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Session 1: Algebra 1
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Domain and range

1. Notice that $x^2-4x+1=(x-2)^2-3$. The minimum is thus 2^{-3} and it is unbounded, the range is thus $[1/8,+\infty)$.

2. For the domain, $x^2-10x+29=(x-5)^2+4\geq 4$, thus there is no restriction for the square root. The denominator cannot be 0, thus the radical cannot be $2/5$, but this is impossible. The domain is $(-\infty,+\infty)$.

   From above, the radical can be anything in $[2,+\infty)$. The maximum is when the radical is 2, giving $3/4$. As the radical grows larger, it approaches 0. The range is $(0,3/4]$.

3. We have $25-x^2-y^2\geq 0, |x|-y\geq 0$. The first is a circle with radius 5, the second is an absolute value function. The intersection is a sector with angle $270°$, which has area $75\pi/4$.

4. $\lfloor x^2-x-2\rfloor$ will be 0 if $0\leq x^2-x-2<1$. Solving yields $(\dfrac{1-\sqrt{13}}{2},-1]\cup[2,\dfrac{1+\sqrt{13}}{2})$.

5. For f, as x approaches $-\infty$, 3^{-x} approaches $+\infty$ and the fraction approaches 2. As x approaches $+\infty, 3^{-x}$ approaches 0 and the fraction approaches $1/2$. The range of f is thus $(-\infty,1/2)\cup(2,\infty)$. Similarly the range of g is $(-3,4)$.

6. Solving for y yields $y=\dfrac{12e^x+3}{3e^x+1}$. By a similar argument as number $5, m=3$.

7. We have $f^4(x)>0, f^3(x)>1, f^2(x)>e, f(x)>e^e, x>e^{e^e}$. The domain is $(e^{e^e},+\infty)$.

8. When $x=a,b,c$, f is 1. Since the degree of f is at most 2, and we have three distinct values of f, by interpolating, $f(x)=1$. The range is $\{1\}$.

Logarithms

1. The sum is $1\times 3+\cdots+20\times 22$. This is equal to $(2^2-1)+\cdots+(21^2-1)$, which we can evaluate using the sum of squares formula as 3290.

2. Raising both sides to the base, we have $4=(x^2-3x)^2$. Thus $x^2-3x=+2,-2$. We see that the negative case is impossible after substituting in the original equation. Thus $x^2-3x=2$, which has two real roots.

3. We have $\left|\log_{\frac{1}{2}}|x|\right|-1=0$. Thus $\log_{\frac{1}{2}}|x|=\pm 1$, or $|x|=\frac{1}{2},2$. This has four real solutions, thus the graph crosses the x-axis four times.

4. After noting that $x>0$ from the $\log_{2014}x$ in the exponent, taking the base-x logarithm of both sides yields $\log_x\sqrt{2014}+\log_{2014}x=2014$. Substituting $u=\log_{2014}x$ and using the fact that $\log_x\sqrt{2014}=\dfrac{1}{2u}$, we see that $2u^2-4028u+1=0$. Suppose that the roots of this are $u_1=\log_{2014}x_1, u_2=\log_{2014}x_2$ and thus by Vieta's and the product rule for logarithms we have $u_1+u_2=2014=\log_{2014}(x_1x_2)$. The product of the roots x_1 and x_2 to the original equation is thus 2014^{2014} which has units digit 6.

5. Multiplying the three given equations yields $(xyz)^2=10^{a+b+c}$, taking the logarithms of both sides yields $\log x+\log y+\log z=\dfrac{a+b+c}{2}$.

6. Note that $a=\log_{14}16=4\log_{14}2$. Thus $\log_{14}2=a/4$. Thus $\log_8 14=\dfrac{1}{\log_{14}8}=\dfrac{1}{3\log_{14}2}=\dfrac{4}{3a}$.

Exponents

1. a) Note that $4^3 = 2^6$. Equating exponents, $2^x = 6$, and thus $x = \log_2 6$.

   b) We see that $x = 1$ is a solution. Equating exponents yields $x = 2$. Thus $x = 1, 2$.

   c) Equating exponents, $x^x = x^2$. From b, we have $x = 1, 2$. Thus $x = 1, 2$.

   d) Again, we see that $x = 1$ is a solution. Equating exponents yields $x = \pm \sqrt[2010]{2010}$. Thus $x = 1, \pm \sqrt[2010]{2010}$.

2. Taking hundredth roots yields $n^3 > 3^5 = 243$. The smallest integral n that satisfies this is 7.

3. First, compare 11^{16} and $25^{12} = 5^{24}$ by taking the eighth root, reducing the comparison to 11^2 and 5^3. It is clear that the former is lesser. Compare $25^{12} = 5^{24}$ and $16^{14} = 2^{56}$ by taking the eighth root, reducing the comparison to 5^3 and 2^7. It is clear that the former is lesser. From least to greatest, we have $11^{16}, 25^{12}, 16^{14}$.

4. We factor the LHS as $(9^{2x-1})(9-1) = 8\sqrt{3}$, by equating exponents, we have $2x - 1 = \frac{1}{2}$. Thus $(2x-1)^{2x} = \sqrt{2}/8$.

More logarithms

1. We see that $2^3 < 3^2$, thus $2 < 3^{2/3}, \log_3 2 < 2/3$. Since $625^2 < 75^3, 625^{2/3} < 75, 2/3 < \log_{625} 75$. Finally, we see that $\log_{625} 75 = \dfrac{\log_5 75}{4} < \log_5 3$. Thus from least to greatest, we have $\log_3 2, 2/3$, $\log_{625} 75, \log_5 3$.

2. After solving, we see $x = 1/2$. The infinite geometric series evaluates to 2.

3. Simplifying, we see that this is equivalent to $1 - \log_a b + 1 - \log_b a$. The minimum value of $\log_a b + \log_b a$ is 2 by AM-GM, thus the maximum value of the expression is 0.

4. Simplifying, we see $5^k 2^m = 400^n = (5^2 2^4)^n$. We have $k = 2n, m = 4n$. Since the greatest common divisor must be 1, we have $n = 1, k = 2, m = 4, k + m + n = 7$.

5. After trial and error, we find $m = 5$ works.

6. Let $u = 5^{\frac{1}{2x}}$. Simplifying, we have $u^2 + 125 < 30u$ which factors into $(u-5)(u-25) < 0$, thus $u \in (5, 25)$ and $x \in (1/4, 1/2)$.

7. We have $x \geq 2(x-1)$, thus $x \leq 2$. But from the argument of $\log(x-1)$ we have $x > 1$. Combining, we see all $x \in (1, 2]$ work.

Floor, ceiling, fractional

1. The equation is $2\lfloor x \rfloor = \lfloor x \rfloor + \{x\} + 2\{x\}$, which is $\lfloor x \rfloor = 3\{x\}$. As $\{x\} \in [0, 1)$, the only values for which $3\{x\}$ is an integer is $\{x\} \in \{0, 1/3, 2/3\}$. These give solutions $x = 0, 4/3, 8/3$.

2. Note that x must be nonnegative. We do casework on $\lfloor x \rfloor$. When $\lfloor x \rfloor = 0$, clearly $x = 0$. When $\lfloor x \rfloor = 1$ then $2x(x-1) = 1$, which has solution $\dfrac{1+\sqrt{3}}{2}$. When $\lfloor x \rfloor = 2$, then $2x(x-2) = 4$, which has solution $1 + \sqrt{3}$. If $\lfloor x \rfloor \geq 3$, then examining the discriminant reveals there is no solution. Thus $x = 0, \dfrac{1+\sqrt{3}}{2}, 1 + \sqrt{3}$.

3. In the interval $(1/4^2, 1/4]$, y is 1, its length is $1/4 - 1/4^2$. In the interval $(1/4^4, 1/4^3]$, y is 3, its length is $1/4^3 - 1/4^4$. Continuing the pattern, the desired sum is $1/4 - 1/4^2 + 1/4^3 - 1/4^4 + \cdots$, an infinite geometric series with sum $1/5$.

Value-finding

1. Letting $x = 0$, we see $f(0) = 2$. Similarly, we see $f(7) = 383$. The difference is 381.

2. We set $f(a) = 1$ and subtract $f(1)$ on both sides. We see that $f(b)^2 = 1$ for all b. Thus $f(1) - f(-1)$ can be anything in $\{-2, 0, 2\}$.

3. We substitute $x = 0$ and $x = 3$ to get the system of equations $2f(0)-2f(3) = -18, -f(3)-2f(0) = -30$. Solving, we get $f(0) = 7$.

Cauchy functional equation

Note: if we have $f(x + y) = f(x) + f(y)$, the solution from $\mathbb{Q} \to \mathbb{R}$ is $f(x) = kx$. Similarly, the solution to $f(x + y) = f(x)f(y)$ is $f(x) = k^x$ and the solution to $f(xy) = f(x) + f(y)$ is $f(x) = \log_k x$.

1. Letting $y = 0$ in the second equation and cancelling $f(0)$ on both sides yields $f(x) = 0$ for all x. Thus $f(\pi^{2013}) = 0$.

2. As per the note, the solution is $f(x) = kx$. We see that $k = 3/2$ and thus $f(2009) = 3013.5$.

3. As per the note, the solution is $f(x) = k^x$. We see that $k = 5$ and $3f(-2) = 3/25$.

Other functional equations

1. Letting $x = y = 0$ gives $f(0) = 1/2009$. Letting $x = y$ gives $f(x) = \pm 1/2009$. The negative case fails, thus $f(\sqrt{2009}) = 1/2009$.

2. Let $x = 0$ to get $f(-1) = f(y) - 2y - 2$. Let $y = 0$ to get $f(-1) = -1$. Equating gives us $f(y) = 2y + 1$ for all y.

3. Let $y = 0$ to get $f(0) = 0$. Let $x = 0$ to get f is odd. Switch x and y and equate to the original, use $f(y - x) = -f(x - y)$; rearrange to get
$$f(x + y)/(x + y) = f(x - y)/(x - y).$$
Thus $f(a)/a$ is a constant k for all a, and $f(a) = ka$. We have $k = 3/5$ and thus $f(2015) = 1209$.

4. Let $g(x) = (x + 2009)/(x - 1)$. The given is $x + f(x) + 2f(g(x)) = 2010$. Replace x with $g(x)$ to get $g(x) + f(g(x)) + 2f(x) = 2010$. Solving, $f(x) = \dfrac{x^2 + 2007x - 6028}{3x - 3}$.

5. Let $f(0) = a$, set $x = 0$ to get $f(a) = 1$. Set $x = a$ to get $f(1) = 1 - a$, set $x = 1$ to get $f(1 - a) = a$. Set $x = 1 - a$ to get $f(a) = 1 - a + a^2$. We get either $a = 0, 1$, either of which make a contradiction. Thus no f exists.

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Session 2: Trigonometry
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Circular functions

1. As $\cos x = -\cos(180^\circ - x)$, the sum is 0.
2. Rearranging, $x/y = 5/3 = \tan\theta$. Thus $\sin\theta = 5/\sqrt{34}$.
3. The line is the terminal side of an angle θ. Note that $\tan\theta = \tan 75^\circ$, so the angle is 75°. The tangent line to the unit circle makes an angle of 165° with the origin, so its slope is $\tan 165^\circ = -2 + \sqrt{3}$.
4. We let $x = 1$ to get the sum of the coefficients as $\cos(2\cos^{-1}(0)) = -1$.

Identities

1. The half-angle identity gives $\cos 15^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.
2. We wish to evaluate $\log_2 \sin(\pi/8)\cos(15\pi/8)$. By the product-to-sum identity, this is $\log_2(1/2)(\sin(2\pi) + \sin(7\pi/4)) = -3/2$.
3. We use the fact that $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ to get $\tan x \tan y = \dfrac{1}{2}$. Then $\cot^2 x + \cot^2 y = \dfrac{(\tan x + \tan y)^2 - 2\tan x \tan y}{\tan^2 x \tan^2 y} = 96$.
4. Note that $\cot(37^\circ + 8^\circ) = \dfrac{\cot 37^\circ \cot 8^\circ - 1}{\cot 37^\circ + \cot 8^\circ} = 1$, so $\cot 37^\circ \cot 8^\circ - 1 = \cot 37^\circ + \cot 8^\circ$. This rearranges to $(1 - \cot 37^\circ)(1 - \cot 8^\circ) = 2$.
5. We see $\cot(\cot^{-1} 2 + \cot^{-1} 3) = \dfrac{2 \cdot 3 - 1}{2 + 3} = 1$. Similarly, $\cot(\cot^{-1} 4 + \cot^{-1} 5) = 19/9$. Finally, $\cot(\cot^{-1} 1 + \cot^{-1} 19/9) = 5/14$.
6. Note that $\tan\theta^\circ \cos 1^\circ + \sin 1^\circ = \dfrac{\sin\theta^\circ \cos 1^\circ + \sin 1^\circ \cos\theta^\circ}{\cos\theta^\circ} = \dfrac{\sin(\theta^\circ + 1^\circ)}{\cos\theta^\circ}$. The product telescopes using cofunctions and the result is $\dfrac{1}{\sin 1^\circ} = \csc 1^\circ$.
7. Interpret this with the unit circle: there is a right triangle with legs of length $\sec\alpha$ and $\csc\alpha$, and its hypotenuse is $\tan\alpha + \cot\alpha$. The area of the triangle is equal to half the product of its legs, or $\frac{1}{2}\sec\alpha\csc\alpha$. It is also equal to half the product of the hypotenuse and the altitude to the hypotenuse, or $\frac{1}{2}(\tan\alpha + \cot\alpha)$. The answer is $\sqrt{14}$.

Equations

1. (The equation holds for all x.) By phase shift, $2\sin 3x = 2\cos\left(3x - \dfrac{\pi}{2} + 2k\pi\right) = -2\cos\left(3x + \dfrac{\pi}{2} + 2k\pi\right)$ for some $k \in \mathbb{Z}$. The product ac in both cases is $(4k-1)\pi$.
2. Square both sides to yield $1 - 2\sin 2\theta \cos 2\theta = 1 - \sin 4\theta = 3/2$, giving $\sin 4\theta = -1/2$. Since $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, it follows $4\theta \in (-2\pi, 2\pi)$. In this interval, $\sin 4\theta$ becomes $-1/2$ four times, so the equation has four solutions.
3. Square both sides and substitute $\cos^2\theta = 1 - \sin^2\theta$ to yield $5\sin^2\theta + 2\sin\theta - 3 = (5\sin\theta - 3)(\sin\theta + 1) = 0$. Either $\sin\theta = 3/5$ or $\sin\theta = -1$, but we can eliminate the latter as $0 < \theta < \pi/2$. Thus $\sin\theta = 3/5$.

