VCSMS PRIME

Session 8: Algebra 3
compiled by Carl Joshua Quines
October 14, 2016

Manipulation

1. The first equation is $\dfrac{x^2+y^2}{xy} = \dfrac{(x+y)^2-2xy}{xy} = 4$, giving $(x+y)^2 = 18$. Then $xy(x+y)^2 - 2(xy)^2 = 3 \cdot 18 - 2 \cdot 3^2 = 36$.

2. The required expression is $2\left(x^2+y^2+z^2+xy+yz+zx\right)$. Squaring the first equation and transposing yz gives $x^2+yz = 2013$, similarly, $y^2+zx = 2014$ and $z^2+xy = 2015$. Addding all expressions and multiplying by 2 gives the answer, 12084.

3. Substitute $2n \to k$ to get $m^3 - 3mk^2 = 40$ and $k^3 - 3m^2k = 20$, we are looking for m^2+k^2. It reminds us of the triple angle formulas for sine and cosine, so substitute $m = r\cos\theta$ and $k = r\sin\theta$, now we are looking for r^2. The equations become $r^3\left(cos^3\theta - 3\sin^2\theta\cos\theta\right) = 40$ and $r^3\left(sin^3\theta - 3\sin\theta\cos^2\theta\right) = 20$. It is a good idea to write each in terms of only one trigonometric function: substituting the Pythagorean identity shows us that the first equation is actually $r^3\left(4\cos^3\theta - 3\cos\theta\right) = r^3\cos 3\theta = 40$. Similarly, the second equation is $r^3\sin 3\theta = 20$. Squaring both equations and adding gives $r^6 = 2000$, from whence $r^2 = \sqrt[3]{2000} = 10\sqrt[3]{2}$.

   More motivated but more high-powered: after substituting, notice $m^3 - 3mk^2 = 40$ and $k^3 - 3m^2k = 20$ look like the expressions from $(m-k)^3$, except the middle terms. We can fix this by making it $(m-ki)^3$; multiply the second equation by i and add to the first to get $m^3 - 3m^2ki - 3mk^2 + k^3i = 40 + 20i = (m-ki)^3$. Taking the modulus of both sides and using de Moivre's gives $|m-ki|^3 = \sqrt{40^2+20^2}$, so $m^2+k^2 = |m-ki|^2 = 10\sqrt[3]{2}$.

4. Abuse degrees of freedom by setting $x = y$. The condition is $x^2+2x-1 = 0$, and the expression needed is $x^2 + \dfrac{1}{x^2} - 2$. From the condition, $x^2 = 1-2x$ and dividing both sides of the condition by x^2, $\dfrac{1}{x^2} = 1 + \dfrac{2}{x}$, so the expression is now $\dfrac{2}{x} - 2x = 2\left(\dfrac{1}{x} - x\right) = 2\left(\dfrac{1-x^2}{x}\right)$. But from the condition, $1 - x^2 = 2x$, so $2\left(\dfrac{1-x^2}{x}\right) = 4$.

   (The legit solution is to clear denominators, factor the numerator, expand to get it as $(xy+x+y+1)(xy-x-y+1)$. The first term is 2, the second term, when divided by xy, is the condition divided by xy.)

5. Cross-multiply the condition and divide both sides by a to get $a + \dfrac{1}{a} = 3$. Divide both numerator and denominator of the expression by a^3; the numerator becomes 1 and the denominator becomes $\left(a^3+\dfrac{1}{a^3}\right) + \left(a^2+\dfrac{1}{a^2}\right) + \left(a+\dfrac{1}{a}\right) + 1$. But from $a + \dfrac{1}{a} = 3$, we get $a^2 + \dfrac{1}{a^2} = 7$ after squaring both sides, and $a^3 + \dfrac{1}{a^3} = 18$ after cubing and subtracting the original expression. The denominator is thus $18+7+3+1 = 29$, so the fraction is $\dfrac{1}{29}$.

