VCSMS PRIME

Program for Inducing Mathematical Excellence
Session 10: Polynomials
October 13, 2017

Lecture problems

1. What is $(1+i)^{2017}$?
2. A robot's first move is to go east one unit. For its $n+1$st move, it turns $45°$ counterclockwise and travels half the distance of the nth move. How far is the robot from where it started after 2017 moves?
3. Complex numbers x, y, z satisfy $|x| = |y| = |z| = xyz = 1$ and $x+y+z = 0$. Find $|(2+x)(2+y)(2+z)|$.
4. (MMC) A third-degree polynomial satisfies $P(0) = -3$ and $P(1) = 4$. When $P(x)$ is divided by x^2+x+1 the remainder is $2x-1$. In the same division, what is the quotient?
5. (USAMO) Let a, b, c, d be real numbers such that $b-d \geq 5$ and the zeroes x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the minimum of $\left(x_1^2+1\right)\left(x_2^2+1\right)\left(x_3^2+1\right)\left(x_4^2+1\right)$.
6. Prove that $\sin(\pi/n)\sin(2\pi/n)\cdots\sin((n-1)\pi/n) = n/2^{n-1}$.
7. (QII3) Let $f(x)$ be a polynomial of degree 4 with integer coefficients, leading coefficient 1, and having $\sqrt{10}+\sqrt{11}$ as one of its zeroes. What is the sum of its coefficients?
8. (MMC) Find the x-coefficient of the fifth-degree polynomial satisfying $P(3^k) = k$ for $k = 0, 1, \ldots, 5$.
9. (AIME II 2005/13) Let $P(x)$ be a polynomial with integer coefficients satisfying $P(17) = 10$ and $P(24) = 17$. Given $P(n) = n+3$ has two distinct integer solutions, find their product.
10. (QI4) Suppose that r_1 and r_2 are the roots of the equation $4x^2 - 3x - 7 = 0$. What is the sum of the squares of the reciprocals of r_1 and r_2?
11. (Mandelbrot) Find the area of the triangle whose side lengths are the roots of $x^3 - 4x^2 + 5x - 1.9$.
12. (AIME 2005/8) Find the sum of the roots of the equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$.

Completing the reals

- The algebraic completion of the reals are the complex numbers. That is, any polynomial with real coefficients has only complex roots. The complex numbers are algebraically closed.
- We can represent as $z = a + bi$ and think of it as Cartesian coordinates, or $re^{i\theta} = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$ and think of it as polar coordinates. The latter is often more helpful.
- We write $|z| = r$ and $\arg z = \theta$. Its conjugate is written as $\bar{z} = a - bi$ and has the important property $|z|^2 = z\bar{z}$. Conjugation distributes over arithmetic.
- Addition is vectorially or end-to-end. Multiplication is rotation and scaling: the radii multiply, the directions add because of $re^{i\theta}$. This gives de Moivre's theorem.
- Problem 1: It would be hard to expand, so instead use polar coordinates: $\left(\sqrt{2}\operatorname{cis}45°\right)^{2017}$.
- Problem 2: Represent each move with complex numbers, it's the previous move times $\frac{\sqrt{2}}{4}(1+i)$. It's a geometric series with first term 1 and 2017 terms. Compute its sum using problem 1.

Roots of unity

- The nth roots of unity are the n complex roots to $x^n - 1 = 0$. The square roots of unity are 1 and -1, the cube roots are 1, cis $\frac{2\pi}{3}$, and cis $\frac{4\pi}{3}$, the fourth roots are 1, i, -1, and $-i$. Generally, nth roots of unity are cis $\left(\frac{2\pi}{n}k\right)$ for nonnegative k less than n.
- Typically: $x^3 = 1$ but $x \neq 1$. Then if you let ω be the first cube root, ω^2 is the second cube root. Also $x^3 - 1 = (x-1)(x^2+x+1) = 0$, but $x \neq 1$, so $\omega^2 + \omega + 1 = 0$ as it is a root.
- Problem 3: If you see symmetric relations involving complex numbers, guess roots of unity. Which works here, luckily: $x = 1, y = \omega, z = \omega^2$ works. Then expand.

Polynomial division

- The division algorithm is $P(x) = Q(x)D(x) + R(x)$ with obvious relations between degrees. The remainder theorem and factor theorem follow from $D(x) = x - a$ and substitution.
- Problem 4: Write $P(x) = Q(x)(x^2+x+1) + (2x-1)$, then $Q(x)$ is linear, substitute $x = 0, 1$.

Factored form

- An nth degree polynomial can be written as $\prod(x - r_i)$, where r_i are its roots. This goes with substitution. Interpreting factors gives the table of signs, relative maxima and minima, etc.
- Problem 5: Thinking $P(x) = \prod(x - x_i)$ and factoring $\prod\left(x_i^2+1\right) = \prod(x_i - i)(x_i + i)$, we get the inspiration to write the product as $P(i)P(-i)$. The product is at least 16 from the inequality, and it's achievable when all the roots are 1.
- Problem 6: Let $\omega = e^{i\pi/n}$. Recall from session 5 that $\sin(k\pi/n) = (\omega^k - \omega^{-k})/2i$. Then the product is $\prod_{k=1}^{n-1} \frac{\omega^k - \omega^{-k}}{2i} = \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{\omega^{-k}}{i}\left(1 - \omega^{2k}\right)$. The left factor is just $\omega^{-n(n-1)/2}$ divided by i^{n-1}, which cancels. The right factor is substituting 1 in the roots of unity polynomial $x^n - 1 = 0$, $x \neq 1$.

Root theorems

- Polynomials with real coefficients have conjugate non-real roots, polynomials with rational coefficients have conjugate irrational roots. Also, rational root theorem and synthetic division for root-finding.
- Problem 7: The roots are $\pm\sqrt{10} \pm \sqrt{11}$. Its sum of coefficients is $f(1)$, think of its factored form. Or, do $x = \sqrt{10} + \sqrt{11}$ and square twice.

Rewriting polynomials

- Problem 8: Consider the polynomial $P(3x) - P(x) - 1$. It is a fifth-degree polynomial with roots $k = 1, \ldots, 5$, so we can write it as $A(x-1)(x-2)\cdots(x-5)$ for some constant A. Substituting $x = 0$ gives the constant, and we only need to find the x-coefficient.
- Problem 9: We have $P(x) - x + 7 = A(x)(x-17)(x-24)$, also $P(x) - x - 3 = B(x)(x - n_1)(x - n_2)$. Then n_1 and n_2 are the roots of $A(x)(x-17)(x-24) = 10$, trial and error gives $n = 19, 22$.

Roots and coefficients

- Vieta's, rule of signs, discrete Fourier: "find the sum of the coefficients with exponent divisible by 3."
- Problem 10: Directly, $1/r_1^2 + 1/r_2^2 = (r_1 + r_2/r_1r_2)^2 - 2/(r_1r_2)$. There is also an important trick: for a bijection f, the substitution $x \to f^{-1}(x)$ changes the roots to $f(r_i)$. So here, we can try $x \to 1/x$.
- Problem 11: Recall Heron's and factored form. The semiperimeter is half the sum of the roots, or 2.
- Problem 12: Substitute $2^{111x} \to y$. The sum of x is the sum of $\frac{1}{111}\log_2 y$, so Vieta's on products.