4. Substituting $\sec^2 x = \tan^2 x + 1$ and simplifying gives the quadratic equation $\tan^2 x + 6\tan x - 16 = (\tan x + 8)(\tan x - 2) = 0$, thus $x \in \{\tan^{-1} 2 \pm k\pi, \tan^{-1}(-8) \pm k\pi | k \in \mathbb{Z}\}$.

5. Transpose $\dfrac{1}{\cos x}$ and square both sides. Substitute $\sin^2 x = 1 - \cos^2 x$ and then $\cos x = u$ to get the equation $\dfrac{3}{1-u^2} = 16 + \dfrac{1}{u^2} - \dfrac{8}{u}$. Clear the denominators to get $16u^4 - 8u^3 - 12u^2 + 8u - 1 = 0$.

   By inspection, $u = \dfrac{1}{2}$ works; dividing through gives $8u^3 - 6u + 1 = 0$. This reminds one of the triple angle formula $\cos 3x = 4\cos^3 x - 3\cos x$. We rewrite the equation as $4u^3 - 3u = -\dfrac{1}{2} = \cos 3x$. Keeping in mind $x \in (-\pi/2, 0)$, we let $3x = -\dfrac{4\pi}{3}$ and get $x = -\dfrac{4\pi}{9}$.

6. Transpose the first term of the left hand side, use the double angle formulae, and then use cofunctions to get $\cos(2x + b) = \sin(2ax - \pi) = \cos(3\pi/2 - 2ax)$. We can see that there are two cases: when $a = 1$ and $b = \pi/2 + 2k\pi, k \in \mathbb{Z}$, or when $a = -1$ and $b = 3\pi/2 + 2k\pi, k \in \mathbb{Z}$.

7. Substitute $\cot\alpha = \dfrac{1}{\tan\alpha}$ and simplify to get $\tan\beta = \dfrac{1-\tan\alpha}{1+\tan\alpha}$. Cross-multiply and rearrange the terms to get $\tan\alpha + \tan\beta = 1 - \tan\alpha\tan\beta$, which is $\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tan(\alpha+\beta) = 1$, so $\alpha + \beta = \pi/4$.

8. Note $\cos 8\theta = 2\cos^2 4\theta - 1$, so $\dfrac{1}{2} + \dfrac{1}{2}\cos 8\theta = \cos^2 4\theta$. Taking the positive root and repeating gives $\cos\theta$. Thus $\cos 4\theta, \cos 2\theta$ and $\cos\theta$ must all be at least 0. This is when $\theta \in \left[0, \dfrac{\pi}{8}\right] \cup \left[\dfrac{15\pi}{8}, 2\pi\right]$.

Triangle laws

1. This is a $45° - 45° - 90°$ triangle, thus $\angle ACD = 60°$ and $\angle CDA = 75°$. By the sine law, $\dfrac{CD}{\sin 45°} = \dfrac{AC}{\sin 75°}$, so $CD = \sqrt{3} - 1$. The altitude of ADC with respect to the base AC has length $CD \sin 60° = \dfrac{1}{2}(3 - \sqrt{3})$, thus the area is $\dfrac{1}{4}(3 - \sqrt{3})$.

2. There is a solution with the sine law, but the synthetic solution involves letting D be the foot of the altitude from C to AB, making ADC a $30° - 60° - 90°$ triangle and BCD a $45° - 45° - 90°$ triangle. AD has length $\dfrac{\sqrt{2}}{2}$ and CD and BD both have length $\dfrac{\sqrt{6}}{2}$. The area is then $\dfrac{3+\sqrt{3}}{2}$.

3. Let $BM = MC = x$. By Apollonius', $AC = \sqrt{2x^2 - 14}$. We use the cosine law to get $\cos\angle BAC = \dfrac{4^2 - \left(\sqrt{2x^2-14}\right)^2 - (2x)^2}{2 \cdot 4\sqrt{2x^2-14}} = \dfrac{1-x^2}{4\sqrt{2x^2-14}}$. We want to maximize this, and upon seeing the numerator being negative, we are inspired to take the negative and minimize using AM-GM. Then $\cos\angle BAC = -\dfrac{1}{4\sqrt{2}}\left(\dfrac{x^2-1}{\sqrt{x^2-7}}\right) = -\dfrac{1}{4\sqrt{2}}\left(\dfrac{x^2-7}{\sqrt{x^2-7}} + \dfrac{6}{\sqrt{x^2-7}}\right) = -\dfrac{1}{4\sqrt{2}}\left(\sqrt{x^2-7} + \dfrac{6}{\sqrt{x^2-7}}\right) \le -\dfrac{1}{4\sqrt{2}} \cdot 2\sqrt{6} = -\dfrac{\sqrt{3}}{2}$ by AM-GM. Thus $\angle BAC \ge 150°$.

4. By the cosine law, $\dfrac{a^2+b^2-c^2}{ab} = 2\cos\gamma$. Since $2\cos\gamma = 2\cos(\pi - \alpha - \beta) = -2\cos(\alpha+\beta)$, we can use the sum formula for cosine to get the answer as $\dfrac{32}{65}$.

5. There is a straightforward solution with the sine law, but we will proceed synthetically. Let A' be the point on the line AB that is not N such that $A'A = 6$. Then $AA' = AC = AN = 6$, thus A is the center of a circle with diameter $A'N$ containing point C, and $\angle A'CN = 90°$. Draw a line through N parallel

to CA' and let it intersect lines CM and CB at P and Q respectively. Since $\triangle A'MC \sim \triangle PMN$ and $\triangle A'BC \sim \triangle NBQ$, we have $PN = \frac{MN}{MA'} \cdot CA'$ and $QN = \frac{BN}{BA'} \cdot CA'$, and substituting the given shows that $PN = QN$, which implies $\triangle CNP \cong \triangle CNQ$, which implies $\angle MCN = \angle NCB$.

6. By the cosine law, $a^2 = b^2 + c^2 - bc$. Factoring, $b^3 + c^3 = (b+c)(b^2 + c^2 - bc) = (b+c)a^2$. Add a^3 to both sides and rearrange to get the desired equality.

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Session 3: Number theory
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Ad hoc

1. Each factor of 5 in 126! has a corresponding factor of 2 to produce a trailing zero, so we only need to count the number of factors of 5. It is well-known to be, by Legendre's formula, $\left\lfloor \frac{126}{5} \right\rfloor + \left\lfloor \frac{126}{25} \right\rfloor + \left\lfloor \frac{126}{125} \right\rfloor = 25 + 5 + 1 = 31$.

2. By Legendre's, $\left\lfloor \frac{27}{2} \right\rfloor + \left\lfloor \frac{27}{4} \right\rfloor + \left\lfloor \frac{27}{8} \right\rfloor + \left\lfloor \frac{27}{16} \right\rfloor = 13 + 6 + 3 + 1 = 23$.

3. The answer is 26. Selecting all 25 even numbers has no two relatively prime; by Pigeonhole, selecting 26 will guarantee two consecutive numbers are selected, which are relatively prime.

4. It is easy to verify the cases $n = 0, 1$ to not produce perfect squares. Suppose $n \geq 2$ and factor out 7^2 to produce $7^2\left(7^{n-2} + 9\right)$. Since the first factor is a perfect square, so should the second.

   Let $m^2 = 7^{n-2} + 9$, so $7^{n-2} = m^2 - 3^2 = (m-3)(m+3)$. Then both $m - 3$ and $m + 3$ are two powers of 7 differing by 6, and since the difference between consecutive powers of 7 increases, the only possible choice is $m = 4$, giving $n = 3$. The answer is 1.

5. Factoring out 2^8 gives $2^8\left(1 + 2^3\right) + 2^n = 2^n + 2^8 3^2$. Let $m^2 = 2^n + 2^8 3^2$, and transposing and using the difference of two squares gives $2^n = (m - 48)(m + 48)$. Then $m - 48$ and $m + 48$ are two powers of two that differ by 96, the only possible pair being 32 and 128, giving $n = 12$.

6. Since $abcde$ is divisible by 5, the only choice for e must be 5. There are only three even-numbered digits, and b, e, f must all be even, so they match to b, e, f in some order. This leaves 1 and 3 for a and c.

   Wishing to maximize, we try $a = 3$. Then $c = 1$, and the number so far is $3b1d5f$. The condition of ab being divisible by 2 is guaranteed, and so is the condition of $abcdef$ being divisible by 6; we are concerned about abc being divisible by 3 and $abcd$ being divisible by 4. The first forces $b = 2$ and the second forces $d = 6$, so the number is 321654.

7. The number N should be the largest power of 2 dividing 10!. By Legendre's formula, the largest power is $\left\lfloor \frac{10}{2} \right\rfloor + \left\lfloor \frac{10}{4} \right\rfloor + \left\lfloor \frac{10}{8} \right\rfloor = 5 + 2 + 1 = 8$, so $N = 2^8$. Thus $2x + y = 2^8$, and we maximize x^2y^2, or $\left(x(2^8 - 2x)\right)^2$. The base is a quadratic with vertex at $x = 2^6$, with value 2^{13}, and its square is thus 2^{26}

8. Since P is divisible by all prime numbers less than 90, for $P + n$ to have a prime factor less than 90, so must n. All $n < 90$ work for trivial reasons, and so do $90, \ldots, 96$, failing at $n = 97$ since it is a prime. Thus the largest N is 96.

Factors

1. The fifth largest divisor corresponds to the fifth smallest divisor upon division. $2{,}015{,}000{,}000 = 2015 \cdot 10^6 = 5 \cdot 13 \cdot 31 \cdot 2^6 \cdot 5^6$, and its smallest divisors are, in order, $1, 2, 4, 5, 8$. Dividing the number by 2^3 leaves $5 \cdot 13 \cdot 31 \cdot 2^3 \cdot 5^6 = 251{,}875{,}000$.

2. The even positive divisors of 1152 are precisely the positive divisors of $1152 \div 2 = 576$ times two, so it remains to find the sum of all its divisors. Since $576 = 2^6 3^2$, the well-known formula for the sum of divisors gives $\left(1 + 2 + \cdots + 2^6\right)\left(1 + 3 + 3^2\right) = \left(2^7 - 1\right)(13) = 1651$, multiplying by 2 gives 3302.

3. By the formula for the number of divisors, the number must either be a product of two primes or the cube of a prime. The first three numbers are $6, 8, 10$, and the fourth is 14.

4. The power of 5 in the LHS is 2, which means that the power of 5 in the RHS is 2 as well, so $y = 2$. Then power of 3 in the RHS is 2, so the power of 3 in the LHS, $2x$, should equal to 2. Thus $x = 1$.

5. The highest power of 7 less than one million is 7^7, so there are 8 factors smaller than a million. The rest of the 10,000 factors are larger, so there are 9992 such factors.

6. Factoring out 5^x gives $5^x(1 + 2 \cdot 5) = 5^x 11$. The number of factors formula gives $(x + 1)2 = 2x + 2$ factors.

7. Multiplying the two equations and taking the square root gives $p = 2^2 \cdot 5^3 \cdot 7^2 \cdot 11$, which has $(2 + 1)(3 + 1)(2 + 1)(1 + 1) = 72$ divisors.

8. For each factor of n^2 less than n, dividing through n^2 gives a corresponding factor greater than n. Thus the number of factors of n^2, minus one to account for n, divided by 2, gives the number of its factors less than n. Then we subtract the number of factors of n.

   In this case, $n^2 = 2^{62}3^{38}$ which has $(62 + 1)(38 + 1) = 2457$ factors, $\dfrac{2457 - 1}{2} = 1228$ of which are less than n. The number n itself has $(31 + 1)(19 + 1) = 640$ factors, so subtracting gives $1228 - 640 = 588$ factors.

9. The number is $300^3 + 1 = (300 + 1)(300^2 - 300 + 1)$. The former, 301, factors as $7 \cdot 43$. The latter factor is $300^2 - 300 + 1 = 300^2 + 600 + 1 - 900 = (300 + 1)^2 - 30^2 = (301 - 30)(301 + 30)$, and both 271 and 331 are prime. The sum is $7 + 43 + 271 + 331 = 652$.

10. Factor out 3^19 from the first two terms to leave $3^19\,(3 + 1) - 12$. Factor out 12 to leave $12\left(3^18 - 1\right)$, which factors by repeatedly using difference of two squares and cubes as $(3 - 1)(3^2 + 3 + 1)(3^6 + 3^3 + 1)(3 + 1)(3^2 - 3 + 1)(3^6 - 3^3 + 1)$. After tedious checking, the factorization is $2^5 \cdot 3 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 757$.

11. The number $360{,}000 = 2^6 \cdot 3^2 \cdot 5^4$ has $(6 + 1)(2 + 1)(4 + 1) = 105$ factors. Since the factors of 360,000 pair up, each of them multiplying to 360,000, and there being $\dfrac{105}{2}$ pairs, the product of all the factors is $(360{,}000)^{\frac{105}{2}}$. Expanding, $\left(2^6 \cdot 3^2 \cdot 5^4\right)^{\frac{105}{2}}$ has sum of exponents $\dfrac{105}{2}(6 + 2 + 4) = 630$.

12. Suppose $f(r) = 0$ for some integer r, and then $f(x) = (x - r)g(x)$ for some polynomial $g(x)$. Let the four integers be a, b, c, d. Substituting a gives $f(a) = p = (a - r)g(a)$, so $a - r$ is a factor of p. Similarly, $b - r, c - r, d - r$ are all factors of p. Since these are all distinct, they must be $-p, -1, 1, p$ in some order.

    Then, from above, $f(-p + r) = p = (-p)g(-p + r)$ implies $g(-p + r) = -1$; similarly, $f(p + r) = p = pg(p + r)$, so $g(p + r) = 1$. However, it is well-known that $a - b$ is a factor of $f(a) - f(b)$; applying this shows $(p + r) - (-p + r) = 2p$ is a factor of $1 - (-1) = 2$, which is impossible.

Divisibility

1. Dividing gives $\dfrac{n + 3}{n - 1} = 1 + \dfrac{4}{n - 1}$, so we must have $n - 1 | 4$. Since 4 has factors $-4, -2, -1, 1, 2, 4$, the number of possible values of n is the same, 6.

2. Dividing gives $2n^2 - n + 1\dfrac{31}{3n + 1}$. As 31 is a prime, $3n + 1$ must equal either $-31, -1, 1$ or 31, which happens only for integers $n = 0, 10$.

3. The greatest common factor of $7^4 - 1 = 2^5 \cdot 3 \cdot 5^2$ and $11^4 - 1 = 2^4 \cdot 3 \cdot 5 \cdot 61$ is $2^4 \cdot 3 \cdot 5$. We show that all $p^4 - 1$ are divisible by $2^4 \cdot 3 \cdot 5$. Note that $p^4 - 1 = (p^2 + 1)(p - 1)(p + 1)$.