6. Dividing both sides by 4 gives $\dfrac{1}{4} + \dfrac{1}{16} + \dfrac{1}{36} + \cdots = \dfrac{\pi^2}{24}$. Subtracting from the original equation gives $1 + \dfrac{1}{9} + \dfrac{1}{25} + \cdots = \dfrac{\pi^2}{8}$.

7. Squaring both sides and subtracting 2 gives $x^2 + x^{-2} = 7$. Repeating gives $x^{2^2} + x^{-2^2} = 47$, etc. The last two digits are $3, 7, 47, 7, 47, \ldots$. The pattern repeats, so the last two digits are 07.

8. Multiply the equations by a, b, c respectively, and subtract pairwise and transpose to get $(a+bc)x = (b+ca)y = (c+ab)z$. The required ratio is $\left(\frac{x}{y}-1\right)\left(\frac{y}{z}-1\right)\left(\frac{z}{x}-1\right)$, to get these we divide the equations with each other and simplify: $\dfrac{(a-1)(b-1)(c-1)(a-b)(b-c)(c-a)}{(a+bc)(b+ca)(c+ab)}$.

Surds

1. Multiplying numerator and denominator by $\sqrt[3]{8}-\sqrt[3]{2}$ and using the difference of two cubes, then cancelling out the factor 6, leaves $2-\sqrt[3]{2}$.
2. Expanding the right-hand side gives $2a^2+3b^2+c^2+2ac\sqrt{2}+2bc\sqrt{3}+2ab\sqrt{6}$. Equating coefficients gives $ac=-2, bc=-3, ab=6$. Multiplying all equations and taking the square root gives $abc=6$, from whence $a=-2, b=-3, c=1$ upon division by the three equations. Then $a^2+b^2+c^2=14$.
3. Squaring both sides gives $2x+2\sqrt{x^2-3x-6}=36$, or $\sqrt{x^2-3x-6}=18-x$. Squaring both sides again gives $x^2-3x-6=x^2-36x+324$, whence $x=10$.
4. Cubing both sides and using the binomial theorem, the terms which would end up with $\sqrt{5}$ in the expansion would have odd exponent for $\sqrt{5}$. If this were negative, then it would multiply out – so the value must be $12-\sqrt{5}$.
5. Note that $a=4+\sqrt{15}$ and $b=4-\sqrt{15}$ after rationalizing denominators. Then $a+b=8$ and $ab=1$. However, $a^4+b^4=\left(a^2+b^2\right)^2-2\left(ab\right)^2=\left((a+b)^2-2ab\right)^2-2\left(ab\right)^2$. Substituting everything yields 7938.
6. Observe $2=\left(1+\sqrt[n]{2}-1\right)^n \geq 1+\binom{n}{2}\left(\sqrt[n]{2}-1\right)$ by the binomial theorem. The inequality follows.

Sequences

1. If there were perfect squares, the 150th term would be 150; except we skipped 12 terms, so it should be 162.
2. Abuse degrees of freedom: one such sequence is $0, 2, 2, 4, 4, \ldots, 98, 98, 100$, so the average of the first and hundredth terms is 50.

