{2}{n(n-1)}}$ for all positive integers $n \ge 2$.

Sequences

1. (14AI4) The sequence $2, 3, 5, 6, 7, 8, 10, 11, \ldots$ is an enumeration of the positive integers that are not perfect squares. What is the 150th term of this sequence?
2. (14QIII1) Let $\{a_n\}$ be a sequence such that the average of the first and second terms is 1, the average of the second and third terms is 2, the average of the third and fourth terms is 3, and so on. Find the average of the first and hundredth terms.
3. (11QII3) If $b_1 = \dfrac{1}{3}$ and $b_{n+1} = \dfrac{1 - b_n}{1 + b_n}$ for $n \ge 2$, find $b_{2010} - b_{2009}$.

4. (13AI8) Let $3x, 4y, 5z$ form a geometric sequence while $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ form an arithmetic sequence. Find the value of $\frac{x}{z} + \frac{z}{x}$.

5. (16AII2) The numbers from 1 to 36 are written in a counterclockwise spiral as follows:

$$\begin{array}{ccccc} \boxed{13} & 12 & 11 & 10 & 25 \\ 14 & \boxed{3} & 2 & 9 & 24 \\ 15 & 4 & \boxed{1} & 8 & 23 \\ 16 & 5 & 6 & \boxed{7} & 22 \\ 17 & 18 & 19 & 20 & \boxed{21} \end{array}$$

In the figure above, all the terms on the diagonal beginning from the upper left corner have been enclosed in a box, and these entries sum up to 45.

Suppose this spiral is continued all the way until 2015, leaving an incomplete square. Find the sum of all the terms on the diagonal beginning from the upper left corner of the resulting (incomplete) square.

6. (14NE10) In the sequence $\{a_n\}, a_1 = 1, a_{n+1} = \dfrac{a_n}{1 + ca_n}$ for some constant c. If $a_{12} = \dfrac{1}{2014}$, find c.

7. (9N1) In the sequence $\{a_n\}, n(n+1)a_{n+1} + (n-2)a_{n-1} = n(n-1)a_n$ for every positive integer n, where $a_0 = a_1 = 1$. Calculate the sum $\dfrac{a_0}{a_1} + \dfrac{a_1}{a_2} + \cdots + \dfrac{a_{2008}}{a_{2009}}$.

Series

1. (10QIII4) A sequence of consecutive positive integers beginning with 1 is written on the blackboard. A student came along and erased one number. The average of the remaining numbers is $35\frac{7}{17}$. What number was erased?

2. (14NA3) Let $n \leq m$ be positive integers such that the first n numbers in $\{1, 2, 3, \ldots, m\}$ and the last $m - n$ numbers in the same sequence have the same sum 3570. Find m.

3. (14QII7) If the sum of the infinite geometric series $\dfrac{a}{b} + \dfrac{a}{b^2} + \dfrac{a}{b^3} + \cdots$ is 4, then what is the sum of $\dfrac{a}{a+b} + \dfrac{a}{(a+b)^2} + \dfrac{a}{(a+b)^3} + \cdots$?

4. (13NA4) Find the value of the infinite sum $1 + 1 + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^4 + \cdots$

5. (11NE8) Evaluate $\dfrac{1}{3} + \dfrac{1}{15} + \dfrac{1}{35} + \dfrac{1}{63} + \dfrac{1}{99} + \dfrac{1}{143} + \dfrac{1}{195}$.

6. (16AI10) Find the largest number N so that $\displaystyle\sum_{n=5}^{N} \frac{1}{n(n-2)} < \frac{1}{4}$.

7. (16QII10) Find the sum of $\displaystyle\sum_{i=1}^{2015} \left\lfloor \frac{\sqrt{i}}{10} \right\rfloor$.

8. (13AI7) Define $f(x) = \dfrac{a^x}{a^x + \sqrt{a}}$ for any $a > 0$. Evaluate $\displaystyle\sum_{i=1}^{2012} f\left(\frac{i}{2013}\right)$.

9. (11NA6) Find the sum $\displaystyle\sum_{k=1}^{19} k\binom{19}{k}$.

Inequalities

1. (13AI18) Find all k so that the inequality $k(x^2+6x-k)(x^2+x-12)>0$ has solution set $(-4,3)$.
2. (16NE14) Find the smallest k such that for all real x, y, z, $(x^2+y^2+z^2)^2 \le k(x^4+y^4+z^4)$.
3. (14NE14) Find the greatest k such that for any positive real a_1, a_2, a_3, a_4, a_5, with sum S, we have: $(S-a_1)(S-a_2)(S-a_3)(S-a_4)(S-a_5) \ge k(a_1a_2a_3a_4a_5)$.
4. (12N3) If $ab>0$ and $0<x<\frac{\pi}{2}$, prove that
$$\left(1+\frac{a^2}{\sin x}\right)+\left(1+\frac{b^2}{\cos x}\right) \ge \frac{(1+\sqrt{2}ab)^2\sin 2x}{2}.$$
5. (9N4) Let k be a positive integer such that $\frac{1}{k+a}+\frac{1}{k+b}+\frac{1}{k+c} \le 1$ for any positive real numbers a, b, c with $abc=1$. Find the minimum value of k.

Single-variable extrema

1. (13QIII2) Find the maximum of $y=(7-x)^4(2+x)^5$ when x lies strictly between -2 and 7.
2. (14NE7) Find the maximum of $4x-x^4-1$.
3. (14NA10) Find the maximum of $\sqrt{(x-4)^2+(x^3-2)^2}-\sqrt{(x-2)^2+(x^3+4)^2}$.
4. (16NA2) Find the minimum of x^2+4y^2-2x, where x, y are reals that satisfy $2x+8y=3$.
5. (13ND4) Find the minimum of $a^6+a^4-a^3-a+1$.

Multi-variable extrema

1. (10QI7) If $|2x-3| \le 5$ and $|5-2y| \le 3$ find the minimum of $x-y$.
2. (15AI16) Find the maximum of $\sum_{i=1}^{2014}(\sin\theta_i)(\cos\theta_{i+1})$, where $\theta_1=\theta_{2015}$.
3. (8ND2) If a and b are positive real numbers, what is the minimum of $\sqrt{a+b}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)$?
4. (14QIII5) Find the minimum of $2a^8+2b^6+a^4-b^3-2a^2-2$, where a and b are real numbers.
5. (10NA5) Let $x, y \in \mathbb{R}^+$ such that $x+2y=8$. Determine the minimum value of $x+y+\frac{3}{x}+\frac{9}{2y}$.
6. (15AI19) Find the maximum of $(1-x)(2-y)(3-z)\left(x+\frac{y}{2}+\frac{z}{3}\right)$ where $x<1, y<2, z<3$, and $x+\frac{y}{2}+\frac{z}{3}>0$.
7. (16AI18) Given $f(1-x)+(1-x)f(x)=5$ for all real numbers x, find the maximum of $f(x)$.
8. (16ND2) Suppose $\frac{1}{2} \le x \le 2$ and $\frac{4}{3} \le y \le \frac{3}{2}$. Determine the minimum of $\frac{x^3y^3}{x^6+3x^4y^2+3x^3y^3+3x^2y^4+y^6}$.
9. (13N5) Let r and s be positive real numbers that satisfy $(r+s-rs)(r+s+rs)=rs$. Find the minimum of $r+s-rs$ and $r+s+rs$.