   Since p is odd, $p^2 + 1$ is even, and $p - 1, p + 1$ are consecutive even integers, so their product is divisible by 8. When divided by 3, p gives a remainder of 1 or 2; in the former, $3|p - 1$, in the latter, $3|p + 1$. Similarly, it is always divisible by 5, as $5|p - 1$ and $5|p + 1$ when it has remainder 1 or 4, and $5|p^2 + 1$ otherwise. The greatest common factor is thus $2^4 \cdot 3 \cdot 5 = 240$.

4. Rationalizing the denominator gives $\dfrac{2013ab - bc + \left(b^2 - ac\right)\sqrt{2013}}{2013b^2 - c^2}$. For this to be rational, the irrational part must be zero, so $b^2 = ac$. Thus a, b, c are in geometric sequence. Rewrite a, b, c as a, ar, ar^2.

   Then $\dfrac{a^2 + b^2 + c^2}{a + b + c} = \dfrac{a^2 + a^2r^2 + a^2r^4}{a + ar + ar^2} = a(r^2 - r + 1)$ after long division. Similarly, $\dfrac{a^3 - 2b^3 + c^3}{a + b + c} = a^2(r^4 - r^3 - r + 1)$. These are both integers.

5. Multiply both sides by $x + y$ and transpose to obtain $xy - 1000x - 1000y = 0$. Add 1,000,000 to both sides and factor to get $(x - 1000)(y - 1000) = 1{,}000{,}000$. It is easy to rule out the case where both factors in the LHS are negative: they cannot both be -1000, and one must be smaller than -1000, meaning either x or y must be negative.

   Thus both are positive, and each factor of $1{,}000{,}000 = 2^6 \cdot 5^6$ corresponds to one positive integer pair. Since it has $6 + 1)(6 + 1) = 49$ factors, then there are 49 pairs.

6. It is well-known that all primes greater than 3 are either 1 or -1 modulo 6. Note that a number that is -1 modulo 6 cannot be divisible by 2 or 3. If none of its prime factors were -1 modulo 6, then all of its prime factors are 1, and their product would be 1 as well, contradiction. Therefore there must be a prime that is -1 modulo 6 that divides it.

   Suppose finitely many primes existed that are -1 modulo 6; multiplying them and adding either 4 or 6 (depending on number of primes) produces a new number that is also -1 modulo 6. This number must be composite, and by the above, divisible by a prime that is -1 modulo 6. But when divided by any such prime, it leaves a remainder of either 4 or 6, contradiction.

Diophantine equations

1. Since both $2x$ and 100 are even, so is $5y$, and thus y is even as well. Any even y produces an integer solution, the ones that give positive solutions are $y = 2, 4, \ldots, 18$. Thus there are 9 ordered pairs.

2. Since $2^{3x} + 5^{3y} = (2^x + 5^y)\, 2^{2x} - 2^x \cdot 5^y + 5^{2y} = 189$. The factors of 189 are $1 \cdot 189, 3 \cdot 63, 7 \cdot 27, 9 \cdot 21$. The only pair that works is $9 \cdot 21$, giving the only values $x = 2$, $y = 1$.

3. Adding twice the second equation to the first gives $5x = 56 - 3a$, and subtracting the second equation from twice the first gives $5y = 4a - 13$. Since $56 - 3a$ and $4a - 13$ are integers divisible by 5, their sum, $a + 43$, is divisible by 5, so a is an integer as well, and it is 2 modulo 5. Both $56 - 3a$ and $4a - 13$ have to be positive, so a is at least 4 and at most 18. The only integers in this range that are 2 modulo 5 are $7, 12, 17$.

4. This is $2xy - 2x + y = 43$ and subtracting 1 to both sides completes the rectangle, giving $(2x + 1)(y - 1) = 42$. Then $2x + 1$ is an odd factor of 42, so it is either $3, 7, 21$, giving $x = 1, 3, 10$, with corresponding $y = 15, 7, 3$. The largest $x + y$ is thus 16.

5. Adding 1 to both sides in each equation completes the rectangle, making $(a + 1)(b + 1) = 16, (b + 1)(c + 1) = 100$, and $(c + 1)(a + 1) = 400$. Taking the product of all equations and its square root gives $(a + 1)(b + 1)(c + 1) = 800$. Dividing with second equation gives $a + 1 = 8$, so $a = 7$. Similarly, $b = 1$ and $c = 49$.

6. Adding twice the first equation to the second gives $16x + 13y = 77$, which has only one nonnegative integer solution, $x = 4, y = 1$. Substituting to either equation gives $z = 2$.

7. Cheat: it must be constant. One such soution is $(3, -4)$, and $\lfloor y/x \rfloor = -1$. In fact, the rest of the solutions are $(3 - 4k, 7k - 4)$ for integral k, and indeed $\lfloor y/x \rfloor = -1$.

8. Dividing both sides by xyz gives $x^{y^z - 1} y^{z^x - 1} z^{x^y - 1} = 3$. One of x, y, z must be 3, so WLOG $x = 3$. Then $y^z - 1 = 1$, which only happens for $y = 2$ and $z = 1$, giving $(3, 2, 1)$, which works, and so does its cycles, giving 3 triples.

9. Note that $\frac{15}{2013} = \left(1 - \frac{1}{x_1}\right) \cdots \left(1 - \frac{1}{x_n}\right) \geq \left(1 - \frac{1}{2}\right) \cdots \left(1 - \frac{1}{n+1}\right) = \frac{1}{n+1}$, showing $n \geq 134$. To prove this is achievable, set $x_1, \ldots, x_{133}$ to $2, \ldots, 134$ and $x_{134} = 671$. This gives us the value $\left(1 - \frac{1}{2}\right) \cdots \left(1 - \frac{1}{134}\right)\left(1 - \frac{1}{671}\right) = \frac{1}{134} \cdots \frac{670}{671} = \frac{15}{2013}$. The minimum value is thus 134.

Modulo

1. The highest power of 5 dividing 16 is, by Legendre's, $\left\lfloor \frac{16}{5} \right\rfloor = 3$, so we take out 8 and three factors of 5 and compute modulo 100 the product $1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 6 \cdot 7 \cdot 1 \cdot 9 \cdot 2 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 3 \cdot 16 = 96$.

2. (The remainder when divided by 5 should be 4.) Since $n + 5 \equiv 3 \pmod 4$, $n \equiv 3 - 5 \equiv -2 \equiv 2 \pmod 4$. Similarly, $n \equiv 0 \pmod 5$. We check $5, 10, 15$ if any give a remainder of 2 when divided by 4, and 10 works. Then $10 + 6 \equiv 16 \pmod 20$, so the remainder is 16.

3. Note that $n = 1$ works, but we require it to be greater than one. By CRT, the solutions to any linear system of moduli differ by the LCM of the moduli. The LCM of $3, 4, 5, 6$ is 60, so the next solution is $1 + 60 = 61$.

4. Taking modulo 11, by Fermat's Little Theorem, we only need to consider the exponent modulo 10. However, $5! \equiv 0 \pmod 10$, so by Fermat's Little Theorem, $3!^{5!^{\cdots}} \equiv \left(3!^{10}\right)^{\cdots} \equiv 1^{\cdots} \equiv 1 \pmod{11}$. The remainder is 1.

5. Since $96 = 3 \cdot 32$, we take modulo 3 and modulo 32. Modulo 3 the expression is $1^{15} - (-1)^{15} - 1^{15} - (-1)^{15} - 1^{15} \equiv 1$. Modulo 32, everything evaporates except for $-1^{15} \equiv -1$. It is 1 modulo 3 and -1 modulo 32, combining both gives the expression as 31 modulo 96.

6. Since $7, 8, 9$ are relatively prime, $739ABC$ is divisible by 504. It is $739000 + ABC \equiv 136 + ABC \equiv 0 \pmod{504}$, giving only the choices $ABC = 368, 872$.

7. If $p \mid a^p$, then $p \mid a^p \mid a^q$. Suppose $p \mid a^q$ and $p \nmid a^p$, then there exists some prime power r^n such that $r^n \mid p$ and $r^n \nmid a^p$. Then $r^n \mid p \mid a^q$ so $r \mid a$, and $r^n \mid a^n$. However, since $r^n \nmid a^p$, then $p < n$. Then $r^p \mid r^n \mid p$, but this implies $r^p < p$, contradiction.

VCSMS PRIME

Session 4: Combinatorics 1

compiled by Carl Joshua Quines

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Ad hoc

1. The single-digit numbers account for $1+2+\cdots+9=45$ of the digits, the remaining $2015-45=1970$ digits are accounted for by two-digit numbers, which have two digits each. Thus we must have $2\left(10+11+\cdots+n\right)\leq 1970$, the maximum value is $n=44$. Thus the 1936th digit onwards is $454545\ldots$, so the 2015th is 5.

2. There are only four possible sets: $\{6,1,0\},\{5,2,0\},\{4,3,0\},\{4,2,1\}$. Multiplying by 3! to account for permutations gives 24.

3. A vertically arranged block is determined by its topmost letter, which can appear anywhere in the top 10×12 part of the array, so there are 120 of them. Similarly, the horizontal blocks are determined by its leftmost letter, anywhere among the left $10\times 12=120$ letters. Similarly for the diagonal blocks, but for two 10×10 blocks depending on its orientation. The total is $120+120+100+100=440$.

4. The cardinalities are $1,3,\ldots$, so we must have $1+3+\cdots+(2n-1)\leq 2009$, which has the largest possible value of $n=44$. Thus 2009 appears in A_{45}.

5. Each diagonal has 5 skew diagonals, and there are 12 diagonals. We divide by 2 for overcounting: $5\times 12\div 2=30$.

6. Each number appears in 2^15 subsets, depending on whether each of the other 15 numbers appear or no. Thus the sum is $2^15\left(1+2+\cdots+16\right)=4456448$.

7. The one-digit numbers take 9 digits and the two-digit numbers take 180 digits, so we stop at $\dfrac{2016-189}{3}+99=708$.

   From 0 to 99, in the ones place the sum is $10(0+1+\cdots+9)$ and in the tens place the sum is $10(0+1+\cdots+9)$, so the total sum is $20(45)=900$. From 100 to 699, there are 6 0 to 99s and 100 occurences of 0 to 6 in the hundreds place, so $100(0+1+\cdots+6)+6(900)=7500$. Then $700,\ldots,708$ have a sum of 99, so the total is $900+7500+99=8499$.

8. On the main diagonal is 0, then above and below there are two diagonals, each with $n-1$ 1s, above and below are two diagonals, each with $n-2$ 2s, etc. The summation is $\sum 2(n-i)i$ from $i=1$ to n, or $2\left(n\sum i-\sum i^2\right)=\dfrac{1}{3}n^3-\dfrac{1}{3}n$. Thus $n^3-n=7980$, and we observe that only $n=20$ works.

9. The first digit has to be 1, then onwards, the digits have to be either 0 or 9. The only choices are $1999,1099,1009,1000$, and the smallest is $\dfrac{1099}{19}$.

10. Burnside's, or bloody casework. We do casework on the number of white vertices. For 0 or 1 white vertices there is clearly one different way each. For 2 white vertices there are three ways: both connected by an edge, both on the same face but not adjacent, and on opposite vertices. For 3 white vertices there are four ways, all on the same face, all on opposite vertices, and two when two are connected. For 4 white vertices there are six ways: all on the same face, four where three share the same face, and one with two pairs opposite each other.

    The $5,6,7,8$ white vertices are analogous to there being $3,2,1,0$ black vertices, so there are the same number of ways. That makes a total of $1+1+3+4+6+4+3+1+1=24$.

Inclusion-Exclusion

1. Of the 100 people, 60 claim to be good, so $100 - 60 = 40$ deny to be good. Of these, 30 correctly deny, so the rest must be people who are good at math but refuse to admit it: $40 - 30 = 10$.

2. There are $\left\lfloor \frac{2015}{3} \right\rfloor = 671$ numbers less than 2015 divisible by 3. Of these, $\left\lfloor \frac{671}{5} \right\rfloor = 134$ are divisible by 5 and $\left\lfloor \frac{671}{7} \right\rfloor = 95$ are divisible by 7, with $\left\lfloor \frac{671}{35} \right\rfloor = 19$ divisible by 35. By PIE, the answer is $671 - 134 - 95 + 19 = 461$.

3. This is $\phi(10000) - 1 = 10000\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) - 1 = 3999$.

4. There are 638 numbers divisible by 3, 239 divisible by 8, 79 divisible by 24, and 319 divisible by 6 in the range $[100, 2015]$. The answer is $638 + 239 - 79 - 319 = 479$.

5. By PIE: $\left\lfloor \frac{999}{10} \right\rfloor + \left\lfloor \frac{999}{15} \right\rfloor + \left\lfloor \frac{999}{35} \right\rfloor + \left\lfloor \frac{999}{55} \right\rfloor - \left\lfloor \frac{999}{30} \right\rfloor - \left\lfloor \frac{999}{70} \right\rfloor - \left\lfloor \frac{999}{110} \right\rfloor - \left\lfloor \frac{999}{105} \right\rfloor - \left\lfloor \frac{999}{165} \right\rfloor - \left\lfloor \frac{999}{385} \right\rfloor + \left\lfloor \frac{999}{210} \right\rfloor + \left\lfloor \frac{999}{330} \right\rfloor + \left\lfloor \frac{999}{770} \right\rfloor = 146$.

Permutations

1. Casework: only 100 has a sum of 1 and 999 has a sum of 27. By balls and urns, there are $\binom{10}{8} = 45$ that sum to 8, except 9 of these start with 0. The sum is $1 + 1 + 45 - 9 = 38$.

2. There are $\frac{10!}{2!}$ permutations, since I is repeated. Except we overcount: we want to consider only AIIU out of its $\frac{4!}{2!} = 12$ possible permutations. So we divide by 12. This gives $\frac{10!}{12 \cdot 2!} = 151200$.

3. In a line, this is $2 \times 6! \times 6!$ – pick either girl or boy to go first and alternate, then multiply by the number of ways to permute per gender. Divide by 12 to account for rotation in a circle: $2 \times 6! \times 6! \div 12 = 86400$.

4. There are $\frac{7!}{2!} = 2520$ permutations since A is repeated. However, we want to consider only AEA out of its 3 permutations, so divide by 3 to get 840.

5. Since $33750 = 2 \cdot 3^3 \cdot 5^4$, we split to four cases: all prime numbers, which is $\frac{8!}{4!3!} = 280$, one of them is 6, which is $\frac{7!}{4!2!} = 105$, one of them is 9, which is $\frac{7!}{4!} = 210$, and when both 6 and 9 are present, $\frac{6!}{4!} = 30$. The sum is $280 + 105 + 210 + 30 = 625$.

6. There are 4! starting with A, 4! with M, 4! with R, a total of 72. Then 3! start with SA, a total of 78. The first starting with SM is SMART, so that must be the 79th.