   The legit method is to write $a_1+a_2=2,\ a_2+a_3=4,\ \ldots,\ a_{99}+a_{100}=198$. Take the sum of the odd-numbered equations to find $a_1+a_2+\cdots+a_{100}$ and the sum of the even-numbered equations to find $a_2+a_3+\cdots+a_{99}$; taking their difference yields $a_1+a_{100}=100$, so the average is 50.
3. Add 1 to both sides of the recursion to get $b_{n+1}+1=\dfrac{2}{1+b_n}$, or $\left(b_n+1\right)\left(b_{n+1}+1\right)=2$. So the terms alternate $\frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \ldots$, so $b_{2010}-b_{2009}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$.
4. From the geometric sequence, $16y^2=15xz$ and $\frac{2}{y}=\frac{1}{x}+\frac{1}{z}$, or $\frac{2}{y}=\frac{x+z}{xz}$. Substituting the first equation gives $\frac{32}{15}y=x+z$. The desired expression is $\dfrac{x^2+z^2}{xz}=\dfrac{(x+z)^2-2xz}{xz}=\dfrac{(x+z)^2}{xz}-2$. Substituting the previous values for xz and $x+z$ makes the y cancel, giving $\frac{34}{15}$.
5. It is clear that the terms in the sequence $1, 3, 7, 13, 21$ are quadratic. The method of differences or Newton interpolation yields the formula n^2-n+1, and continuing to 2015 means the sum is taken from $n=1$ to 45. The sum is then $\sum n^2-\sum n+45$, or 30405.
6. The condition is equivalent to $\dfrac{1}{a_{n+1}}=\dfrac{1}{a_n}+c$, so the reciprocals of the terms are arithmetic. With this in mind, $c=183$.

7. It can be easily proven, say, with induction, that $a_n = \dfrac{1}{n!}$. Or prove $a_{n-1}/a_n = n$ with induction. The required sum is $1 + 2 + \cdots + 2009 = 2019045$.

Series

1. There were $17n+1$ numbers on the board originally, making the original sum $602n$ plus whatever number was erased. Estimate $1+2+\cdots+17n+(17n+1) \geq 602n$ to get $n = 4$, the sum is $1+2+\cdots+69 = 2415$, and $602n = 2408$. The erased number was $2415 - 2408 = 7$.

2. Adding the first n and the last $m-n$ numbers gives the sum of the first m numbers being 7140. Solving $1 + 2 + \ldots + m = \dfrac{m(m+1)}{2} = 7140$ is to estimate $\sqrt{2 \times 7140} = \sqrt{14280} \approx 120$, checking, $m = 119$ works.

3. The sum of the first series is $\dfrac{\frac{a}{b}}{1 - \frac{1}{b}} = \dfrac{a}{b-1} = 4$, so $a = 4b-4$. The second series is $\dfrac{\frac{a}{a+b}}{1 - \frac{1}{a+b}} = \dfrac{a}{a+b-1}$. Substituting $a = 4b - 4$, factoring out $b - 1$, and cancelling gives its value as $\dfrac{5}{4}$.

4. Let the sum be S. Then $2S = 2+2+3\left(\frac{1}{2}\right)+4\left(\frac{1}{2}\right)^2+5\left(\frac{1}{2}\right)^3+\cdots$, and subtracting the original equation from it yields $S = 2 + (2-1) + \left(3\left(\frac{1}{2}\right) - 1\right) + \left(4\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right)^2\right) + \left(5\left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right)^3\right) + \cdots$, or $S = 2 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$. Then S is an infinite geometric series, with sum $S = \dfrac{2}{1 - \frac{1}{2}} = 4$.

5. This is $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \cdots + \dfrac{1}{13 \times 15}$, which telescopes as $\frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \frac{1}{2}\left(\frac{1}{13} - \frac{1}{15}\right)$. The sum is $\dfrac{7}{15}$.

6. The telescope is $\dfrac{1}{n(n-2)} = \dfrac{1}{2}\left(\dfrac{1}{n-2} - \dfrac{1}{n}\right)$. Multiply both sides of the sum by 2 and expand the two telescopes to get $\dfrac{1}{3} + \dfrac{1}{4} - \dfrac{1}{N-1} - \dfrac{1}{N} < \dfrac{1}{2}$, or $\dfrac{1}{N-1} + \dfrac{1}{N} > \dfrac{1}{12}$. The maximum that satisfies this is when $N = 24$.