7. There are $\frac{7!}{2!2!2!} = 630$ ways without restriction. There are $\frac{5!}{2!} = 60$ ways where PHI appears and also $\frac{5!}{2!} = 60$ ways where ILL appears. For strings with both PHI and ILL, it can be either as PHILL, I, P which is $3! = 6$ ways, or as PHI, ILL and P for another $3! = 6$ ways. By PIE, there are $630 - 60 - 60 + 6 + 6 = 522$ ways.

8. We use PIE. There are $\frac{6!}{2!2!2!} = 90$ ways to arrange MURMUR. By symmetry, when two Ms, Us, or Rs are together, there are $\frac{5!}{2!2!} = 30$ ways. Similarly, if two pairs of letters are together, there are $\frac{4!}{2!} = 12$ ways. Finally there are 3! ways when each pair is together. By PIE, there are $90 - 30 - 30 - 30 + 12 + 12 + 12 - 6 = 30$ ways.

Combinations

1. There are 2^n subsets, 1 with no elements and n with one element. Thus $2^n - n - 1 = 57$, which only has the solution $n = 6$.

2. One of the numbers chosen has to be 7, another has to be either $3, 6$ or 9, the last number can be anything. This gives $1 \cdot 3 \cdot 7 = 21$, but we overcounted when the last number is also divisible by 3, which happens 6 ways, when we only want to count it 3 times. So $21 - 6 + 3 = 18$.

3. Order the boys arbitrarily, then there are $6! = 720$ ways to arrange the girls to form pairs.

Balls and urns

1. Let Amy, Bob and Charlie receive a, b, c cookies respectively; we have $a+b+c = 15$, or $(a-4)+b+c = 11$. Now each of $a - 4, b, c$ are positive integers, so this is balls and urns. There are 10 slots in between and we pick 2 of them, so $\binom{10}{2} = 45$ ways.

2. Balls and urns directly: $\binom{12}{2} = 66$.

3. Consider $(x - 1000) + (y - 600) + (z - 400) = 16$, where each variable is now a positive integer, so by balls and urns there are $\binom{15}{2} = 105$ ways.

4. Consider the 6 integers that are left. You are placing four integers such that no two are adjacent, so you have to place them in-between, in front, or behind the 6 integers, giving 7 slots. Each slot can only go to one integer so no two are adjacent, giving $\binom{7}{4} = 35$ ways.

5. Consider the 7 books that are left. You are placing 5 books so that no two of them are adjacent, so you have to place them in-between, in front, or behind the 7 books, giving 8 slots. Each slot can only go to one book so no two are adjacent, so that's $\binom{8}{5} = 56$ ways.

6. Similar as above, there are $\binom{8}{5} = 56$ ways in a row. However, we overcount: we don't want to count the $\binom{6}{3} = 20$ ways when a book is placed in front and behind. This is $56 - 20 = 36$ ways.

7. Bloody casework on $a + b$. When $a + b = 0$, there is 1 solution for (a, b). Then either $c + d + e = 0$, with $\binom{2}{2}$ solutions, $c + d + e = 1$ with $\binom{3}{2}$ solutions, etc., up to $c + d + e = 4$ with $\binom{6}{2}$ solutions, all by balls and urns. This is thus $\binom{2}{2} + \binom{3}{2} + \cdots + \binom{6}{2} = \binom{7}{3} = 35$.

   Similarly, $a + b = 1$ has $\binom{2}{1} = 2$ solutions, and the hockeystick sum is $\binom{2}{2} + \cdots + \binom{5}{2} = \binom{6}{3} = 20$, so there are $2 \times 20 = 40$ ways. When $a + b = 2$ then the sum is $\binom{3}{1} \times \left(\binom{2}{2} + \cdots + \binom{4}{2}\right) = 30$. The sum of all cases is 105.

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Session 5: Algebra 2
compiled by Carl Joshua Quines
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Equations

1. 2/3 of the work needs to be completed in 12 days, so multiplying the amount of work by 2 doubles the men, and multiplying the amount of days by 2/3 multiplies the amount of men by 3/2. There should be 180 men to do the work, so there should be $180 - 60 = 120$ more workers.

2. Either $2 - x^2 = 1$ or $x^2 - 3\sqrt{2}x + 4 = 0$. The former when $x = \pm 1$, the latter when $x = \sqrt{2}, 2\sqrt{2}$. However, $\sqrt{2}$ makes the base zero and thus undefined. The solutions are $x = -1, 1, 2\sqrt{2}$.

3. This is a linear equation, and by inspection $x = a + b + c$ satisfies, so it must be the only solution.

4. Observe that $\left(\sqrt{2014} + \sqrt{2013}\right)^{-\tan^2 x} = \left(\dfrac{1}{\sqrt{2014} + \sqrt{2013}}\right)^{\tan^2 x} = \left(\sqrt{2014} - \sqrt{2013}\right)^{\tan^2 x}$ after rationalizing the denominator. Equating the exponents gives $\tan^2 x = 3$, which is satisfied by $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$.

5. Cross-multiplying and simplifying shows $x = \dfrac{2m - 6}{m - 5}$, which has its denominator is zero, or when $x = 2, 6$. This happens when $m = 5, 6$.

6. Let $u = 2015^x$ and note that $2015^{-x} = \dfrac{1}{u}$. Cross-multiplying and solving for u yields $\sqrt{\dfrac{1 - 3k}{4 - k}}$. The fraction has to be positive, which happens when $k < \dfrac{1}{3}$ or $k > 4$.

7. Observe $x + 3 - 4\sqrt{x - 1} = \left(\sqrt{x - 1} - \sqrt{4}\right)^2$ and $x + 8 - 6\sqrt{x - 1} = \left(\sqrt{x - 1} - \sqrt{9}\right)^2$, so the LHS is $\left|\sqrt{x - 1} - 2\right| + \left|\sqrt{x - 1} - 3\right|$. There are three cases: $\sqrt{x - 1} \geq 3$, $\sqrt{x - 1} \leq 2$, and $2 \leq \sqrt{x - 1} \leq 3$. Solving each case and taking the union gives any $x \in [5, 10]$ works.

8. Let $u = \sqrt{2} - 1$. Observe that $\dfrac{1}{u} = \dfrac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$ upon rationalizing the denominator. Then $u^x + 8u^{-x} = 9$, or multiplying both sides by u^x, $u^{2x} + 8 = 9u^x$. This is quadratic in u^x, with solutions 1 and 8. Now $u^x = 1, 8$ yields $x = 0, \log_u 8$ or $\log_{\sqrt{2}-1} 8$.

Systems of equations

1. Taking the product of all equations and taking the cube root gives $wxyz = 30$. Dividing by the third and last equations gives $y = \dfrac{2}{3}$ and $w = \dfrac{5}{2}$. Thus $w + y = \dfrac{19}{6}$.

2. The first equation is $2xy - x + y - 6 = 0$, the second is $xy + x - y - 2 = 0$. Adding the two equations gives $xy = \dfrac{8}{3}$, or $y = \dfrac{8}{3x}$. Substituting in either equation and solving for x and y gives $\left(-2, -\dfrac{4}{3}\right), \left(\dfrac{4}{3}, 2\right)$.

3. Let $u = \sqrt{x + y}$. From $(x - y)(x + y) = 9$ we get $\sqrt{x - y} = \dfrac{3}{u}$, substituting in the first equation gives $u + \dfrac{3}{u} = 4$ or $u^2 - 4u + 3 = 0$ which has positive solution $u = 1$. Thus $x + y = 1$ and $x - y = 9$; adding and subtracting gives $(a, b) = (5, -4)$. Then $\dfrac{ab}{a + b} = -20$.

4. Subtracting the first from second equation gives $3w + 5x + 7y + 9z = 1$ and the second from third equation gives $5w + 7x + 9y + 11z = 5$. Subtracting these two from each other and dividing by 2 gives $w + x + y + z = 2$.

5. Dividing both sides of the first equation by $4xy$ gives $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$. Similarly, $\frac{1}{y}+\frac{1}{z}=\frac{1}{4}$ and $\frac{1}{z}+\frac{1}{x}=\frac{1}{8}$. Adding all the equations and dividing by 2 gives $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{7}{16}$, subtracting the second equation gives $\frac{1}{x}=\frac{3}{16}$, so $x=\frac{16}{3}$.

Complex numbers

1. The polynomial factors as $(x+1)(x^2+1)=0$ with roots $-1, i, -i$. Else, multiply both sides by $x-1$ to get $x^4=1$, the roots are the fourth roots of unity except for 1, so $-1, i, -i$.
2. Since $i^4=1$, then $\frac{1}{i}\cdot\frac{i^3}{i^3}=i^3=-i$. Similarly, the sum becomes $(1-i-1+i+\cdots)$, with the pattern repeating. However, $1-i-1+i=0$, so every four terms cancel out, until $\frac{1}{i^{2012}}$, leaving $(1-i-1)^2=-1$.
3. This is $1+\frac{2}{z^4-1}=\frac{i}{\sqrt{3}}$, and thus $z^4=-\frac{1}{2}-\frac{i\sqrt{3}}{2}=\operatorname{cis}240°$. By de Moivre's, $z=\operatorname{cis}60°, \operatorname{cis}150°, \operatorname{cis}240°, \operatorname{cis}330°$. Written out, $z=\frac{1}{2}+\frac{i\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}+\frac{i}{2}, -\frac{1}{2}-\frac{i\sqrt{3}}{2}, \frac{\sqrt{3}}{2}-\frac{i}{2}$.
4. Since $z\neq 1$, we have $z^2+z+1=0$. Dividing both sides by z shows $z+1+\frac{1}{z}=0$, adding three to both sides shows $z+\frac{1}{z}+4=3$.

Polynomials

1. We $P(-7)=a(-7)^7+b(-7)^3+c(-7)-5$, or $7+5=-a(7^7)-b(7^3)-c(7)$. Then $P(7)=a(7^7)+b(7^3)+c(7)-5=-(7+5)-5=-17$.
2. By Vieta's, making a mistake in the constant term means the sum of the roots is conserved, similarly making a mistake in the linear term means the product of the roots is conserved. The sum of the roots is 10 and the product is 9, meaning one such original equation can be $x^2-10x+9=0$.
3. If the roots are r, s, the distance of the roots is $p=|r-s|$. This is hard to work with, so we square the distance instead: $p^2=(r-s)^2$. We can rewrite this in terms of the sum and product: $p^2=(r+s)^2-4rs$, and then in terms of the coefficients using Vieta's: $p^2=\left(\frac{b}{a}\right)^2-\frac{4c}{a}$.

   Turning $p\to 2p$ makes $p^2\to 4p^2$, which is $4p^2=4\left(\frac{b}{a}\right)^2-\frac{16c}{a}$. We also want to write this in terms of a shift in c, from the original. Suppose the roots are translated k downward, then $c\to c-k$, making $4p^2=\left(\frac{b}{a}\right)^2-\frac{4(c-k)}{a}$. Equating and solving gives $k=\frac{3b^2}{4a}-3c$.
4. The discriminant should be less than zero, which is $(4p)^2-4(4)(1-q^2)<0$, or $p^2+q^2<1$. This is a circle of area π over a square of area 4, so the probability is $\frac{\pi}{4}$.
5. We can form x^5 through $(x)(x^2)^2$ or $(x)^3(x^2)$. The former term is $\binom{4}{1,1,2}(2)^1(-x)^1(x^2)^2=-\frac{4!}{1!1!2!}2x^5=-24x^5$. The latter term is $\binom{4}{0,3,1}(2)^0(-x)^3(x^2)^1=-4x^5$, and their sum gives the coefficient -28.
6. Consider $Q(x)=P(x)-3$, which is a degree-four polynomial that attains its maximum value of 0 at $x=2, x=3$. Thus 2 and 3 are both roots, and since they are maximums, they should have multiplicity 2. Thus $Q(x)=a(x-2)^2(x-3)^2$ for some constant a, since it is quartic. Then $P(x)=a(x-2)^2(x-3)^2+3$ and plugging in $x=1$ gives $a=-\frac{3}{4}$. Finally $P(5)=-24$.

7. Plugging in $x = 1$ gives $5^{2009} - 5^{2009} = 0$, the sum of the coefficients. Plugging in $x = -1$ gives $1^{2009} + 1^{2009} = 2$, which is the difference between the coefficients of terms with even exponents and coefficients of terms with odd exponents. So subtracting them will cancel out the terms with even exponents, and dividing by two gives the answer: $\frac{0-2}{2} = -1$.

8. Treating this is $(x + (y + z))^{2015} + (x - (y + z)^{2015}$ and expanding by the binomial theorem cancels out the terms with odd $(y+z)$ exponent, leaving $2x^{2015} + 2\binom{2015}{2}x^{2014}(y+z)^2 + \cdots + 2\binom{2015}{2014}x(y+z)^{2014}$. The first term has 1 term, the second term has 3 terms, etc.; none of these terms combine because they have different powers of x. All in all, there are $1 + 3 + \cdots + 2015 = 1016064$ terms.

Polynomial factors

1. Since $x^2 - x - 2 = (x-2)(x+1)$ and it divides $ax^4 + bx^2 + 1$, it must have 2 and -1 as roots. Thus $16a + 4b + 1 = a + b + 1 = 0$. Solving yields $a = -\frac{1}{4}$ and $b = -\frac{3}{4}$.

2. The remaining factor must be $x^2 + cx + d$ for some c, d. Multiplying out gives $x^4 + (2+c)x^3 + (2c + d + 5)x^2 + (5c + 2d)x + 5d$. Equating the cubic and linear coefficients shows $2 + c = 0$ and $5c + 2d = 0$, so $c = -2$ and $d = 5$. Then $a = 6$ and $b = 25$, so $a + b = 31$.

   Alternatively, note that since $x^4 + ax^2 + b$ can be rewritten as $\left(x^2 - h\right)^2 - k$, which is likely to be a difference of two squares. We guess it as $\left((x^2+5) + 2x\right)\left((x^2+5) - 2x\right)$, which gives the product $\left(x^2+5\right)^2 - 4x^2$, which fits the form. Substituting 1 gives the sum of the coefficients, $6^2 - 4 = 32$, and subtracting the leading coefficient 1 gives the answer 31.

3. From $a^3 + b^3 + c^3 = 3abc$ if $a + b + c = 0$, we have $(r-s)^3 + (s-t)^3 + (t-r)^3 = 3(r-s)(s-t)(t-r)$. Alternatively, use the factor theorem by substituting $r = s$, to get $r - s$ is a factor, etc.

4. By the fundamental theorem of algebra, $x^{2015} + 18 = (x - r_1)(x - r_2)\cdots(x - r_{2015})$ for some complex roots $r_1, \dots, r_{2015}$. Any combination of linear factors produces a factor of $x^{2015} + 18$.

   Since each linear factor either appears or does not appear in the factor, there are 2^{2015} factors. However, we overcounted since one of them is where none of the linear factors appear, so there are $2^{2015} - 1$ factors.

5. Substituting $x = 5$ gives $p(4) = 0$. Then substituting $x = 4, 3, 2, 1$ in turn gives $p(3) = p(2) = p(1) = p(0) = 0$. Thus the polynomial $p(x) = x(x-1)(x-2)(x-3)(x-4)q(x)$ for some polynomial q. Substituting $p(x)$ in the original gives $q(x-1) = q(x)$, so $q(x)$ is constant. Substituting $x = 6$ gives the constant $\frac{1}{6}$, so only $p(x) = \frac{1}{6}x(x-1)(x-2)(x-3)(x-4)$ works.

Remainder theorem

1. By the remainder theorem, $P(r) = 2$. The division algorithm says $P(x) = (2x^2 + 7x - 4)(x - r)Q(x) + \left(-2x^2 - 3x + 4\right)$. Substituting r gives $P(r) = 2 = -2r^2 - 3r + 4$, which has the solutions $r = -2, \frac{1}{2}$.

2. By the remainder theorem, $f\left(-\frac{3}{2}\right) = 4$ and $f\left(-\frac{4}{3}\right) = 5$. By the division algorithm, $f(x) = (2x+3)(3x+4)Q(x) + R(x)$, and $R(x)$ is a linear polynomial. Substituing $x = -\frac{3}{2}, -\frac{4}{3}$ gives $R\left(-\frac{3}{2}\right) = 4$ and $R\left(-\frac{4}{3}\right) = 5$, which is the line $R(x) = 6x + 13$.

3. By a similar solution as above, the remainder is $R(x) = -x + 118$.

Root-finding

1. We solve the equation $2x^4 - 7x^3 + 2x^2 + 7x + 2 = 0$. Dividing both sides by x^2 and grouping terms gives $2\left(x^2 + \frac{1}{x^2}\right) - 7\left(x - \frac{1}{x}\right) + 2 = 0$. Letting $u = x - \frac{1}{x}$, observe $u^2 = x^2 - 2 + \frac{1}{x^2}$; substituting in gives $2\left(u^2 + 2\right) - 7u + 2 = 0$. This has roots $u = \frac{3}{2}, 2$.

   The roots of $x - \frac{1}{x} = \frac{3}{2}$ are $-\frac{1}{2}$ and 2, while the roots of $x - \frac{1}{x} = 2$ are $1 - \sqrt{2}$ and $1 + \sqrt{2}$. The smallest root is $-\frac{1}{2}$ and the largest is $1 + \sqrt{2}$, and their difference is $\frac{3}{2} + \sqrt{2}$.

2. For the polynomial to have equal roots, the discriminant must be zero. Thus $(2(1+3m))^2 - 4(7)(3+2m) = 0$, or $4(m-2)(9m+10) = 0$, giving $m = -\frac{10}{9}, 2$. Substituting either shows that the equal root is 7.

   Alternatively, if the root is r, the polynomial must be $(x-r)^2 = x^2 - 2rx + r^2$. Equating coefficients gives $r = 1 + 3m$ and $r^2 = 7(3+2m)$. Then $(1+3m)^2 = 7(3+2m)$, again giving $m = -\frac{10}{9}, 2$, showing that $r = 7$.

3. The polynomial factors as $f(x-5) = -3(x-3)(x-12)$. Substituting $x = 3, 12$ shows that $f(-2) = f(7) = 0$, so its roots are $-2, 7$.

4. Suppose the roots are $a-d, a, a+d$. By Vieta's, the sum of the roots $3a = 6p$, so $a = 2p$, and our roots are $2p-d, 2p$ and $2p+d$. Using Vieta's on the linear and constant terms gives $-44 = p(4p^2 - d^2)$ and $5p = 12p^2 - d^2$. Solving for d and equating, we get $4p^2 + \frac{44}{p} = 12p^2 - 5p$, or $8p^3 - 5p^2 - 44 = 0$. This factors as $(p-2)(8p^2 + 11p + 22) = 0$, and the only real root is $p = 2$.

5. By Vieta's, we know the sum of the roots is zero. Let the roots be $a, a, -2a$. The product of the roots is $-2a^3 = 128$, so $a = -4$. Then the sum of the pairwise products of the roots is $k = a^2 - 2a^2 - 2a^2 = -3a^2 = -48$.

6. By Vieta's, the product of the roots is 2, so let the roots be $p, p, \frac{2}{p^2}$. Then the sum of pairwise products is $-3 = p^2 + \frac{2}{p} + \frac{2}{p}$, or $p^3 + 3p + 4 = 0$, which has the only real root $p = -1$. Thus the roots are $-1, -1, 2$ and $a = -(-1-1+2) = 0$.

Vieta's

1. If the number is x, then $x - \frac{1}{x} = 2$, or $x^2 - 2x - 1 = 0$. By Vieta's, their product is -1.

2. WLOG the leading coefficient of the polynomial is one. Then $f(x) = x^{2016} - Sx^{2015} + \cdots$, so $f(2x-3) = (2x-3)^{2016} - S\,(2x-3)^{2015} + \cdots$. Expanding and looking at the first two terms gives $(2x)^{2016} - 2016(2x)^{2015}(-3) - S(2x)^{2015} + \cdots$, or $2^{2016}x^{2016} - \left(2016 \cdot 2^{2015} \cdot 3 + S \cdot 2^{2015}\right)x^{2015}$. By Vieta's, the new sum of the roots is $\frac{S + 2016 \cdot 3}{2} = \frac{1}{2}S + 3024$.

   Alternatively, $f(x)$ factors as $(x - r_1)\cdots(x - r_{2016})$ for its roots. Then $f(2x-3) = (2x - 3 - r_1)\cdots(2x - 3 - r_{2016})$, which has roots $x = \frac{r_1 + 3}{2}, \frac{r_2 + 3}{2}, \ldots, \frac{r_{2016} + 3}{2}$. The new sum of the roots is then $\frac{1}{2}S + 3024$.

3. Suppose the roots are r and s. Then $|r - s| = 75$, and squaring both sides gives $(r-s)^2 = 75^2$, or $r^2 - 2rs + s^2 = (r+s)^2 - 4rs = 75^2$. By Vieta's, $r + s = 51$ and so $rs = -\frac{75^2 - 51^2}{4} = -756$. Then $r^2 + s^2 = (r+s)^2 - 2rs = 4113$.

4. Suppose the roots are r and s. Then $r+s=-4$ and $rs=8$ by Vieta's. The sum of the reciprocals of the roots is $\frac{1}{r}+\frac{1}{s}=\frac{r+s}{rs}=-\frac{1}{2}$, and the product of the reciprocals is $\frac{1}{r}\cdot\frac{1}{s}=\frac{1}{8}$. One such quadratic polynomial is thus $x^2+\frac{1}{2}x+\frac{1}{8}$ by Vieta's, to make its coefficients integral we multiply by 8 to get $8x^2+4x+1=0$.

   Alternatively, the transformation $x\to\frac{1}{x}$ makes the new polynomial have roots as reciprocals, giving $\frac{1}{x^2}+\frac{4}{x}+8=0$. Multiplying by x^2 gives $8x^2+4x+1=0$.

5. If the roots are a,b,c,d, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{abc+abd+acd+bcd}{abcd}$. By Vieta's, this is $\frac{-\frac{2}{4}}{\frac{-6}{4}}=\frac{1}{3}$.

   Alternatively, the substitution $x\to\frac{1}{x}$ changes the roots to their reciprocals. The new polynomial is $\frac{4}{x^4}-\frac{3}{x^3}-\frac{1}{x^2}+\frac{2}{x}-6=0$; multiplying both sides by x^4 gives $4-3x-x^2+2x^3-6x^4=0$. The sum of the roots is $-\frac{2}{-6}=\frac{1}{3}$.

6. Both roots are real, so $b^2\geq 4c$. In order to maximize c, we must have equality: let $b^2=4c$. We want to minimize $b+c=b+\frac{b^2}{4}=\left(\frac{b}{2}+1\right)^2-1\geq 1$, which attains its minimum when $b=-2$.

7. Note that $x^3-4x+1=0$ implies that each root satisfies $x^3+1=4x$. Substituting in the denominator and cancelling makes the sum $\frac{3abc}{4}$, which by Vieta's is $-\frac{3}{4}$.

Coordinate plane

1. The first equation is two intersecting lines: $y=x$ and $y=-x$. The second equation is a circle of radius zero centered at $(a,0)$, so it only consists of that point. Intersecting the line $y=0$ with the first equation gives $x=0$, so the only value of a is $a=0$.

2. The focus of $y=ax^2$ is $\left(0,\frac{1}{4a}\right)$, so the focus of $y=x^2-1$ is $\left(0,\frac{1}{4}\right)$ shifted downward by one unit to $\left(0,-\frac{3}{4}\right)$. Similarly, its vertex $(0,0)$ is shifted downward to $(0,-1)$. We can rotate clockwise to $\left(-\frac{1}{4},-\frac{3}{4}\right)$.

3. Since the parabola points upward, there must be either one or no intersections of the parabola with the line. Substituting $y=-12x+5$ to $y=x^2-2px+p+1$ gives $x^2-(2p-12)x+(p-4)=0$, and since it has at most one real root, its discriminant must be nonpositive. Thus $(2p+12)^2-4(p-4)\leq 0$, or $(p-5)(p-8)\leq 0$, which has solutions $p\in[5,8]$.

4. Since the two circles are congruent, the two points of intersection must pass through the perpendicular bisector of their centers, $(0,0)$ and $(16,16)$. This perpendicular bisector is $x+y=16$. Since (a,b) and (c,d) lie on this line, $a+b=16$ and $c+d=16$, so $a+b+c+d=32$.

5. The locus of points with $PA+PB=10$ is an ellipse, the locus of point with $|PC-PD|=6$ is a hyperbola. The top-most point of the ellipse is $(0,4)$ and the bottom-most point of the hyperbola is $(0,3)$, so the upper arm of the hyperbola intersects it in two points. By symmmetry, so does the lower arm. Then there are 4 points satisfying both.

6. The line passes through $P(0,5)$ and is tangent to the circle with center $O(0,0)$ and radius 3. The point of tangency is Q, and since $\angle PQO=90^\circ$ due to tangency, PQO is a $3-4-5$ right triangle.

We want to find the slope of PQ. Drop the perpendicular from Q to PO at point R, then the slope is $\frac{PR}{RQ}$, since it is rise over run. By similarity, $\frac{PR}{RQ} = \frac{PQ}{QO} = \frac{4}{3}$.

7. Divide the plane in quadrants. We only need to consider $x > -15$, so this produces four equations: $x + y = \frac{1}{4}(x+15)$ in quadrant I, $-x + y = \frac{1}{4}(x+15)$ in quadrant II, $-x - y = \frac{1}{4}(x+15)$ in quadrant III, and $x - y = \frac{1}{4}(x+15)$ in quadrant IV.

   This produces a kite with vertices at the intercepts: $(5,0)$, $\left(0, \frac{15}{4}\right)$, $(-3,0)$ and $\left(0, -\frac{15}{4}\right)$. Thus its area is 30.

8. Suppose $y = m(x-1)$. Substituting to the curve $4x^2 - y^2 - 8x = 12$ gives $(4-m^2)x^2 + (2m^2-8)x - (m^2+12) = 0$. Since this should not have a solution, its discriminant should not be positive. Its discriminant is $(2m^2-8)^2 - 4(4-m^2)(m^2+12) = 16(4-m^2)$, which is positive when $m < 2$. The least positive m is 2, making the line $y = 2(x-1)$.

9. Let $A(-2,1), B(2,5)$ and $C(5,2)$. After either finding the side lengths or looking at the slopes, we notice that $\angle ABC = 90°$. The incenter must lie on the angle bisector of $\angle ABC$, which is a vertical line, so the incenter must be $(2,h)$ for some h. Since it is a right triangle, we can find its inradius using the formula $\frac{a+b-c}{2} = \sqrt{2}$.

   This means the equation of the circle is $(x-2)^2 + (y-h)^2 = 2$ for some h. Intersecting it with line BC should only produce one intersection, since it is tangent. Line BC is $y = 7 - x$, substituting gives $2x^2 + (2h-18)x + (h^2 - 14h + 51) = 0$, with discriminant $-4(h-3)(h-7)$. There should only be one solution, so the discriminant should be zero: we reject $h = 7$ because it is above the triangle. Thus the circle is $(x-2)^2 + (y-3)^2 = 2$.

10. Reflecting the point about the x-axis gives $P(-3,-7)$; the problem is now to find the shortest path from that point to the circle with center $O(5,8)$. The segment PO intersects the circle at Q, the shortest path has to be PQ. Then since QO is a radius and has length 5, we have $PQ = PO - QO = \sqrt{(-3-5)^2 + (-7-8)^2} - 5 = 12$.

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Session 6: Combinatorics 2
compiled by Carl Joshua Quines
October 7, 2016

Random variable

1. Count: 10 has $\{6,3,1\}, \{6,2,2\}, \{5,4,1\}, \{5,3,2\}, \{4,4,2\}, \{4,3,3\}$, multiplying by the number of permutations gives 27. There are 6^3 tuples, so $\frac{27}{216} = \frac{1}{8}$.
2. Consider all 6^3 tuples of dice rolls. There are $3^3 = 27$ with numbers from 1 to 3, but of these, $2^3 = 8$ have no threes, leaving $27 - 8 = 19$ with the greatest being 3. Thus the probability is $\frac{19}{216}$.
3. Sherlock wins if and only if the sequence is TTT, with probability $\frac{1}{8}$, and cannot win otherwise. Since the game must terminate, Mycroft wins with probability $\frac{7}{8}$, and thus Mycroft has a higher probability of winning.
4. Let the probability of obtaining F be f. The probability of obtaining the side opposite F is thus $\frac{1}{6} - f$. So getting a sum of 13 has probability $2f\left(\frac{1}{6} - f\right) + 10 \cdot \frac{1}{12} \cdot \frac{1}{12} = \frac{29}{384}$; solving the quadratic equation gives $f = \frac{1}{48}, \frac{7}{48}$. Since $f > \frac{1}{12}$, then the probability is $\frac{7}{48}$.