7. From $i = 1$ to 99, the value is 0. From $i = 100$ to 399, the value is 1, so the subtotal is 300. From $i = 400$ to 899, the value is 2, the subtotal is 1000. From $i = 900$ to 1599, the value is 3, the subtotal is 2100. From $i = 1600$ to 2015, the value is 4, so the subtotal is 1664. The total sum is $300 + 1000 + 2100 + 1664 = 5064$.

8. Expand to prove $f(x) + f(1-x) = 1$, so pairing up terms in the series gives 1006.

9. Take the derivative of both sides of $(1+x)^{19} = \sum \binom{19}{k} x^k$ to get $19(1+x)^{18} = \sum k\binom{19}{k} x^{k-1}$. Substitute $x = 1$ to get $19 \cdot 2^{18}$.

   Alternatively, there is a combinatorial proof involving choosing a subset of 19 people and making choosing 1 to be the leader: either you pick the subset first and choose 1 then, giving the sum, or you pick one to be the leader first and each of the 18 others are either in the subset or not.

Inequalities

1. The inequality $x^2 + x - 12 > 0$ is $(x+4)(x-3) > 0$. For it to have solution set $(-4, 3)$, the sign should be reversed – so we must have $k(x^2 + 6x - k) < 0$ for all x. Then k should be negative and $x^2 + 6x - k$ should have negative discriminant, or $k < -9$. Thus $k \in (-\infty, 9]$ works.

2. By Cauchy–Schwarz, $(x^2 + y^2 + z^2)^2 \leq (1^2 + 1^2 + 1^2)(x^4 + y^4 + z^4)$, giving $k = 3$.

3. From AM-GM, $S - a_1 = a_2 + a_3 + a_4 + a_5 \geq 4\sqrt[4]{a_2a_3a_4a_5}$, taking the cyclic product gives $k = 4^5 = 1024$.

4. By Cauchy–Schwarz, $\left(1^2 + \left(\frac{a}{\sqrt{\sin x}}\right)^2\right) + \left(1^2 + \left(\frac{b}{\sqrt{\cos x}}\right)^2\right) \geq \left(1 + \left(\frac{ab}{\sqrt{\sin x \cos x}}\right)^2\right)$, and using the equality $\sin 2x = 2 \sin x \cos x$, the right-hand side can be manipulated to give the right-hand side of the inequality.

5. The inequality clearly does not hold when $k < 2$, for example, when $a = b = c = 1$. To show it is true for $k = 2$, it is equivalent to $(2 + a)(2 + b) + (2 + b)(2 + c) + (2 + c)(2 + a) \leq (2 + a)(2 + b)(2 + c)$ after clearing denominators. Expanding and cancelling many terms, then using $1 = abc$, gives $ab + bc + ca \geq 3$ which is true by AM-GM as follows: $ab + bc + ca \geq 3\left\{a^2b^2c^2\right\} = 3$. The steps are reversible.

Single-variable extrema

1. By AM-GM, since both terms are positive, $(7-x)^4(2+x)^5 \leq \left(\frac{(7-x) + \cdots + (7-x) + (2+x) + \cdots + (2+x)}{9}\right)^9$. The numerator simplifies to $38 + x$, and since we want equality, we let $7 - x = 2 + x$ or $x = 2.5$, making the maximum $(4.5)^9$.

2. We have $4x - x^4 - 1 = -(x^4 - 2x^2 + 1) - 2x^2 + 4x - 1 + 1 = -(x^2 - 1)^2 - 2(x^2 - 2x + 1) + 2 = -(x^2 - 1)^2 - 2(x - 1)^2 + 2 \leq 2$ by the trivial inequality, equality at $x = 1$. Thus the maximum is 2.

3. Let $A(4, 2), B(2, -4)$, and O be a point on $y = x^3$. We then wish to maximize $AO - BO$, which occurs when O lies on the line AB past either end, which does indeed intersect the graph of $y = x^3$. Then $AO - BO = AB$, and the distance is $2\sqrt{10}$.