Random selection

1. Each of the $2^6 - 1 = 63$ possible subsets of six colors are equally likely, and only 1 uses only her favorite color; the probability is $\frac{1}{63}$.
2. There is one cube with no red sides, 6 cubes with one red side, 12 cubes with two red sides and 8 cubes with three. A cube with one red side has $\frac{1}{6}$ probability, etc., so the probability is $\frac{6}{27} \cdot \frac{1}{6} + \frac{12}{27}\frac{1}{3} + \frac{8}{27}\frac{1}{2} = \frac{1}{3}$.
3. Modulo 2, the tuples $(a, b, c) = (0,0,0), (0,1,0), (1,0,0)$ and $(1,1,1)$ work. Since there is an equal probability of being either odd or even, then the probability is $\frac{4}{2^3} = \frac{1}{2}$.
4. The probability of getting a different color *and* a different number is $\frac{8}{14}$, since among the 14 chips left 10 are of different colors but 2 have the same number. So the probability is $1 - \frac{8}{14} = \frac{3}{7}$.
5. We count the number that does not contain any 2s. Replace 5000 with 0000. The thousands digit can be anything from 0 to 4, the hundreds to ones digit can be 0 to 9, except 2. This gives $4 \times 9 \times 9 \times 9 = 2916$, so the probability is $\frac{2916}{5000} = \frac{729}{1250}$.
6. Use casework, or be witty: equivalent to Josh just picking two chips from all together without replacement. This is because, suppose we permute the six chips in a row, with the first three going to urn 1, and the second three going to urn 2, and Josh picked the first and fourth chips, which is equivalent. The probability both are red is $\frac{4}{6} \cdot \frac{3}{5} = \frac{2}{5}$.

7. The first draw must not all be red or not all be green. It is all red with probability $\frac{3}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{20}$, and by symmetry all green with probability $\frac{1}{20}$. The first draw is not all red and not all green with probability $1 - \frac{1}{20} - \frac{1}{20} = \frac{9}{10}$.
   The bag now has one of one color, two of the other color, and three white, so the probability they are all different in the second draw is $\frac{1 \cdot 2 \cdot 3}{\binom{6}{3}} = \frac{3}{10}$. The product is $\frac{27}{100}$.

8. There are $\binom{23}{3} = 1771$ ways to pick three non-adjacent people in a row of 25, subtract the 21 ways in which the front and back people are placed, for a total of 1750 ways. There are $\binom{25}{3} = 2300$ ways to pick three people randomly, so the probability is $1 - \frac{1750}{2300} = \frac{11}{46}$.

9. Induction on k. Base case is $1 - \frac{1}{3} - \frac{2}{3} = \frac{2}{3}$, as wanted. Suppose $k = n - 1$ is true, then there are two cases: when the sum to $n - 1$ is even and when the sum to $n - 1$ is odd.
   For the former, the probability this happens is $\frac{1}{2} + \frac{1}{2 \cdot 3^{n-1}}$ by inductive hypothesis; for the whole sum to be even, $a_n b_n$ has to be even too, with probability $\frac{2}{3}$. The whole probability for this case is thus $\frac{2}{3}\left(\frac{1}{2} + \frac{1}{2 \cdot 3^{n-1}}\right)$.
   Similarly, the probability for the other case is $\frac{1}{3}\left(\frac{1}{2} - \frac{1}{2 \cdot 3^{n-1}}\right)$. Taking their sum and simplifying yields the expression we want.

Geometric probability

1. Suppose the AB has length ℓ. Then $\frac{AP}{\ell - AP} < r$ so $AP < \frac{r\ell}{r+1}$. The segment of success has length $\frac{r\ell}{r+1}$ divided by the whole segment with length ℓ, giving the probability $\frac{r}{r+1}$.
2. Scale by 1/5000. Let the prices of the gifts be x, y pesos. Then the region of the plane is the square with $0 \le x, y \le 5$ and we must have $x + y \le 9$. The failure region is $x + y > 9$, which intersects the square at the triangle with vertices $(4,5), (5,5)$ and $(5,4)$. Its area is $\frac{1}{2}$. The whole area of consideration is 25, so the probability is $1 - \frac{\frac{1}{2}}{25} = \frac{49}{50}$.
3. Factoring, $x^2 - 3xy + 2y^2 > 0$ if $x > y$ or $x < 2y$. Intersecting with the square $0 \le x, y \le 1$ produces a region with area $\frac{3}{4}$; since the area of the square is 1, the probability is $\frac{3}{4}$.
4. The intervals where the sum is 5 are when the first number is $(0.5, 1)$, $(1.5, 2)$, $(2.5, 3)$ and $(3.5, 4)$. Each interval has length 0.5 and the whole interval has length 4.5, so the probability is $\frac{4 \cdot 0.5}{4.5} = \frac{4}{9}$.

Existence combinatorics

1. By PHP two of them are the same modulo 6 and thus have a difference that is zero, so probability 1.
2. Modulo 2 the points are $(0,0)$, $(0,1)$, $(1,0)$ or $(1,1)$, by PHP two points are the same modulo 2 and thus have a midpoint with integer coordinates, so probability 1.
3. For the former, consider a regular pentagon: by PHP three of them are the same color and form an isosceles triangle. For the latter, color half the circle red and the other half blue, no such equilateral triangle exists.

4. Modulo 3, the number of blue, red, and yellow chips cycles $(1, 2, 0) \to (0, 1, 2) \to (2, 0, 1) \to (1, 2, 0)$. All the chips being the same color is $(2, 2, 2)$, which is impossible.

5. For $n = 1005$ the sequence $0, 1, 2, \ldots, 1004$ trivially does not have two whose sum or difference is divisible by 2009. For $n = 1006$, consider modulo 2009, if no two have a difference that is 0 then they must all be distinct, but by PHP one of $\{0\}, \{-1, 1\}, \{-2, 2\}, \ldots, \{-1004, 1004\}$ has two, which then have a sum divisible by 2009.

6. The sequence 1 to 2011^{2011} has at least 2012 prime numbers since $17489 < 2011^{2011}$. Then all numbers from $\left(2011^{2011} + 1\right)! + 2$ to $\left(2011^{2011} + 1\right)! + 2011^{2011} + 1$ are composite. Now move the left endpoint one upward and the right endpoint one upward: either the number of primes is increased by 1, decreased by 1, or stays the same. Since it starts from ≥ 2012 and eventually becomes 0, it will hit 2011 some time.

7. Let A be the set $\{1, 2, \ldots, n\}$ and B be the set $\{n + 1, \ldots, 2n\}$. Since the numbers are arranged on a circle, there are two adjacent points from opposite sets, join them with a chord. Remove them from the circle and keep connecting points with chords in this manner, you end up with n non-intersecting chords. The sum is the sum of all the elements in set B minus the sum of all the elements in set A, which is n^2.

8. Consider a matrix with 120 rows and 10 columns, and write a 1 on each entry if the student corresponding to the row does *not* follow the celebrity for that column. Suppose that the hypothesis is not true, that is, each pair of students has at least one celebrity that both do not follow. This translates to each pair of rows having a column where both are 1.

   We count the number of pairs of 1s in each column. Vertically, since each column has at most $120 - 85 = 35$ ones, the sum is at most $10\binom{35}{2} = 5950$. Horizontally, each of the $\binom{120}{2}$ pairs of rows has at least one pair of 1s, so the sum is at least 7140. Contradiction.

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Session 7: Geometry 1
compiled by Carl Joshua Quines
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Circles

1. Let the centers of the circles be A, B, one internal tangent be CD tangent to circle A at C, and to circle B at D, and let E be the intersection of the two tangents.

   Since $\angle E$ is right, and $\angle C$ is right as well, then ACE must be an isosceles right triangle. Thus AE, BE are $4\sqrt{2}$ and $2\sqrt{2}$, so AB is $6\sqrt{2}$.

2. Since $\angle QPR + \angle QSR = 180°$ then quadrilateral $PQRS$ is cyclic, so C_1 and C_2 are the same circle, and they intersect at infinitely many points.

3. Drop the perpendicular from C to AB at point D. Then $CD = 5$, and $CB = 13$, so by Pythagorean, $DB = 12$. Similarly, $AD = 12$, so the perimeter is $12 + 12 + 13 + 13 = 50$.

4. Extend CB to meet the circle again at F. By power of a point, we get $CF = 12$, and so $AF = 5$. By Stewart's on triangle OBF we find $OB = BF = 2\sqrt{6}$. Pythagorean on OCD gives $OC = 2\sqrt{15}$.

5. Let C_2 have center O, the smaller circle have center P, tangent to C_1, C_2 and AB at I, J, K, respectively. Let the smaller circle have radius r and let $OK = s$.

   Then Pythagorean on APK gives $AP^2 = AK^2 + PK^2$, or $(AI + IP)^2 = (AO + OK)^2 + PK^2$, or $(12+r)^2 = (12+s)^2+r^2$. Pythagorean on OPK gives $OP^2 = PK^2+OK^2$, or $(OJ-JP)^2 = PK^2+OK^2$, or $(12 - r)^2 = r^2 + s^2$. The r^2 term cancels in both equations, and we can equate $24r$ in both to get $144 - s^2 = s^2 + 24s$. Thus $s = 6\sqrt{3} - 6$ and $r = 3\sqrt{3}$.

6. From power of a point on E we get $AE = BE$ so ABE is equilateral. Thus $\angle ABC = 120°$. By the law of cosines $AC = 2\sqrt{7}$, and by the extended law of sines $2R = \dfrac{AC}{\sin 120°}$, so the circumradius is $\dfrac{2\sqrt{21}}{3}$.

Angles

1. Let $\angle ABD = \angle DBC = x°$. We know that $\angle ADB = \angle DBC + \angle BCD$ since it is an exterior angle, however $\angle ADB = \dfrac{180° - \angle ABD}{2}$ as triangle ABD is isosceles. Equating gives $x + 36 = \dfrac{180 - x}{2}$, or $x = 36°$. Thus since triangle ABD is isosceles, $\angle ADB = \dfrac{180° - \angle ABD}{2} = 72°$; since triangle ADE is isosceles $\angle ADE = \dfrac{180° - \angle DAB}{2} = 54°$, and so $\angle BDE = \angle ADB - \angle ADE = 17°$.

2. Let Q be the midpoint of BC. Then $\angle ABP = \angle APB = 52°$ by triangle angle sum on ABP, so $AB = BP$. Then $ABQP$ is a rhombus. Then AQ is an angle bisector since it is a diagonal, so $\angle AQP = 38°$. But $PC||AQ$ and $PQ||CD$ so $\angle PCD = \angle AQP = 38°$.

3. Note $\angle CBD = \angle ADB - \angle DCB$ upon considering exterior $\angle ADB$. But $\angle ADB = \angle ABD = \angle ABC - \angle CBD$ through isosceles triangle ABD. Substituting, $\angle CBD = (\angle ABC - \angle CBD) - \angle DCB = (\angle ABC - \angle ACB) - \angle CBD = 45° - \angle CBD$. Thus $\angle CBD = 22.5°$.

4. Since in $AFGE$ we have $\angle AFG + \angle AEG = 90° + 90° = 180°$, it is a cyclic quadrilateral. Similarly, since in $BDEF$ we have $\angle BED = \angle BFD = 90°$ then it is also cyclic. Thus $\angle GAB = \angle GAF$, and $\angle GAF = \angle GEF$ by cyclic quadrilateral $AFGE$, and $\angle GEF = \angle BEF = \angle BDF$ by cyclic quadrilateral $BDEF$. However, $\angle BDF + \angle FDE = \angle CED$ since $BCDE$ is a rectangle. Thus $\angle GAB = \angle BDF = 17°$.

5. From $CA \perp CG$ and $BG \perp CG$ we have $CA||BG$. Then $\angle ABG+\angle CAB = 180°$, whence $\angle ABG = 78°$. Then $\angle ABG = \angle EBG = 2\angle EFG = 2\angle DFG$, so $\angle DFG = 39°$.

Three-dimensional

1. By Euler's formula, $V - E + F = 2$, so $V = 34$.

2. It is a regular tetrahedron of edge 1. Drop the height from the top vertex to the base, which hits its center. It forms a right triangle with one edge as the hypotenuse, the other leg is from the length from a vertex to the center. The other leg is 2/3 the median, so its length is $\frac{\sqrt{3}}{3}$. This gives its height as $\sqrt{1^2 - \left(\frac{\sqrt{3}}{3}\right)^2} = \frac{\sqrt{6}}{3}$. Its volume is one-third the area of the base times its height, or $\frac{1}{3} \cdot \frac{\sqrt{3}}{4} \cdot \frac{\sqrt{6}}{3} = \frac{\sqrt{2}}{12}$.

3. We stack the $7 \times 9 \times 11$ boxes in a $2 \times 3 \times 3$ fashion, making it take up $14 \times 27 \times 33$, which fits in the $17 \times 27 \times 37$ box. This makes the maximum number 18.

4. Let the sides of the prism be x, y, z; we have $xyz = 120$ and $(x-2)(y-2)(z-2) = 24$. WLOG z is divisible by 5. Then if $z = 5$, we see $(6, 4, 5)$ works. The surface area is then $2(6 \cdot 5 + 5 \cdot 4 + 4 \cdot 6) = 148$.

5. The centers of the spheres form a regular tetrahedron of edge 3. Through similar logic as number 2 in this section, its height is $\sqrt{3^2 - \sqrt{3}^2} = \sqrt{6}$. The overall height is the height of the tetrahedron plus two radii, so its height is $3 + \sqrt{6}$.

Areas

1. The area consists of two $150°$ sectors of a circle with radius 10, one on either side of the horse. Wrapping around the equilateral triangle gives two more $120°$ sectors, of radius $10 - 8 = 2$. The total area is thus $2 \cdot \frac{150°}{360°}\pi \cdot 10^2 + 2 \cdot \frac{120°}{360°}\pi \cdot 2^2 = 86\pi$.

2. Drop the altitude from E to AB and CD, which are parallel, so the altitude is the same line. The length of the altitude from E to AB has to be 20 for the area of AEB to be 60. Since $AB||CD$ we have $EAB\ simEDC$ and thus the length of the altitude from E to CD has to be $\frac{80}{3}$. Thus the distance between lines AB and CD is $\frac{80}{3} - 20 = \frac{20}{3}$, which is also the length of the altitude from D to AB. Thus $[BAD] = \frac{1}{2} \cdot 6 \cdot \frac{20}{3} = 20$.

3. Note that $\triangle AEB$ and $\triangle AEF$ share the same base and altitude, so they have the same area. Subtracting $[AEG]$ from both gives $[ABG] = [EFG] = 9$. Similarly, $[CDH] = [EFH] = 15$. Thus $[EGFH] = [EFG] + [EFH] = 24$.

4. (Should have E as intersection of diagonals.) Note that AEB and CED are similar with ratio $6 : 15$. Then $EB : ED = 6 : 15$ as well, as AED and AEB share the same altitude from A, their areas are in the ratios of their bases, so $[AEB] : [AED] = 6 : 15$. Thus $[AEB] = 12$.

5. Let the triangle be ABC intersecting the circle with center O at B' and C' lying on AB and AC, respectively. The required region is quadrilateral $AB'OC'$ minus the sector with arc $B'C'$. This is twice the area of a unit equilateral triangle minus the unit sector of $60°$, or $2 \cdot \frac{\sqrt{3}}{4} - \frac{1}{6}\pi = \frac{3\sqrt{3} - \pi}{6}$.

6. In rectangle $ABMN$ with area 2, triangles APM and BPN form half the area, so the sum of their areas is 1. P is vertically halfway between AM and BN, so its distance to DC is $\frac{3}{2}$. The area of DPC is thus $\frac{1}{2} \cdot 2 \cdot \frac{3}{2} = \frac{3}{2}$. Then triangles PQR and DCP are similar, but the height from P to QR is the

distance from P to AB, which is $\frac{1}{2}$. Thus the ratio of similarity is $1:3$, so the ratio of their areas is $1:9$, thus the area of PQR is $\frac{1}{6}$. The sum is $\frac{8}{3}$.

7. It is simplest to Cartesian bash. Set $M(0,0)$, $B(0,18)$, $I(16,0)$. Thus $H(0,8)$ and $A(6,0)$. Line BA is $\frac{x}{6}+\frac{y}{18}=1$ in intercept form, also line IH is $\frac{x}{16}+\frac{y}{8}=1$. Equating gives $\frac{x}{6}-\frac{x}{16}=\frac{y}{8}-\frac{y}{18}$ or $\frac{10x}{6\cdot 16}=\frac{10y}{8\cdot 18}$, cancelling gives $3x=2y$. Substituting back to either equation gives $T(4,6)$. Using the shoelace formula on $MATH$ gives its area as 34.

8. It is also simple to Cartesian bash: set $C(0,0)$, $B(0,16)$, $A(13,16)$ and $D(11,0)$. Then E is a midpoint so $E(12,8)$. The slope of AD is 8 so the slope of EF is $-\frac{1}{8}$. Point F lies on BC so its x-coordinate is zero; it lies on EF so $F(0,9.5)$. Using the shoelace formula gives 91.

9. Suppose that point C is C' after folding, and DC' and EC' intersect AB at A' and B' respectively. Drop altitudes H from C to DE and M from C to AB. Clearly C,H,M,C' are collinear. The ratio $[A'B'C']:[ABC]=16:100$ is given, thus the ratio $C'M:CM=4:10$ due to similarity. Also, $CH=C'H$ since they are the same altitude after folding. Since $CH+C'H=CM+C'M$ due to collinearity, $2CH=CM+\frac{2}{5}CM$ from earlier. By similarity, $CH:CM=7:10=DE:AB$, so $DE=\frac{56}{5}$.

10. Let x be the side of the square. The Pythagorean theorem on right CEH gives $\left(r-\frac{x}{2}\right)^2+x^2=r^2$, so $x=\frac{4}{5}r$. Thus $\angle HCE=\tan^{-1}\frac{4}{3}$. The required area is equal to $[CHGF]$ minus the sector with arc HM; the former is $\frac{1}{2}\left(r+\frac{x}{2}+x\right)x$ while the latter is $\frac{1}{2}r^2\tan^{-1}\frac{4}{3}$. Simplifying yields $r^2\left(\frac{22}{25}-\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$.

11. Official solution uses algebra and whatever. We use Cartesian. Take an affine transformation to $A(0,1)$, $B(1,0)$, $C(0,0)$ which preserves the problem, and let $P(a,b)$. It is easy to bash $D\left(-\frac{a}{b-1},0\right)$, $E\left(0,-\frac{b}{a-1}\right)$, $F\left(\frac{a}{a+b},\frac{b}{a+b}\right)$. Then $[DBP]=[ECP]=[FAP]$ and bashing gives $a=b=\frac{1}{3}$, which is as required.

VCSMS PRIME

Session 8: Algebra 3
compiled by Carl Joshua Quines
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Manipulation

1. The first equation is $\dfrac{x^2+y^2}{xy} = \dfrac{(x+y)^2-2xy}{xy} = 4$, giving $(x+y)^2 = 18$. Then $xy(x+y)^2 - 2(xy)^2 = 3 \cdot 18 - 2 \cdot 3^2 = 36$.

2. The required expression is $2\left(x^2+y^2+z^2+xy+yz+zx\right)$. Squaring the first equation and transposing yz gives $x^2+yz = 2013$, similarly, $y^2+zx = 2014$ and $z^2+xy = 2015$. Adddding all expressions and multiplying by 2 gives the answer, 12084.

3. Substitute $2n \to k$ to get $m^3 - 3mk^2 = 40$ and $k^3 - 3m^2k = 20$, we are looking for m^2+k^2. It reminds us of the triple angle formulas for sine and cosine, so substitute $m = r\cos\theta$ and $k = r\sin\theta$, now we are looking for r^2. The equations become $r^3\left(cos^3\theta - 3\sin^2\theta\cos\theta\right) = 40$ and $r^3\left(sin^3\theta - 3\sin\theta\cos^2\theta\right) = 20$. It is a good idea to write each in terms of only one trigonometric function: substituting the Pythagorean identity shows us that the first equation is actually $r^3\left(4\cos^3\theta - 3\cos\theta\right) = r^3\cos 3\theta = 40$. Similarly, the second equation is $r^3\sin 3\theta = 20$. Squaring both equations and adding gives $r^6 = 2000$, from whence $r^2 = \sqrt[3]{2000} = 10\sqrt[3]{2}$.

   More motivated but more high-powered: after substituting, notice $m^3 - 3mk^2 = 40$ and $k^3 - 3m^2k = 20$ look like the expressions from $(m-k)^3$, except the middle terms. We can fix this by making it $(m-ki)^3$; multiply the second equation by i and add to the first to get $m^3 - 3m^2ki - 3mk^2 + k^3i = 40 + 20i = (m-ki)^3$. Taking the modulus of both sides and using de Moivre's gives $|m-ki|^3 = \sqrt{40^2+20^2}$, so $m^2+k^2 = |m-ki|^2 = 10\sqrt[3]{2}$.

4. Abuse degrees of freedom by setting $x = y$. The condition is $x^2+2x-1 = 0$, and the expression needed is $x^2 + \dfrac{1}{x^2} - 2$. From the condition, $x^2 = 1-2x$ and dividing both sides of the condition by x^2, $\dfrac{1}{x^2} = 1 + \dfrac{2}{x}$, so the expression is now $\dfrac{2}{x} - 2x = 2\left(\dfrac{1}{x} - x\right) = 2\left(\dfrac{1-x^2}{x}\right)$. But from the condition, $1 - x^2 = 2x$, so $2\left(\dfrac{1-x^2}{x}\right) = 4$.

   (The legit solution is to clear denominators, factor the numerator, expand to get it as $(xy+x+y+1)(xy-x-y+1)$. The first term is 2, the second term, when divided by xy, is the condition divided by xy.)

5. Cross-multiply the condition and divide both sides by a to get $a + \dfrac{1}{a} = 3$. Divide both numerator and denominator of the expression by a^3; the numerator becomes 1 and the denominator becomes $\left(a^3 + \dfrac{1}{a^3}\right) + \left(a^2 + \dfrac{1}{a^2}\right) + \left(a + \dfrac{1}{a}\right) + 1$. But from $a + \dfrac{1}{a} = 3$, we get $a^2 + \dfrac{1}{a^2} = 7$ after squaring both sides, and $a^3 + \dfrac{1}{a^3} = 18$ after cubing and subtracting the original expression. The denominator is thus $18+7+3+1 = 29$, so the fraction is $\dfrac{1}{29}$.

6. Dividing both sides by 4 gives $\dfrac{1}{4} + \dfrac{1}{16} + \dfrac{1}{36} + \cdots = \dfrac{\pi^2}{24}$. Subtracting from the original equation gives $1 + \dfrac{1}{9} + \dfrac{1}{25} + \cdots = \dfrac{\pi^2}{8}$.

7. Squaring both sides and subtracting 2 gives $x^2 + x^{-2} = 7$. Repeating gives $x^{2^2} + x^{-2^2} = 47$, etc. The last two digits are $3, 7, 47, 7, 47, \ldots$. The pattern repeats, so the last two digits are 07.

8. Multiply the equations by a, b, c respectively, and subtract pairwise and transpose to get $(a+bc)x = (b+ca)y = (c+ab)z$. The required ratio is $\left(\frac{x}{y}-1\right)\left(\frac{y}{z}-1\right)\left(\frac{z}{x}-1\right)$, to get these we divide the equations with each other and simplify: $\dfrac{(a-1)(b-1)(c-1)(a-b)(b-c)(c-a)}{(a+bc)(b+ca)(c+ab)}$.

Surds

1. Multiplying numerator and denominator by $\sqrt[3]{8}-\sqrt[3]{2}$ and using the difference of two cubes, then cancelling out the factor 6, leaves $2-\sqrt[3]{2}$.

2. Expanding the right-hand side gives $2a^2+3b^2+c^2+2ac\sqrt{2}+2bc\sqrt{3}+2ab\sqrt{6}$. Equating coefficients gives $ac=-2, bc=-3, ab=6$. Multiplying all equations and taking the square root gives $abc=6$, from whence $a=-2, b=-3, c=1$ upon division by the three equations. Then $a^2+b^2+c^2=14$.

3. Squaring both sides gives $2x+2\sqrt{x^2-3x-6}=36$, or $\sqrt{x^2-3x-6}=18-x$. Squaring both sides again gives $x^2-3x-6=x^2-36x+324$, whence $x=10$.

4. Cubing both sides and using the binomial theorem, the terms which would end up with $\sqrt{5}$ in the expansion would have odd exponent for $\sqrt{5}$. If this were negative, then it would multiply out – so the value must be $12-\sqrt{5}$.

5. Note that $a=4+\sqrt{15}$ and $b=4-\sqrt{15}$ after rationalizing denominators. Then $a+b=8$ and $ab=1$. However, $a^4+b^4=\left(a^2+b^2\right)^2-2\left(ab\right)^2=\left((a+b)^2-2ab\right)^2-2\left(ab\right)^2$. Substituting everything yields 7938.

6. Observe $2=\left(1+\sqrt[n]{2}-1\right)^n \geq 1+\binom{n}{2}\left(\sqrt[n]{2}-1\right)$ by the binomial theorem. The inequality follows.

Sequences

1. If there were perfect squares, the 150th term would be 150; except we skipped 12 terms, so it should be 162.

2. Abuse degrees of freedom: one such sequence is $0, 2, 2, 4, 4, \ldots, 98, 98, 100$, so the average of the first and hundredth terms is 50.

   The legit method is to write $a_1+a_2=2$, $a_2+a_3=4$, $\ldots$, $a_{99}+a_{100}=198$. Take the sum of the odd-numbered equations to find $a_1+a_2+\cdots+a_{100}$ and the sum of the even-numbered equations to find $a_2+a_3+\cdots+a_{99}$; taking their difference yields $a_1+a_{100}=100$, so the average is 50.

3. Add 1 to both sides of the recursion to get $b_{n+1}+1=\dfrac{2}{1+b_n}$, or $\left(b_n+1\right)\left(b_{n+1}+1\right)=2$. So the terms alternate $\frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \ldots$, so $b_{2010}-b_{2009}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$.

4. From the geometric sequence, $16y^2=15xz$ and $\frac{2}{y}=\frac{1}{x}+\frac{1}{z}$, or $\frac{2}{y}=\frac{x+z}{xz}$. Substituting the first equation gives $\frac{32}{15}y=x+z$. The desired expression is $\dfrac{x^2+z^2}{xz}=\dfrac{(x+z)^2-2xz}{xz}=\dfrac{(x+z)^2}{xz}-2$. Substituting the previous values for xz and $x+z$ makes the y cancel, giving $\frac{34}{15}$.

5. It is clear that the terms in the sequence $1, 3, 7, 13, 21$ are quadratic. The method of differences or Newton interpolation yields the formula n^2-n+1, and continuing to 2015 means the sum is taken from $n=1$ to 45. The sum is then $\sum n^2-\sum n+45$, or 30405.

6. The condition is equivalent to $\dfrac{1}{a_{n+1}}=\dfrac{1}{a_n}+c$, so the reciprocals of the terms are arithmetic. With this in mind, $c=183$.

7. It can be easily proven, say, with induction, that $a_n = \dfrac{1}{n!}$. Or prove $a_{n-1}/a_n = n$ with induction. The required sum is $1 + 2 + \cdots + 2009 = 2019045$.

Series

1. There were $17n+1$ numbers on the board originally, making the original sum $602n$ plus whatever number was erased. Estimate $1+2+\cdots+17n+(17n+1) \geq 602n$ to get $n = 4$, the sum is $1+2+\cdots+69 = 2415$, and $602n = 2408$. The erased number was $2415 - 2408 = 7$.

2. Adding the first n and the last $m - n$ numbers gives the sum of the first m numbers being 7140. Solving $1 + 2 + \ldots + m = \dfrac{m(m+1)}{2} = 7140$ is to estimate $\sqrt{2 \times 7140} = \sqrt{14280} \approx 120$, checking, $m = 119$ works.

3. The sum of the first series is $\dfrac{\frac{a}{b}}{1-\frac{1}{b}} = \dfrac{a}{b-1} = 4$, so $a = 4b-4$. The second series is $\dfrac{\frac{a}{a+b}}{1-\frac{1}{a+b}} = \dfrac{a}{a+b-1}$. Substituting $a = 4b - 4$, factoring out $b - 1$, and cancelling gives its value as $\dfrac{5}{4}$.

4. Let the sum be S. Then $2S = 2+2+3\left(\dfrac{1}{2}\right)+4\left(\dfrac{1}{2}\right)^2+5\left(\dfrac{1}{2}\right)^3+\cdots$, and subtracting the original equation from it yields $S = 2 + (2-1) + \left(3\left(\dfrac{1}{2}\right) - 1\right) + \left(4\left(\dfrac{1}{2}\right)^2 - 3\left(\dfrac{1}{2}\right)^2\right) + \left(5\left(\dfrac{1}{2}\right)^3 - 4\left(\dfrac{1}{2}\right)^3\right) + \cdots$, or $S = 2 + 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots$. Then S is an infinite geometric series, with sum $S = \dfrac{2}{1-\frac{1}{2}} = 4$.

5. This is $\dfrac{1}{1\times 3}+\dfrac{1}{3\times 5}+\cdots+\dfrac{1}{13\times 15}$, which telescopes as $\dfrac{1}{2}\left(1-\dfrac{1}{3}\right)+\dfrac{1}{2}\left(\dfrac{1}{3}-\dfrac{1}{5}\right)+\cdots+\dfrac{1}{2}\left(\dfrac{1}{13}-\dfrac{1}{15}\right)$. The sum is $\dfrac{7}{15}$.