4. By Cauchy–Schwarz, $(2(x-1) + 4(2y))^2 \leq (2^2+4^2)\left((x-1)^2 + 4y^2\right)$. The left-hand-side is $2x+8y-2 = 1$, so we get $x^2 + 4y^2 - 2x \geq -\frac{19}{20}$.

5. Scrapped.

Multi-variable extrema

1. x and y are independent, so we want to minimize x and maximize y. This happens when $x = -1$ and $y = 4$, whence $x - y = -5$.

2. Clearly we must want all the terms to be positive, by AM-GM the sum is at least $2014 \sqrt[2014]{\prod_{i=1}^{2014} \sin\theta_i \cos\theta_i} = 2014 \sqrt[2014]{\prod_{i=1}^{2014} \frac{1}{2}\sin 2\theta_i} \geq 2014 \sqrt[2014]{\prod_{i=1}^{2014} \frac{1}{2}} = 1007$, the last inequality from $\sin\theta \geq 1$. Equality is achievable when $\sin 2\theta_i = 1$, or when all the $\theta_i = 45°$, giving the maximum as 1007.

3. Distributing the product and the square root shows it is equivalent to $\sqrt{1 + \frac{b}{a}} + \sqrt{1 + \frac{a}{b}}$, which by AM-GM is at least $2\sqrt[4]{2 + \frac{b}{a} + \frac{a}{b}}$, and by AM-GM again is at least $2\sqrt[4]{2+2} = 2\sqrt{2}$.

4. This is $\left(2a^8 + a^4 - 2a^2\right) + \left(2b^6 - b^3 - 2\right)$, so it suffices to minimize each independently. This can be done through calculus, the legit way is slower. Take $u = a^2$ and the derivative, to get minimum as $-\frac{5}{8}$; the second is just a quadratic with vertex at $-\frac{17}{8}$. Their sum is $-\frac{11}{4}$.

5. The legit solution is to manipulate cleverly and use AM–GM. The cheating solution is to convert it to a single-variable problem by substituting $x = 8 - 2y$ and using calculus, the minimum is attained at $y = 3$, giving the value 8.

6. We factor out the 2 from $2-y$ and the 3 from $3-z$ to get $6(1-x)\left(1-\frac{y}{2}\right)\left(1-\frac{z}{3}\right)\left(x+\frac{y}{2}+\frac{z}{3}\right)$ which by AM-GM is at most $6\left(\frac{(1-x)+\left(1-\frac{y}{2}\right)+\left(1-\frac{z}{3}\right)+\left(x+\frac{y}{2}+\frac{z}{3}\right)}{4}\right)^4=\frac{3^5}{2^7}=\frac{243}{128}$.

7. Substituting $x\to 1-x$ gives a system of linear equations, from which $f(x)=\frac{5(x-1)}{x^2-x+1}=\frac{5}{(x-1)+1+\frac{1}{x-1}}$, and by AM-GM this is maximized when $x-1=\frac{1}{x-1}$ or $x=2$. Then $f(2)=\frac{5}{3}$.

8. The denominator is $(x^2+y^2)^3+3x^3y^3$, dividing numerator and denominator by x^3y^3 and simplifying makes the expression $\frac{1}{\left(\frac{x}{y}+\frac{y}{x}\right)^3+3}$. We need to maximize $\frac{x}{y}+\frac{y}{x}$, which occurs when $x=\frac{1}{2}$ and $y=\frac{3}{2}$, making the minimum $\frac{27}{1081}$.

9. Let $r+s=a$ and $rs=b$. The given is $(a-b)(a+b)=b$, so $b^2+b=a^2\geq 4b$ by AM-GM. Hence $b\geq 3$ and $a\geq 2\sqrt{3}$, which makes the minimum of $r+s-rs=a-b$ as $2\sqrt{3}-3$ and the minimum of $r+s+rs=a+b$ as $2\sqrt{3}+3$, which are achievable.