6. The telescope is $\dfrac{1}{n(n-2)} = \dfrac{1}{2}\left(\dfrac{1}{n-2} - \dfrac{1}{n}\right)$. Multiply both sides of the sum by 2 and expand the two telescopes to get $\dfrac{1}{3} + \dfrac{1}{4} - \dfrac{1}{N-1} - \dfrac{1}{N} < \dfrac{1}{2}$, or $\dfrac{1}{N-1} + \dfrac{1}{N} > \dfrac{1}{12}$. The maximum that satisfies this is when $N = 24$.

7. From $i = 1$ to 99, the value is 0. From $i = 100$ to 399, the value is 1, so the subtotal is 300. From $i = 400$ to 899, the value is 2, the subtotal is 1000. From $i = 900$ to 1599, the value is 3, the subtotal is 2100. From $i = 1600$ to 2015, the value is 4, so the subtotal is 1664. The total sum is $300 + 1000 + 2100 + 1664 = 5064$.

8. Expand to prove $f(x) + f(1-x) = 1$, so pairing up terms in the series gives 1006.

9. Take the derivative of both sides of $(1+x)^{19} = \sum \binom{19}{k} x^k$ to get $19(1+x)^{18} = \sum k\binom{19}{k} x^{k-1}$. Substitute $x = 1$ to get $19 \cdot 2^{18}$.

   Alternatively, there is a combinatorial proof involving choosing a subset of 19 people and making choosing 1 to be the leader: either you pick the subset first and choose 1 then, giving the sum, or you pick one to be the leader first and each of the 18 others are either in the subset or not.

Inequalities

1. The inequality $x^2 + x - 12 > 0$ is $(x+4)(x-3) > 0$. For it to have solution set $(-4, 3)$, the sign should be reversed – so we must have $k(x^2 + 6x - k) < 0$ for all x. Then k should be negative and $x^2 + 6x - k$ should have negative discriminant, or $k < -9$. Thus $k \in (-\infty, 9]$ works.

2. By Cauchy–Schwarz, $(x^2 + y^2 + z^2)^2 \leq (1^2 + 1^2 + 1^2)(x^4 + y^4 + z^4)$, giving $k = 3$.

3. From AM-GM, $S - a_1 = a_2 + a_3 + a_4 + a_5 \geq 4\sqrt[4]{a_2a_3a_4a_5}$, taking the cyclic product gives $k = 4^5 = 1024$.

4. By Cauchy–Schwarz, $\left(1^2 + \left(\frac{a}{\sqrt{\sin x}}\right)^2\right) + \left(1^2 + \left(\frac{b}{\sqrt{\cos x}}\right)^2\right) \geq \left(1 + \left(\frac{ab}{\sqrt{\sin x \cos x}}\right)^2\right)$, and using the equality $\sin 2x = 2\sin x \cos x$, the right-hand side can be manipulated to give the right-hand side of the inequality.

5. The inequality clearly does not hold when $k < 2$, for example, when $a = b = c = 1$. To show it is true for $k = 2$, it is equivalent to $(2+a)(2+b) + (2+b)(2+c) + (2+c)(2+a) \leq (2+a)(2+b)(2+c)$ after clearing denominators. Expanding and cancelling many terms, then using $1 = abc$, gives $ab+bc+ca \geq 3$ which is true by AM-GM as follows: $ab + bc + ca \geq 3\left\{a^2b^2c^2\right\} = 3$. The steps are reversible.

Single-variable extrema

1. By AM-GM, since both terms are positive, $(7-x)^4(2+x)^5 \leq \left(\frac{(7-x) + \cdots + (7-x) + (2+x) + \cdots + (2+x)}{9}\right)^9$. The numerator simplifies to $38 + x$, and since we want equality, we let $7 - x = 2 + x$ or $x = 2.5$, making the maximum $(4.5)^9$.

2. We have $4x - x^4 - 1 = -(x^4 - 2x^2 + 1) - 2x^2 + 4x - 1 + 1 = -(x^2-1)^2 - 2(x^2 - 2x + 1) + 2 = -(x^2-1)^2 - 2(x-1)^2 + 2 \leq 2$ by the trivial inequality, equality at $x = 1$. Thus the maximum is 2.

3. Let $A(4, 2), B(2, -4)$, and O be a point on $y = x^3$. We then wish to maximize $AO - BO$, which occurs when O lies on the line AB past either end, which does indeed intersect the graph of $y = x^3$. Then $AO - BO = AB$, and the distance is $2\sqrt{10}$.

4. By Cauchy–Schwarz, $(2(x-1) + 4(2y))^2 \leq (2^2+4^2)\left((x-1)^2 + 4y^2\right)$. The left-hand-side is $2x+8y-2 = 1$, so we get $x^2 + 4y^2 - 2x \geq -\frac{19}{20}$.

5. Scrapped.

Multi-variable extrema

1. x and y are independent, so we want to minimize x and maximize y. This happens when $x = -1$ and $y = 4$, whence $x - y = -5$.

2. Clearly we must want all the terms to be positive, by AM-GM the sum is at least $2014\sqrt[2014]{\prod_{i=1}^{2014} \sin\theta_i \cos\theta_i} = 2014\sqrt[2014]{\prod_{i=1}^{2014} \frac{1}{2}\sin 2\theta_i} \geq 2014\sqrt[2014]{\prod_{i=1}^{2014} \frac{1}{2}} = 1007$, the last inequality from $\sin\theta \geq 1$. Equality is achievable when $\sin 2\theta_i = 1$, or when all the $\theta_i = 45°$, giving the maximum as 1007.

3. Distributing the product and the square root shows it is equivalent to $\sqrt{1 + \frac{b}{a}} + \sqrt{1 + \frac{a}{b}}$, which by AM-GM is at least $2\sqrt[4]{2 + \frac{b}{a} + \frac{a}{b}}$, and by AM-GM again is at least $2\sqrt[4]{2+2} = 2\sqrt{2}$.

4. This is $\left(2a^8 + a^4 - 2a^2\right) + \left(2b^6 - b^3 - 2\right)$, so it suffices to minimize each independently. This can be done through calculus, the legit way is slower. Take $u = a^2$ and the derivative, to get minimum as $-\frac{5}{8}$; the second is just a quadratic with vertex at $-\frac{17}{8}$. Their sum is $-\frac{11}{4}$.

5. The legit solution is to manipulate cleverly and use AM–GM. The cheating solution is to convert it to a single-variable problem by substituting $x = 8 - 2y$ and using calculus, the minimum is attained at $y = 3$, giving the value 8.

6. We factor out the 2 from $2-y$ and the 3 from $3-z$ to get $6(1-x)\left(1-\frac{y}{2}\right)\left(1-\frac{z}{3}\right)\left(x+\frac{y}{2}+\frac{z}{3}\right)$ which by AM-GM is at most $6\left(\frac{(1-x)+\left(1-\frac{y}{2}\right)+\left(1-\frac{z}{3}\right)+\left(x+\frac{y}{2}+\frac{z}{3}\right)}{4}\right)^4 = \frac{3^5}{2^7} = \frac{243}{128}$.

7. Substituting $x \to 1-x$ gives a system of linear equations, from which $f(x) = \frac{5(x-1)}{x^2-x+1} = \frac{5}{(x-1)+1+\frac{1}{x-1}}$, and by AM-GM this is maximized when $x-1 = \frac{1}{x-1}$ or $x=2$. Then $f(2) = \frac{5}{3}$.

8. The denominator is $(x^2+y^2)^3 + 3x^3y^3$, dividing numerator and denominator by x^3y^3 and simplifying makes the expression $\frac{1}{\left(\frac{x}{y}+\frac{y}{x}\right)^3+3}$. We need to maximize $\frac{x}{y}+\frac{y}{x}$, which occurs when $x = \frac{1}{2}$ and $y = \frac{3}{2}$, making the minimum $\frac{27}{1081}$.

9. Let $r+s=a$ and $rs=b$. The given is $(a-b)(a+b)=b$, so $b^2+b=a^2 \geq 4b$ by AM-GM. Hence $b \geq 3$ and $a \geq 2\sqrt{3}$, which makes the minimum of $r+s-rs=a-b$ as $2\sqrt{3}-3$ and the minimum of $r+s+rs=a+b$ as $2\sqrt{3}+3$, which are achievable.

VCSMS PRIME

Session 9: Geometry 2

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Ad hoc

1. Let $BC = 1$, $AB = 2$. Then $AC = \sqrt{5}$, and $CD = DA = BD = \frac{\sqrt{5}}{2}$ by Thales's. Since $CD = DA$ and they share the same altitude from B, $[BCD] = [BDA] = \frac{1}{2}[ABC] = \frac{1}{2}$. But $[BCD] = \frac{1}{2}CE \cdot BD$, so $CE = \frac{2\sqrt{5}}{5}$. Using the Pythagorean theorem gives BE and ED, then $BE : ED = 2 : 3$.

2. Let the center of the circle be O, the intersection of the diagonals of the square $ABCD$ be E. Let the tangents from A to the circle be AR and AS, with R lying on AE. Let EO intersect the square at T, and let $RE = x$.

   Then $AR = \sqrt{2} - x$ as $AE = \sqrt{2}$, and $AS = AR$ as they are both tangents from A. But clearly $ATOS$ is a rectangle, so $AS = TO$, whence $EO = ET + TO = 1 + \sqrt{2} - x$. From Pythagorean on ERO, we have $EO^2 = RE^2 + RO^2$ or $\left(1 + \sqrt{2} - x\right)^2 = x^2 + 1$, giving $x = 1$ by inspection.

   Then $TO = \sqrt{2} - 1$, and $PO = 1$, so by Pythagorean $PT = \sqrt{2\sqrt{2} - 2}$. PQ is double this, or $2\sqrt{2\sqrt{2} - 2} = \sqrt{8\left(\sqrt{2} - 1\right)}$.

3. Let the perpendicular bisectors of AP and BP intersect at O, and let OP intersect CD again at F. Then $\angle CPF = \angle APO$ due to vertical angles. However, $\angle ABP = \frac{1}{2}\angle AOP = \frac{1}{2}\left(180^\circ - 2\angle APO\right)$ since $AO = OP$ due to it being the circumcenter, and thus AOP is isosceles. This makes $\angle ABP = 90^\circ - \angle APO = 90^\circ - \angle CPF$. But $\angle ABP = \angle DCP$ since $\triangle ABP \cong \triangle DCP$ by SAS. Thus $\angle DCP = \angle FCP = 90^\circ - \angle CPF$, so $\angle FCP + \angle CPF = 90^\circ$ and thus $\angle PFC = 90^\circ$, which is what we wanted.

4. Let $AB = a, BC = b, CD = c, DA = d, PD = p$. Then $[CPD] = \frac{1}{2}cp\sin D$, and $[ABCP] = [ABC] + [ACD] - [CPD] = \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D - \frac{1}{2}cp\sin D$, but $\sin B = \sin D$ since it is a cyclic quadrilateral. Factoring out, $[CPD] = [ABCP]$ implies $cp = ab + cd - cp$, or $2cp = ab + cd$. Equal perimeters imply $2p = a + b - c + d$, substituting yields $ac + bc - c^2 + cd = ab + cd$, which factors as $(c - a)(c - b) = 0$. Thus either $c = a$ or $c = b$.

5. There is a solution using similar triangles, as the official solution: from $PBC \sim PDB$ implies $BC/BD = BP/DP$ and from $PAC \sim PDA$ implies $AC/AD = AP/DP$. Since $AP = BP$, we get $BC/AC = BD/AD$. But from $AEB \sim ABC$, $BC/AC = BE/AB$ and from $AFB \sim ABD$ we get $BD/AD = BF/AB$. Thus $BE/AB = BF/AB$ and $BE = BF$.

   But projective is much nicer. Since AA, BB and CD concur, then $ACBD$ is a harmonic quadrilateral, and $-1 = (A, B; C, D)$. Taking a perspectivity through A to line EF gives us $-1 = (T, B; E, F)$, where T is the point on infinity on EF, from whence B is the midpoint.

Triangles

1. We can construct a lot of altitudes, but trigonometry is cleaner: $DE^2 = DC^2 + EC^2 - 2DC \cdot EC\cos\angle DCE$, but $\cos\angle DCE = \cos\angle ACB = \frac{4}{5}$. Thus $CE = \frac{8}{3}$, so the perimeter of $ABED$ is $\frac{28}{3}$.

2. Let $BC = x$, from which $AB = AF = 2x$ as they are both tangents, $BC = CD = x$ as they are both tangents. For the perimeter to be 36, we must have $EF = DF = 18 - 3x$. Using Pythagorean on ACE gives $x = 0, 3$, where 0 is obviously extraneous. Then $CE = 18 - 2x = 12$.

3. Since $AQC \sim QEC$, we get $AC/QC = QC/EC$, or $QC^2 = EC \cdot AC$. Similarly, $PC^2 = DC \cdot BC$. As $\angle AEB = \angle ADB = 90°$ then $ABDE$ is cyclic and $EC \cdot AC = DC \cdot BC$ by power of a point through C, whence $PC^2 = QC^2$ and $PC = QC$.

4. WLOG $AB < AC$. Use Ptolemy's, Pythagorean, and the given identity to show that $2 \cdot DF(AB+AC) = BC \cdot AC - BC \cdot AB$. Since $EF = EC - FC$, we can find EC using angle bisector theorem and FC is half of BC. Simplifying shows $DF = EF$.

5. Let Z be the midpoint of BC. Since $XYZ \sim ABC$, then $\angle XZY = \angle BAC = \angle XDY$ so $XDZY$ is cyclic. But $\angle XDB = 180° - \angle XDZ = \angle XYZ = \angle ABC$ again since $XYZ \sim ABC$. This implies $XA = XB = XD$, and thus AB is a diameter of (ABD), from which $\angle ADB = 90°$.

Coordinate geometry

1. Let the center of the circle be $Q(0, 2)$ and let P be a point on the circle. From the equation, it has radius 1. When P is on the upper semicircle, the tangent line clearly intersects the y-axis above the circle, so it has a positive y-intercept.

   Consider the point P such that the tangent line through Q passes through the origin $O(0, 0)$. Since it is a tangent, $\angle QPO = 90°$, since it is a radius, $QP = 1$ and we know the distance $QO = 2$. Thus triangle QPO is a $30 - 60 - 90$ triangle. Then $\angle PQO = 60°$.

   There is a $60°$ arc from either side in the lower half, and in this arc everything has non-negative y-intercept. There is the whole upper half from earlier, which makes a total of $60° + 60° + 180° = 300°$. The length of the arcs is thus $\frac{300°}{360°} 2\pi r = \frac{5}{3}\pi$.

2. Shoelace formula gives 144.

3. Assign a mass of $1A$, $1B$ and $2C$. Let E be the midpoint of AB, and G be the intersection of CE and AP. Then $1A + 1B = 2E$, and since $BP : PC = 2 : 1$, we have $1B + 2C = 3P$. Then $4G = 1A + 3P = 2E + 2C$, making G the midpoint of EC.